*Bull. London Math. Soc.* 36 (2004) 671–682 
DOI: 10.1112/S0024609304003510

# TRANSFORMATION FORMULAS FOR GENERALIZED DEDEKIND ETA FUNCTIONS

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Abstract

Transformation formulas are obtained for generalized Dedekind eta functions; these are simpler to apply than Schoeneberg's formulas. As an application, a list is given of the generators of all the function fields associated with torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

## 1. *Transformation formulas for generalized Dedekind eta functions*

Let $\tau$ be a complex number with $\operatorname{Im}\tau > 0$. The ordinary Dedekind eta function is defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}).$$

This function plays a central role in the study of the theory of modular functions and its applications to other areas. One of the most important properties of the eta function is the transformation formula

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = e^{\pi ik/12}\sqrt{c\tau+d}\,\eta(\tau), \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{1}$$

where $k$ is an integer, and where the exact value of $k$ is often crucial in applications. For this purpose, there are two useful expressions for $k$ in the literature (see, for example, [**10**, Chapter 9]).

Let $((x))$ denote the periodic function

$$((x)) = \begin{cases} x-\lfloor x\rfloor - 1/2, & \text{if } x\notin\mathbb{Z},\\ 0, & \text{if } x\in\mathbb{Z},\end{cases}$$

and define the Dedekind sum $s(h,k)$ by

$$s(h,k) = \sum_{r=1}^{k-1}\left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right).$$

Then one has

$$\log\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \log\eta(\tau) + \begin{cases} \dfrac{\pi ib}{12d}, & \text{for } c=0,\\ \dfrac{1}{2}\log\left(\dfrac{c\tau+d}{i}\right) + \pi is(-d,c) + \dfrac{\pi i(a+d)}{12c}, & \text{for } c>0.\end{cases}$$

This formula clearly carries more information than we need in (1). In applications, the following formula is often sufficient, and more convenient. For instance, it

Received 12 June 2003; revised 16 October 2003.

2000 *Mathematics Subject Classification* 11F20 (primary), 20H05 (secondary).

would be difficult to use the above formula directly to give a useful criterion for a product $\prod \eta(a\tau)^b$ of eta functions to be invariant under $\Gamma_0(N)$. However, using the formula below, Newman [**9**] succeeded in constructing such criteria.

LEMMA 1. *For*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

*the transformation formula for $\eta(\tau)$ is given by*

$$\eta(\tau + b) = e^{\pi i b/12}\eta(\tau), \quad \text{for } c = 0,$$

*and by*

$$\eta(\gamma\tau) = \varepsilon_1(a,b,c,d)\sqrt{\frac{c\tau+d}{i}}\eta(\tau), \quad \text{for } c \neq 0,$$

*with*

$$\varepsilon_1(a,b,c,d) = \begin{cases} \left(\dfrac{d}{c}\right) i^{(1-c)/2} e^{\pi i(bd(1-c^2)+c(a+d))/12}, & \text{if } c \text{ is odd}, \\ \left(\dfrac{c}{d}\right) e^{\pi i(ac(1-d^2)+d(b-c+3))/12}, & \text{if } d \text{ is odd}, \end{cases} \tag{2}$$

*where $\left(\frac{d}{c}\right)$ is the Legendre–Jacobi symbol.*

The main object of this paper is to derive the equivalent of Lemma 1 for a class of generalized Dedekind eta functions studied by, for example, Berndt [**1**], Dieter [**3**], Meyer [**6**, **7**], Miao and Tzeng [**8**], and Schoeneberg [**12**]. Following Schoeneberg's notation [**13**, Chapter 8], we let $N$ be a positive integer, and $g$ and $h$ be real numbers. If we set $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i/N}$, the generalized Dedekind eta functions $\eta_{g,h}(\tau)$ of level $N$ are defined by

$$\eta_{g,h}(\tau) = \alpha(g,h)q^{P_2(g/N)/2} \prod_{\substack{m \geqslant 1 \\ m \equiv g \bmod N}} \left(1 - \zeta^h q^{m/N}\right) \prod_{\substack{m \geqslant 1 \\ m \equiv -g \bmod N}} \left(1 - \zeta^{-h} q^{m/N}\right) \tag{3}$$

with

$$\alpha(g,h) = \begin{cases} (1-\zeta^{-h})e^{\pi i P_1(h/N)}, & \text{if } g \equiv 0, h \not\equiv 0 \bmod N, \\ 1, & \text{otherwise}, \end{cases}$$

$$P_1(x) = \{x\} - 1/2, \qquad P_2(x) = \{x\}^2 - \{x\} + 1/6,$$

where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of a real number $x$, and the notation $\zeta^h$ represents $e^{2\pi i h/N}$.

Clearly, the definition of $\eta_{g,h}$ generalizes that of the ordinary Dedekind eta function, since $\eta_{g,h}$ reduces to $\eta^2$ when $g, h \equiv 0 \bmod N$. However, we remark that, unlike the ordinary eta function, which can be used to construct modular forms of weight greater than zero (see [**5**] for example), the functions $\eta_{g,h}$ alone yield only modular functions (of weight 0). (See Corollaries 1–3 and Section 2 for more details.)

In [**13**, Chapter 8] the transformation formula for $\eta_{g,h}$ under

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

is shown to be

$$\log \eta_{g,h}(\gamma\tau) - \log \eta_{g',h'}(\tau) = \begin{cases} \pi i\left\{ \frac{a}{c} P_2\left(\frac{g}{N}\right) + \frac{d}{c} P_2\left(\frac{g'}{N}\right) - 2 \operatorname{sgn} c \cdot s_{g,h}(a,c) \right\}, & \text{if } c \neq 0, \\ \pi i \frac{b}{d} P_2\left(\frac{g}{N}\right), & \text{if } c = 0, \end{cases}$$

where $g' = ag + ch$, $h' = bg + dh$, and $s_{g,h}(a,c)$ is the generalized Dedekind sum

$$s_{g,h}(a,c) = \sum_{r=0}^{c-1} \left(\left(\frac{g+rN}{cN}\right)\right)\left(\left(\frac{g'+arN}{cN}\right)\right).$$

Again, this result contains more information than we usually need. In light of the two transformation formulas for the ordinary Dedekind eta function, it should be possible to obtain a result analogous to Lemma 1 for the generalized eta functions. In the following theorem we show that this is indeed the case.

THEOREM 1. *Let $N$ be a positive integer, and let $g$ and $h$ be arbitrary real numbers not simultaneously congruent to 0 modulo $N$. For $\tau$ with $\operatorname{Im}\tau > 0$, we set $q = e^{2\pi i\tau}$, and we define the generalized Dedekind eta functions $E_{g,h}(\tau)$ by*

$$E_{g,h}(\tau) = q^{B(g/N)/2} \prod_{m=1}^{\infty} \left(1 - \zeta^h q^{m-1+g/N}\right)\left(1 - \zeta^{-h} q^{m-g/N}\right),$$

*where $\zeta = e^{2\pi i/N}$ and $B(x) = x^2 - x + 1/6$. Then the functions $E_{g,h}$ satisfy*

$$E_{g+N,h} = E_{-g,-h} = -\zeta^{-h} E_{g,h}, \qquad E_{g,h+N} = E_{g,h}. \tag{4}$$

*Moreover, let*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

*Then we have*

$$E_{g,h}(\tau + b) = e^{\pi i b B(g/N)} E_{g,bg+h}(\tau), \quad \text{for } c = 0,$$

*and*

$$E_{g,h}(\gamma\tau) = \varepsilon(a,b,c,d) e^{\pi i \delta} E_{g',h'}(\tau), \quad \text{for } c \neq 0,$$

*where*

$$\varepsilon(a,b,c,d) = \begin{cases} e^{\pi i(bd(1-c^2)+c(a+d-3))/6}, & \text{if } c \text{ is odd}, \\ -ie^{\pi i(ac(1-d^2)+d(b-c+3))/6}, & \text{if } d \text{ is odd}, \end{cases}$$

$$\delta = \frac{g^2ab + 2ghbc + h^2cd}{N^2} - \frac{gb + h(d-1)}{N},$$

*and*

$$(g' \ h') = (g \ h) \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From this formula we can deduce sufficient conditions for a product of generalized eta functions to be modular on $\Gamma(N)$.

COROLLARY 1. *Let $(g,h)$ be pairs of integers, and suppose that $e_{g,h}$ are integers such that*

$$\sum_{(g,h)} e_{g,h} \equiv 0 \mod 12, \tag{5}$$

*and*

$$\sum_{(g,h)} g^2 e_{g,h} \equiv \sum_{(g,h)} ghe_{g,h} \equiv \sum_{(g,h)} h^2 e_{g,h} \equiv 0 \mod 2N. \tag{6}$$

*Then the product $f(\tau) = \prod_{(g,h)} E_{g,h}(\tau)^{e_{g,h}}$ is a modular function on $\Gamma(N)$.*

We note that a formula equivalent to that in Theorem 1 for the special case when $\gamma$ is in the principal congruence group $\Gamma(N)$ of level $N$, and $g$ and $h$ are integers, was obtained in [**6**]. Meyer's method utilized the reciprocity law for generalized Dedekind sums.

We also remark that our definition of generalized Dedekind eta functions is slightly different from (3). In particular, when $g \equiv 0 \mod N$, the two definitions differ by an extra factor $e^{\pi i P_1(h/N)}$, in addition to the $(-1)^{\lfloor g/N \rfloor}$ factor. However, using (4) one can easily translate our result to a formula for the standard generalized eta functions.

Our definition of generalized eta functions was largely inspired by the work of Fine [**4**], in which he used the Jacobi theta function $\vartheta_1(z|\tau)$ to study the transformation law for quotients of generalized Dedekind eta functions of the type $E_{4g,0}(N\tau)/E_{2g,0}(N\tau)$. The functions $E_{g,0}(N\tau)$ will also be the subject of our next corollary. For convenience, let us denote, for real numbers $g$ not congruent to 0 modulo $N$, the function $E_{g,0}(N\tau)$ by $E_g(\tau)$. Assume that

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N).$$

(Without the change of variable $\tau \to N\tau$, we would consider the group $\Gamma^0(N)$ instead.) Using the fact that

$$N\gamma\tau = \frac{a(N\tau)+bN}{c(N\tau)+d} = \begin{pmatrix} a & bN \\ c & d \end{pmatrix}(N\tau)$$

and applying Theorem 1, we obtain the following transformation formula for $E_g$.

COROLLARY 2. *Let $N$ be a positive integer, and let $g$ be an arbitrary real number not congruent to 0 modulo $N$. For $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$, we define the generalized Dedekind eta function $E_g(\tau)$ by*

$$E_g(\tau) = q^{NB(g/N)/2} \prod_{m=1}^{\infty} \left(1 - q^{(m-1)N+g}\right)\left(1 - q^{mN-g}\right),$$

*where $q = e^{2\pi i\tau}$ and $B(x) = x^2 - x + 1/6$. The functions $E_g$ satisfy*

$$E_{g+N} = E_{-g} = -E_g. \tag{7}$$

*Moreover, let $\gamma = \left(\begin{smallmatrix} a & b \\ cN & d \end{smallmatrix}\right) \in \Gamma_0(N)$. We have*

$$E_g(\tau + b) = e^{\pi i bNB(g/N)} E_g(\tau), \quad \text{for } c = 0,$$

*and*

$$E_g(\gamma\tau) = \varepsilon(a, bN, c, d) e^{\pi i(g^2 ab/N - gb)} E_{ag}(\tau), \quad \text{for } c \neq 0,$$

*where $\varepsilon(a,b,c,d)$ is defined as in Theorem 1.*

From Corollary 2 we see that if we choose a set of integers $g$ and $e_g$ suitably, then the function $\prod E_g^{e_g}$ is a modular function on a congruence group. We summarize the conditions in the following corollary. (A proof is given in Section 3.)

COROLLARY 3. *Let $N$ be a positive integer, and consider the function $f(\tau) = \prod_g E_g(\tau)^{e_g}$, where $g$ and $e_g$ are integers, and $E_g$ are defined as in Corollary 2. Suppose that one has*

$$\sum_g e_g \equiv 0 \mod 12 \quad \text{and} \quad \sum_g g e_g \equiv 0 \mod 2. \tag{8}$$

*Then $f$ is invariant under the action of*

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

*Moreover, if, in addition to* (8), *one also has*

$$\sum_g g^2 e_g \equiv 0 \mod 2N, \tag{9}$$

*then $f$ is a modular function on*

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

*Furthermore, for the cases where $N$ is a positive odd integer, the conditions* (8) *and* (9) *can be reduced to*

$$\sum_g e_g \equiv 0 \mod 12$$

*and*

$$\sum_g g^2 e_g \equiv 0 \mod N,$$

*respectively.*

We note that we have used only functions of the same level in the above corollary. In general, we can combine functions of different levels to obtain modular functions on $\Gamma(N)$ and $\Gamma_1(N)$. However, for the application in Section 2, functions of the same level are sufficient. Thus, in order to keep the notation simple, we do not state the result in the most general setting. For a more general result, one can consult [**11**], in which Robins gives a criterion for a product of functions of different levels to be invariant under $\Gamma_1(N)$. In the same paper, Robins also provides a formula for essentially the functions $E_g$ in the special cases when

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$$

with $(a, 6) = 1$.

EXAMPLES.

(i) For $N = 5$, the functions $E_2/E_1$ and $(E_2/E_1)^5$ are invariant under the action of $\Gamma(5)$ and $\Gamma_1(5)$, respectively.

(ii) For $N = 7$, the functions $E_2^2 E_3/E_1^3$ and $E_3^3/(E_1^2 E_2)$ are modular on $\Gamma_1(7)$.

(iii) For $N = 10$, the function $E_3 E_4/(E_1 E_2)$ is modular on $\Gamma_1(10)$.

The rest of the paper is organized as follows. In Section 2 we apply our main results, to obtain generators of function fields associated with a class of subgroups of $\mathrm{PSL}_2(\mathbb{R})$; the proofs of the main results will be given in Section 3.

## 2. *Generators of functions fields associated with congruence groups*

Let $\Gamma$ be a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, and denote by $K(\mathbb{H}^*/\Gamma)$ the field of modular functions invariant under the action of $\Gamma$, where $\mathbb{H}$ is the upper half-plane $\mathrm{Im}\,\tau > 0$, and $\mathbb{H}^*$ denotes $\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$. It is well known that if the genus of the Riemann surface $\mathbb{H}^*/\Gamma$ is zero, then the function field $K(\mathbb{H}^*/\Gamma)$ can be generated by a single function. When the genus zero group $\Gamma$ contains $\Gamma_0(N)$ for some $N$, one can usually find generators that are explicitly expressed as products of the Dedekind eta functions (see [**2**, Table 3]). For example, $\eta(\tau)^4/\eta(7\tau)^4$ is a generator of $K(\mathbb{H}^*/\Gamma_0(7))$.

However, to construct generators for function fields associated with groups not containing $\Gamma_0(N)$, the plain Dedekind eta functions are not sufficient, and we will need additional tools. In this section we will demonstrate that the generalized Dedekind eta functions studied in the previous section will be the 'building blocks' in the case of general genus zero congruence groups. In particular, we will find generators of all $K(\mathbb{H}^*/\Gamma)$ associated with torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

In [**14**], Sebbar showed that there are fifteen $\mathrm{PSL}_2(\mathbb{R})$-conjugacy classes of torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$, and that the representatives of those classes are:

$$\begin{array}{ll}\Gamma(5),\\ \Gamma_1(8)\cap\Gamma(2),\\ \Gamma_0(N) & \text{with } N = 4,6,8,9,12,16,18,\\ \Gamma_1(N) & \text{with } N = 5,7,8,9,10,12.\end{array}$$

Among them, generators of $K(\mathbb{H}^*/\Gamma)$ with $\Gamma$ of the form $\Gamma_0(N)$ are well known (see, for example, [**2**, Table 3]). Thus, we need only to consider the cases where $\Gamma$ is $\Gamma(5)$, $\Gamma_1(8)\cap\Gamma(2)$ or $\Gamma_1(N)$.

We first need the following two lemmas, which give sufficient conditions for a product of generalized Dedekind eta functions to be a generator of a function field. Here in the lemmas, the function $P_2(x)$ denotes the second Bernoulli polynomial

$$P_2(x) = \{x\}^2 - \{x\} + 1/6,$$

where $\{x\} = x - \lfloor x\rfloor$ represents the fractional part of a real number $x$.

LEMMA 2. *Let $N$ be a positive integer, and let $E_g$ be defined as in Corollary 2. Then, given a matrix*

$$\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

*the Fourier expansion of $E_g(\gamma\tau)$ starts from $\varepsilon q^\delta$ + (higher powers), where $|\varepsilon| = 1$ and*

$$\delta = \frac{(c,N)^2}{2N} P_2\left(\frac{ag}{(c,N)}\right).$$

*Proof.* Let $E_{g,h}(\tau)$ be defined as in Theorem 1. Using the elementary identity

$$1 - x^N = \prod_{h=0}^{N-1} \left(1 - \zeta^h x\right),$$

where $\zeta = e^{2\pi i/N}$, we see that

$$E_g(\tau) = \prod_{h=0}^{N-1} E_{g,h}(\tau).$$

Thus, by Theorem 1, we have

$$E_g(\gamma\tau) = \varepsilon' \prod_{h=0}^{N-1} E_{ag+ch,bg+dh}(\tau)$$

for some $\varepsilon'$ with $|\varepsilon'| = 1$. It follows from (4) that $E_g(\gamma\tau) = \varepsilon q^\delta + \ldots$, where

$$\delta = \frac{1}{2} \sum_{h=0}^{N-1} P_2\left(\frac{ag+ch}{N}\right)$$

and $|\varepsilon| = 1$. We now show that the $\delta$ in the last expression is the same as that given in the statement of the lemma.

We note that $P_2(x)$ is a periodic function, and its Fourier expansion is easily verified to be

$$P_2(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2}. \tag{10}$$

We therefore have

$$\delta = \frac{1}{2\pi^2} \sum_{h=0}^{N-1} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{2\pi n(ag+ch)}{N}\right).$$

Since the double sum converges absolutely, we can change the order of summation. Moreover, using the fact that

$$\sum_{h=0}^{N-1} \cos\left(\frac{2\pi n(ag+ch)}{N}\right) = \begin{cases} N\cos(2\pi nag/N), & \text{if } N/(c,N) \big| n, \\ 0, & \text{otherwise,} \end{cases}$$

we reduce $\delta$ to

$$\delta = \frac{N}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{(mN/(c,N))^2} \cos\left(\frac{2\pi mag}{(c,N)}\right) = \frac{(c,N)^2}{2N\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos\left(\frac{2\pi mag}{(c,N)}\right),$$

which, in view of (10), is exactly the expression given in the assertion. This proves the lemma. □

LEMMA 3. *Let $\Gamma$ be a congruence subgroup of level $N$, and suppose that the width of the cusp $\infty$ is $m$. Let $f(\tau) = \prod_g E_g^{e_g}$ be a modular function in $K(\mathbb{H}^*/\Gamma)$. Suppose that*

(i) *$f$ has a Fourier expansion $q^{-1/m} + a_0 + a_1 q^{1/m} + \ldots$;*

(ii) *for all $a/c \in \mathbb{Q}$ inequivalent to $\infty$ in $\Gamma$, where $(a,c) = 1$, one has*

$$\sum_g e_g P_2\left(\frac{ag}{(c,N)}\right) \geqslant 0.$$

*Then the genus of $\Gamma$ is zero, and $f$ is a generator of $K(\mathbb{H}^*/\Gamma)$.*

*Proof.* Let

$$\sigma = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $f$ generates $K(\mathbb{H}^*/\Gamma)$ if and only if $f|_\sigma = q^{-1} + \dots$ generates $K(\mathbb{H}^*/\sigma^{-1}\Gamma\sigma)$. Thus, we may assume that $m = 1$. Now, condition (i) implies that $f$ has a pole of order 1 at $\infty$, and, by Lemma 2, condition (ii) implies that $f$ has no poles at cusps inequivalent to $\infty$. Furthermore, when $\operatorname{Im}\tau > 0$, the infinite product defining $f$ converges absolutely. In particular, $f$ has no poles in $\mathbb{H}$. Therefore, $f$ has only one pole of order 1 at $\infty$ in $\mathbb{H}^*/\Gamma$, and $f$ is a homeomorphism from $\mathbb{H}^*/\Gamma$ to the Riemann sphere. It follows that the genus of $\Gamma$ is zero, and $f$ is a generator of $K(\mathbb{H}^*/\Gamma)$. □

With the above lemmas we can now find the generators as follows. Let $\Gamma_1(N)$ be one of the groups mentioned above. Since there are essentially $\nu = \lfloor N/2 \rfloor$ distinct $E_g$ for each $N$, in light of Corollary 3 and Lemma 3, we need only to find solutions of the following equations and inequalities:

$$\sum_{g=1}^{\nu} e_g = 0,$$

$$\sum_{g=1}^{\nu} g^2 e_g \equiv 0 \mod \begin{cases} 2N, & \text{if } N \text{ is even,} \\ N, & \text{if } N \text{ is odd,} \end{cases}$$

$$\frac{N}{2} \sum_{g=1}^{\nu} e_g P_2(g/N) = -1,$$

and

$$\sum_{g=1}^{\nu} e_g P_2\left(\frac{ag}{(c,N)}\right) \geqslant 0$$

for all $c|N$ and $1 \leqslant a \leqslant c-1$ satisfying $(a,c) = 1$ and $(a\ c) \not\equiv (\pm 1\ 0) \mod N$. This gives a method of finding generators of $K(\mathbb{H}^*/\Gamma_1(N))$. A similar method also yields a generator of $K(\mathbb{H}^*/\Gamma(5))$.

For the group $\Gamma_1(8) \cap \Gamma(2)$, we can easily check that we need only to replace the above conditions by

$$\sum_{g=1}^{4} e_g = 0,$$

$$\sum_{g=1}^{4} g^2 e_g \equiv 0 \mod 8,$$

$$4 \sum_{g=1}^{4} e_g P_2(g/8) = -1/2,$$

and

$$\sum_{g=1}^{4} e_g P_2\left(\frac{ag}{(c,8)}\right) \geqslant 0$$

for all $c|8$ and $1 \leqslant a \leqslant c-1$ satisfying $(a,c) = 1$ and $(a\ c) \not\equiv (\pm 1\ 0) \mod 8$. We list our findings in Table 1. Here the generators of $K(\mathbb{H}^*/\Gamma_0(N))$ are taken from [**2**, Table 3], and the notation $\prod a^r$ is an abbreviation for $\prod \eta(a\tau)^r$.

## 3. *Proofs of Theorem* 1 *and Corollaries* 1 *and* 3

To prove our main results, we require the following transformation formula for the Jacobi theta function $\vartheta_1(z|\tau)$. (For a proof of the lemma, see [**10**, Chapter 10].)

LEMMA 4. *For $z, \tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$, let the Jacobi theta function $\vartheta_1(z|\tau)$ be given by*

$$\vartheta_1(z|\tau) = -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{\pi i(2n+1)z},$$

*where $q$ denotes $e^{2\pi i\tau}$. The function $\vartheta_1$ has the infinite product representation*

$$\vartheta_1(z|\tau) = -iq^{1/8} e^{\pi iz} \prod_{m=1}^{\infty} (1-q^m)\left(1-q^m e^{2\pi iz}\right)\left(1-q^{m-1}e^{-2\pi iz}\right). \tag{11}$$

*Moreover, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the Jacobi function $\vartheta_1$ satisfies*

$$\vartheta_1(z|\tau + b) = e^{\pi ib/4}\vartheta_1(z|\tau), \quad \text{for } c = 0;$$

*and*

$$\vartheta_1\left(\frac{z}{c\tau+d}\middle|\gamma\tau\right) = \varepsilon_2(a,b,c,d)\sqrt{\frac{c\tau+d}{i}}\, e^{\pi icz^2/(c\tau+d)}\vartheta_1(z|\tau), \quad \text{for } c \neq 0,$$

*with*

$$\varepsilon_2(a,b,c,d) = -i\varepsilon_1(a,b,c,d)^3, \tag{12}$$

*where $\varepsilon_1(a,b,c,d)$ is given by* (2).

We are now ready to prove our results.

TABLE 1. *Generators of the function fields associated with torsion-free genus zero congruence subgroups of* $\mathrm{PSL}_2(\mathbb{R})$.

| Group | Generator |
|---|---|
| $\Gamma_0(4)$ | $1^8/4^8$ |
| $\Gamma_0(6)$ | $2^3\cdot 3^9/1^3\cdot 6^9$ |
| $\Gamma_0(8)$ | $1^4\cdot 4^2/2^2\cdot 8^4$ |
| $\Gamma_0(9)$ | $1^3/9^3$ |
| $\Gamma_0(12)$ | $4^4\cdot 6^2/2^2\cdot 12^4$ |
| $\Gamma_0(16)$ | $1^2\cdot 8/2\cdot 16^2$ |
| $\Gamma_0(18)$ | $6\cdot 9^3/3\cdot 18^2$ |
| $\Gamma_1(5)$ | $(E_2/E_1)^5,\ N=5$ |
| $\Gamma_1(7)$ | $E_2^2E_3/E_1^3,\ N=7$ |
| $\Gamma_1(8)$ | $(E_3/E_1)^2,\ N=8$ |
| $\Gamma_1(9)$ | $E_2E_4/E_1^2,\ N=9$ |
| $\Gamma_1(10)$ | $E_3E_4/(E_1E_2),\ N=10$ |
| $\Gamma_1(12)$ | $E_5/E_1,\ N=12$ |
| $\Gamma(5)$ | $E_2/E_1,\ N=5$ |
| $\Gamma_1(8)\cap\Gamma(2)$ | $E_3/E_1,\ N=8$ |

*Proof of Theorem* 1. The proof of (4) is straightforward, and we proceed to prove the second claim. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We first consider the easier case, $c = 0$. From the definition of $E_{g,h}$ we have

$$\begin{aligned} E_{g,h}(\tau + b) &= e^{\pi i b B(g/N)} q^{B(g/N)/2} \prod_{m=1}^{\infty} \left(1 - \zeta^{bg+h} q^{m-1+g/N}\right)\left(1 - \zeta^{-bg-h} q^{m-g/N}\right) \\ &= e^{\pi i b B(g/N)} E_{g,bg+h}(\tau). \end{aligned}$$

This proves the assertion for the case $c = 0$.

Now we consider the case $c \neq 0$. Setting $z = -(g\tau + h)/N$ in the infinite product representation (11) for $\vartheta_1$, we obtain

$$\begin{aligned} \vartheta_1\left(-\frac{g\tau+h}{N}\middle|\tau\right) &= -iq^{1/8}\zeta^{-h/2}q^{-g/(2N)} \\ &\quad \times \prod_{m=1}^{\infty}(1-q^m)\left(1-\zeta^{-h}q^{m-g/N}\right)\left(1-\zeta^{h}q^{m-1+g/N}\right) \\ &= -i\zeta^{-h/2}q^{-g^2/(2N^2)}E_{g,h}(\tau)\eta(\tau). \end{aligned} \tag{13}$$

We now apply the modular transformation $\tau \to \gamma\tau$ on the identity above. On the one hand, we have

$$\vartheta_1\left(-\frac{g\gamma\tau+h}{N}\middle|\gamma\tau\right) = \vartheta_1\left(-\frac{g'\tau+h'}{N(c\tau+d)}\middle|\gamma\tau\right)$$

with $g' = ag + ch$ and $h' = bg + dh$. It follows from Lemma 4 that

$$\vartheta_1\left(-\frac{g\gamma\tau+h}{N}\middle|\gamma\tau\right) = \varepsilon_2\sqrt{\frac{c\tau+d}{i}}e^{\pi i c v^2/(c\tau+d)}\vartheta_1(v|\tau),$$

where $\varepsilon_2 = \varepsilon_2(a,b,c,d)$ is the multiplier (12) given in Lemma 4, and

$$v = -\frac{g'\tau+h'}{N} = -\frac{(ag+ch)\tau+(bg+dh)}{N}.$$

Using (13), we thus obtain

$$\vartheta_1\left(-\frac{g\gamma\tau+h}{N}\middle|\gamma\tau\right) = -i\varepsilon_2\sqrt{\frac{c\tau+d}{i}}e^{\pi i c v^2/(c\tau+d)}\zeta^{-h'/2}q^{-(g')^2/(2N^2)}E_{g',h'}(\tau)\eta(\tau).$$

On the other hand, from (13) we also have

$$\begin{aligned} \vartheta_1\left(-\frac{g\gamma\tau+h}{N}\middle|\gamma\tau\right) &= -i\zeta^{-h/2}e^{-\pi i g^2\gamma\tau/N^2}E_{g,h}(\gamma\tau)\eta(\gamma\tau) \\ &= -i\varepsilon_1\zeta^{-h/2}e^{-\pi i g^2\gamma\tau/N^2}\sqrt{\frac{c\tau+d}{i}}E_{g,h}(\gamma\tau)\eta(\tau), \end{aligned}$$

where we have used Lemma 1 and $\varepsilon_1 = \varepsilon_1(a,b,c,d)$ as defined by (2). Combining the two expressions above, we therefore see that

$$E_{g,h}(\gamma\tau) = \frac{\varepsilon_2}{\varepsilon_1}\zeta^{(h-h')/2}e^{\pi i f/N^2}E_{g',h'}(\tau),$$

where

$$f = \frac{cv^2N^2}{c\tau+d} - (g')^2\tau + g^2\gamma\tau = \frac{c(g'\tau+h')^2}{c\tau+d} - (g')^2\tau + g^2\frac{a\tau+b}{c\tau+d}$$
$$= \frac{1}{c\tau+d}\Big\{ \left(g^2(a-a^2d+2abc) + 2ghbc^2 + h^2c^2d\right)\tau + \left(g^2(b+bc^2) + 2ghbcd + h^2cd^2\right)\Big\}.$$

With the condition $ad-bc=1$, we can simplify $f$ to

$$f = \frac{1}{c\tau+d}\Big\{ \left(g^2abc + 2ghbc^2 + h^2c^2d\right)\tau + \left(g^2abd + 2ghbcd + h^2cd^2\right)\Big\}$$
$$= g^2ab + 2ghbc + h^2cd.$$

Moreover, the explicit expressions (2) and (12) for $\varepsilon_1$ and $\varepsilon_2$ show that

$$\frac{\varepsilon_2}{\varepsilon_1} = -i\varepsilon_1^2 = \begin{cases} e^{\pi i(bd(1-c^2)+c(a+d-3))/6}, & \text{if } c \text{ is odd,} \\ -ie^{\pi i(ac(1-d^2)+d(b-c+3))/6}, & \text{if } d \text{ is odd,} \end{cases}$$

while we have $\zeta^{(h-h')/2} = e^{\pi i(-gb+h-hd)/N}$. Therefore, we conclude that

$$E_{g,h}(\gamma\tau) = \varepsilon(a,b,c,d)e^{\pi i\delta}E_{g',h'}(\tau)$$

with $\varepsilon$, $\delta$, $g'$ and $h'$ given as in the statement of the result.

This completes the proof. □

*Proof of Corollary* 1. Let

$$\gamma = \begin{pmatrix} 1+aN & bN \\ cN & 1+dN \end{pmatrix} \in \Gamma(N).$$

By the assumption that $\sum_{(g,h)} e_{g,h} \equiv 0 \mod 12$ and Theorem 1, we have

$$f(\gamma\tau) = \exp\left\{\pi i\left(\frac{(1+aN)b}{N}\sum_{(g,h)} g^2e_{g,h} + \frac{(1+cN)d}{N}\sum_{(g,h)} h^2e_{g,h} - \sum_{(g,h)}(bg+dh)e_{g,h}\right)\right\}\prod_{(g,h)} E_{g+(ag+ch)N,h+(bg+dh)N}(\tau).$$

It follows from (4) and condition (6) that

$$f(\gamma\tau) = \prod_{(g,h)}(-\zeta^{-h})^{(ag+ch)e_{g,h}}E_{g,h}(\tau)^{e_{g,h}} = \prod_{(g,h)} E_{g,h}(\tau)^{e_{g,h}} = f(\tau).$$

This shows that the function $f(\tau)$ is a modular function on $\Gamma(N)$. □

*Proof of Corollary 3.* Let

$$\gamma = \begin{pmatrix} 1+aN & b \\ cN & d \end{pmatrix} \in \Gamma_1(N).$$

The condition $\sum e_g \equiv 0 \mod 12$ implies that

$$f(\gamma\tau) = \exp\left\{\pi i\left((1+aN)b\sum g^2e_g/N - b\sum ge_g\right)\right\}\prod E_{g(1+aN)}(\tau)^{e_g}.$$

Thus, if the integers $b$, $g$ and $e_g$ satisfy

$$\sum ge_g \equiv 0 \mod 2 \quad \text{and} \quad b\sum g^2e_g \equiv 0 \mod 2N,$$

then, by (7), we have

$$f(\gamma\tau) = \prod E_{g+agN}(\tau)^{e_g} = \prod (-1)^{age_g} E_g(\tau)^{e_g} = f(\tau).$$

This gives the conditions (8) and (9) for the function $f$ to be invariant under the action of $\Gamma(N)$ and $\Gamma_1(N)$, respectively.

When $N$ is a positive odd integer, we can use the property (7) to express $f$ as a product of $E_g$ where all indices $g$ are even integers. To be more precise, we have

$$f(\tau) = \prod_{g \text{ even}} E_g^{e_g} \prod_{g \text{ odd}} E_{N-g}^{e_g}.$$

This shows that the condition $\sum ge_g \equiv 0 \mod 2$ can always be fulfilled for any such functions $f$. This completes the proof of the corollary. □

*Acknowledgements.* The author would like to thank Professor M.-L. Lang of the National University of Singapore for suggesting the application described in Section 2, and for providing relevant references.

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