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Journal of Algebra 279 (2004) 22–37

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Equivalence classes of Vogan diagrams ☆

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Received 15 December 2002

Communicated by Geoffrey Mason

Abstract

A Vogan diagram is a Dynkin diagram with an involution, and the vertices fixed by the involution may be painted. They represent real simple Lie algebras, and two diagrams are said to be equivalent if they represent the same Lie algebra. In this article we classify the equivalence classes of all Vogan diagrams. In doing so, we find that the underlying Dynkin diagrams have certain properties in graph painting. We show that this combinatorial property provides an easy classification for most of the simply-laced Dynkin diagrams.

Keywords: Vogan diagram; Dynkin diagram; Simple Lie algebra; Graph painting

1. Introduction

A Vogan diagram [4] is a Dynkin diagram with two extra data: There is an automorphism θ on the diagram with $\theta^2 = 1$, and the vertices fixed by θ may be painted or unpainted. Each Vogan diagram corresponds to a real simple Lie algebra. Two diagrams are said to be equivalent if they represent the same Lie algebra. We are interested in equivalence classes of the Vogan diagrams. In this respect, we can ignore once and for all the diagrams with no painted vertex, as they represent Lie algebras without noncompact imaginary root and so cannot be equivalent to any diagram with painted vertices. Then the Borel–de Siebenthal theorem [3] says that every Vogan diagram is equivalent to one with a single painted vertex. However, it does not give the explicit equivalence. We shall develop algorithms which convert a diagram to an equivalent one with fewer painted vertices. As a result, not only we

☆ This work is supported in part by the National Science Council of Taiwan.
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0021-8693/$ – see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/j.jalgebra.2003.10.011

have reproved the Borel–de Siebenthal theorem, we give the equivalence classes explicitly. We shall label the vertices of the underlying Dynkin diagram with $1, \ldots, n$. Then the Vogan diagram with vertices $i_1, \ldots, i_k$ painted, where $i_1 < \cdots < i_k$ is denoted by $(i_1, \ldots, i_k)$. For diagrams with $\theta = 1$, the equivalence classes are listed in Table 1.

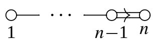
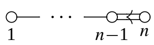
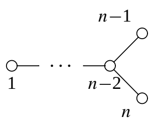
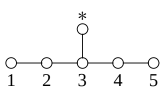
The left column labels the vertices with $1, 2, 3, \ldots$ and so on. The middle column lists the diagrams with single painted vertex, for example, (2) corresponds to the diagram with vertex 2 painted. The right column provides all the Vogan diagrams in their equivalence classes. For example, if we consider $(1, 3, 4)$ in A_5, then the formula $i_3 - i_2 + i_1 = 4 - 3 + 1 = 2$ says that it is equivalent to the diagram with vertex 2 painted.

It turns out that E_n are the most complicated ones. The following methods explain how to use Table 1 for Vogan diagrams of E_n:

(1) Diagrams in the following special cases:

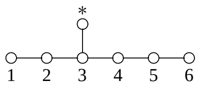
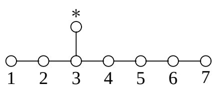
$$
\begin{aligned}
&(2,4), (1,3,4), (3,5), (2,4,*), (1,3,4,*), (3,5,*),\\
&(3,4,6), (3,4,5,6), (3,4,6,*), (3,4,5,6,*) \quad \text{in } E_6 \text{ and } E_7.
\end{aligned} \tag{1.1}
$$

Table 1

Dynkin diagram	Single painted vertex	Equivalent diagrams
A_n 	(N), $1 \leqslant N \leqslant (n+1)/2$	$(i_1, \ldots, i_k)$, $\sum_{p=1}^{k} (-1)^{k-p} i_p = N, n+1-N$
B_n 	$(N), 1 \leqslant N \leqslant n$	$(i_1, \ldots, i_k)$, $\sum_{p=1}^{k} (-1)^{k-p} i_p = N$
C_n 	(n) $(N),\ 1 \leqslant N \leqslant n/2$	$(i_1, \ldots, i_k, n)$ $(i_1, \ldots, i_k),\ i_k \leqslant n-1,\ \sum_{p=1}^{k} (-1)^{k-p} i_p = N,\ n-N$
D_n 	$(N),\ 1 \leqslant N \leqslant n/2$ (n)	$(i_1, \ldots, i_k),\ i_k \leqslant n-2,\ \sum_{p=1}^{k} (-1)^{k-p} i_p = N,\ n-N$ $(i_1, \ldots, i_k, n-1, n),\ \sum_{p=1}^{k} (-1)^{k-p} i_p = N-1,\ n-N-1$ $(n-1),\ (i_1, \ldots, i_k, n-1),\ (i_1, \ldots, i_k, n)$
E_6 	(1) $(*)$	(5), (2, 4), (1, 3, 4), $(1,*)$, $(2,*)$, $(4,*)$, $(5,*)$, $(3,5,*)$ $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1$, $J = \begin{cases} 2-I \text{ and } I+s \text{ is odd}, \\ 4-I \text{ and } I+s \text{ is even}, \\ 1-I. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),\ j_1 = \begin{cases} 4+I \text{ and } I+s \text{ is odd}, \\ 1+I. \end{cases}$ (2), (3), (4), (3, 5), $(3,*)$, $(2,4,*)$, $(1,3,4,*)$ $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1$, $J = \begin{cases} 2-I \text{ and } I+s \text{ is even}, \\ 4-I \text{ and } I+s \text{ is odd}, \\ 3-I. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),\ j_1 = \begin{cases} 4+I \text{ and } I+s \text{ is even}, \\ 3+I. \end{cases}$

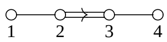
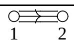
(continued on next page)

Table 1 (*continued*)

Dynkin diagram	Single painted vertex	Equivalent diagrams
	(1)	(2), (3), (5), (3, 5), (3, 4, 6), (3, 4, 5, 6), (4, ∗), (6, ∗), (2, 4, ∗), (1, 3, 4, ∗) $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1,$ $J = \begin{cases} 1 - I,\ 3 - I, \text{ and } I + s \text{ is odd}, \\ 2 - I,\ 4 - I, \text{ and } I + s \text{ is even}. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),$ $j_1 = \begin{cases} 1 + I,\ 2 + I,\ 3 + I,\ 5 + I, \text{ and } I + s \text{ is even}, \\ 4 + I \text{ and } I + s \text{ is odd}. \end{cases}$
E_7 	(6)	(2, 4), (1, 3, 4), (1, ∗), (2, ∗), (5, ∗), (3, 5, ∗), (3, 4, 6, ∗), (3, 4, 5, 6, ∗) $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1,$ $J = \begin{cases} 1 - I \text{ and } I + s \text{ is even}, \\ 2 - I \text{ and } I + s \text{ is odd}. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),\ j_1 = 1 + I,\ 2 + I,\ 5 + I, \text{ and } I + s \text{ is odd}.$
	(∗)	(4), (3, ∗) $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1,$ $J = \begin{cases} 3 - I \text{ and } I + s \text{ is even}, \\ 4 - I \text{ and } I + s \text{ is odd}. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),$ $j_1 = \begin{cases} 3 + I \text{ and } I + s \text{ is odd}, \\ 4 + I \text{ and } I + s \text{ is even}. \end{cases}$
	(7)	(2), (3), (6), (1, ∗), (2, ∗), (5, ∗), (6, ∗) $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1,$ $J = \begin{cases} 1 - I,\ 5 - I, \text{ and } I + s \text{ is even}, \\ 3 - I \text{ and } I + s \text{ is odd}, \\ 2 - I,\ 6 - I. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),$ $j_1 = \begin{cases} 1 + I,\ 5 + I, \text{ and } I + s \text{ is odd}, \\ 3 + I \text{ and } I + s \text{ is even}, \\ 2 + I, 6 + I. \end{cases}$
E_8 	(∗)	(1), (4), (5), (3, ∗), (4, ∗), (7, ∗) $(i_1, \ldots, i_k, j_1, \ldots, j_l, s),\ l \neq 1,$ $J = \begin{cases} 1 - I,\ 5 - I, \text{ and } I + s \text{ is odd}, \\ 3 - I \text{ and } I + s \text{ is even}, \\ 4 - I. \end{cases}$ $(i_1, \ldots, i_k, j_1, s),$ $j_1 = \begin{cases} 1 + I,\ 5 + I, \text{ and } I + s \text{ is even}, \\ 3 + I \text{ and } I + s \text{ is odd}, \\ 4 + I. \end{cases}$

(*continued on next page*)

Table 1 (*continued*)

Dynkin diagram	Single painted vertex	Equivalent diagrams
F_4 	(1)	$(i_1, \ldots, i_k),\ \{i_1, \ldots, i_k\} \cap \{1, 2\} \neq \emptyset$
	(4)	$(i_1, \ldots, i_k),\ \{i_1, \ldots, i_k\} \cap \{1, 2\} = \emptyset$
G_2 	(1)	(2), $(1, 2)$

Obviously we disregard the second row of (1.1) in E_6 because there is no vertex 6. Their equivalence classes can be found directly in Table 1.

(2) Diagrams not in (1.1):
Write it in the form

$$(i_1, \ldots, i_k, j_1, \ldots, j_l, s), \tag{1.2}$$

where $1 \leqslant i_1 < \cdots < i_k \leqslant 3 < j_1 < \cdots < j_l \leqslant n - 1$, and s is either $*$ or empty depending on whether $*$ is painted or not. In this case, let

$$I = \sum_{p=1}^{k} (-1)^{k-p} i_p \quad (I = 0 \text{ if no } i \text{ appears}),$$

$$J = \sum_{p=1}^{l} (-1)^{l-p} j_p \quad (J = 0 \text{ if no } j \text{ appears}). \tag{1.3}$$

In computing the sign of $I + s$, we make the convention that $s = *$ is odd and $s = \emptyset$ is even. Then find the equivalence class in Table 1.

Note that method (2) cannot be used against the diagrams in (1.1), because that would lead to the wrong equivalence classes. The significance of (1.1) will be explained in Proposition 3.2.

For example, consider $(1, 2, 3, 5, *)$ in E_7, which is not in (1.1). We see that $l = 1$,

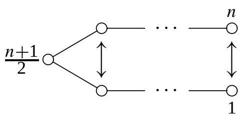
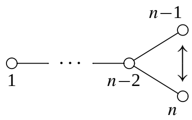
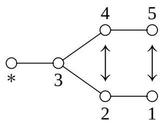
$$I = i_3 - i_2 + i_1 = 3 - 2 + 1 = 2 \quad \text{and} \quad J = j_1 = 5 = 3 + I.$$

Here $s = *$, and $I + s = 2 + *$ is odd. By Table 1, $(1, 2, 3, 5, *) \sim (*)$ in E_7.

We shall prove Table 1, for the classical diagrams in Section 2, and the exceptional diagrams in Section 3. We shall only prove the equivalence of each grouping in Table 1. We need not prove inequivalence of different groupings, since this is done in [4]. For instance [4, p. 355] says that in A_4, (1) is $su(1, 4)$, and (2) is $su(2, 3)$, so (1) and (2) are not equivalent.

Next we consider the Vogan diagrams with nontrivial involutions θ. Here θ imposes a symmetry requirement on the underlying Dynkin diagrams, and the only vertices fixed by θ may be painted. Therefore, such Vogan diagrams are limited. They are listed in Table 2, together with their equivalence classes, where "$\leftrightarrow$" indicates the two-element-orbits of θ.

Table 2

Dynkin diagram		Single painted vertex	Equivalent diagrams
A_n n odd		$(n+1)/2$	
D_n		$(N),\ N \leqslant (n-1)/2$	$(i_1,\dots,i_k),\ i_k \leqslant n-2,$ $\sum_{p=1}^{k}(-1)^{k-p} i_p = N,\ n-N-1$
E_6		$(*)$	$(3),\ (3,*)$

Once again, we ignore the ones without painted vertex, which are obviously not equivalent to any other diagram.

We shall prove Table 2 in Section 4. Tables 1 and 2 confirm the Borel–de Siebenthal theorem. Their proofs use some algorithms $F[i]$ (see (2.1)) which reduce the number of painted vertices to one. In Section 5, we show that these algorithms lead to a necessary condition for a graph to be Dynkin (Corollary 5.2). We shall see that this necessary condition is almost sufficient, thereby providing a very easy classification for almost all simply-laced Dynkin diagrams.

2. Classical diagrams

In this section we consider Vogan diagrams of types A, B, C, D in Table 1, with $\theta = 1$. We label their vertices with $1,\dots,n$ as in Table 1. A Vogan diagram with painted vertices $i_1,\dots,i_k$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, is denoted by $(i_1,\dots,i_k)$. Suppose that $i \in \{i_1,\dots,i_k\}$, so that i is a painted vertex. We introduce an operation $F[i]$ on the Vogan diagram as follows. Let $F[i]$ act on the root system by reflection corresponding to the noncompact simple root i. As a result, it leads an equivalent Vogan diagram. The effect of $F[i]$ on the Vogan diagram is as follows (developed in [1], see also [2, p. 89]):

$$F[i]: \quad \begin{cases} \bullet \text{ The colors of } i \text{ and all vertices not adjacent to } i \text{ remain unchanged.} \\ \bullet \text{ If } j \text{ is joined to } i \text{ by a double edge and } j \text{ is long, the color of } j \\ \quad \text{remains unchanged.} \\ \bullet \text{ Apart from the above exceptions, reverse the colors of all vertices} \\ \quad \text{adjacent to } i. \end{cases} \tag{2.1}$$

For instance, if we apply $F[4]$ to $(1,3,4,7)$, then we reverse the colors of $3,5$ and get $(1,4,5,7)$. Thus $(1,3,4,7)$ is equivalent to $(1,4,5,7)$.

Using the operation $F[i]$, the next lemma shows that a pair of painted vertices can be shifted leftward or rightward.

Lemma 2.1. *Let* $i_1 < \cdots < i_k$.

(a) $(i_1, \ldots, i_k) \sim (i_1, \ldots, i_{r-1}, i_r - c, i_{r+1} - c, i_{r+2}, \ldots, i_k)$ *whenever* $i_{r-1} < i_r - c$.
(b) $(i_1, \ldots, i_k) \sim (i_1, \ldots, i_{r-1}, i_r + c, i_{r+1} + c, i_{r+2}, \ldots, i_k)$ *whenever* $i_{r+1} + c < i_{r+2}$. *We require* $i_{r+1} + c \leqslant n - 1$ *in* C_n *and* $i_{r+1} + c \leqslant n - 2$ *in* D_n.

Proof. We now prove (a). Suppose we want to move i_r, i_{r+1} leftward c steps, where $i_{r-1} < i_r - c$. It is equivalent to moving them 1 step for c times, namely it suffices to show that

$$(i_1, \ldots, i_k) \sim (i_1, \ldots, i_{r-1}, i_r - 1, i_{r+1} - 1, i_{r+2}, \ldots, i_k). \tag{2.2}$$

By applying $F[i_r + 1], F[i_r + 2], \ldots, F[i_{r+1} - 1]$ consecutively to $(i_1, \ldots, i_k)$, we get (2.2), and (a) follows.

The proof of (b) is similar. The restrictions on C_n, D_n are added because $F[n-1]$ does not change the color of n in C_n, and $F[n-2]$ changes the colors of $n-1, n$ in D_n. □

For example, in $(1, 5, 7, 9)$, we can move the pair 5, 7 leftward three steps and get $(1, 5, 7, 9) \sim (1, 2, 4, 9)$. The following lemma provides a way to reduce the number of painted vertices.

Lemma 2.2. *In* A_n, B_n, $(i_1, \ldots, i_k) \sim (i_2 - i_1, i_3, \ldots, i_k)$. *If* $i_2 \leqslant n - 1$, *this is true in* C_n. *If* $i_2 \leqslant n - 2$, *this is true in* D_n.

Proof. We divide the arguments for $(i_1, \ldots, i_k)$ into two cases.

Case 1: $i_1 = 1$. If $i_2 = 2$ then $F[1](1, 2, i_3, \ldots, i_k) = (1, i_3, \ldots, i_k)$ and we are done. So suppose that $i_2 > 2$. Apply $F[1], F[2], \ldots, F[i_2 - 1]$ to $(1, i_2, \ldots, i_k)$, we get $(1, i_2, \ldots, i_k) \sim (i_2 - 1, i_3, \ldots, i_k)$. This solves Case 1.

Case 2: $i_1 > 1$. By Lemma 2.1(a), $(i_1, \ldots, i_k) \sim (1, i_2 - i_1 + 1, i_3, \ldots, i_k)$. This is reduced to case 1, so we get $(1, i_2 - i_1 + 1, i_3, \ldots, i_k) \sim (i_2 - i_1, i_3, \ldots, i_k)$. This solves Case 2.

The extra conditions are imposed to deal with the special cases of $F[n-1]$ in C_n and $F[n-2]$, $F[n-1]$, $F[n]$ in D_n, as explained in Lemma 2.1. This proves the lemma. □

Proposition 2.3. *In* A_n *and* B_n, $(i_1, \ldots, i_k) \sim (\sum_{p=1}^{k} (-1)^{k-p} i_p)$.

Proof. Consider $(i_1, \ldots, i_k)$ in A_n or B_n. By Lemma 2.2,

$$\begin{aligned}(i_1, \ldots, i_k) &\sim (i_2 - i_1, i_3, \ldots, i_k) \sim (i_3 - i_2 - i_1, i_4, \ldots, i_k) \\ &\sim \cdots \sim \left(\sum_{p=1}^{k} (-1)^{k-p} i_p \right).\end{aligned} \tag{2.3}$$

This proves the proposition. □

Obviously $(N) \sim (n+1-N)$ in A_n, by symmetry of the diagram. Therefore, by Proposition 2.3, we have verified the equivalence classes of diagrams of types A and B in Table 1. The next proposition considers the type C diagrams of Table 1. The argument is similar unless the vertex n is painted.

Proposition 2.4. *Consider $(i_1, \ldots, i_k)$ in C_n.*

(a) *If $i_k < n$, then $(i_1, \ldots, i_k) \sim (\sum_{p=1}^{k} (-1)^{k-p} i_p)$.*
(b) *If $i_k = n$, then $(i_1, \ldots, i_k) \sim (n)$.*

Proof. If $i_k < n$, we can repeat the argument as in (2.3) and get the desired result. We now consider the case $i_k = n$, namely $(i_1, \ldots, i_{k-1}, n)$. Let $c = n - 1 - i_{k-1}$. Then

$$\begin{aligned} (i_1, \ldots, i_{k-1}, n) &\sim (i_1, \ldots, i_{k-3}, i_{k-2} + c, n-1, n) && \text{by Lemma 2.1(b)} \\ &\sim (i_1, \ldots, i_{k-3}, i_{k-2} + n - 1 - i_{k-1}, n) && \text{by } F[n]. \end{aligned} \tag{2.4}$$

Thus the number of entries has gone from k to $k-1$. Repeat the applications of Lemma 2.1(b) and $F[n]$ as in (2.4), we end up with (n). $\square$

Most of C_n in Table 1 follow from Proposition 2.4. It remains only to check that if $(i_1, \ldots, i_k)$ with $i_k < n$ satisfies $\sum_{p=1}^{k} (-1)^{k-p} i_p = n - N$, then $(i_1, \ldots, i_k) \sim (N)$. This can be done by modifying (2.3) to $(i_1, \ldots, i_k) \sim (i_1, \ldots, i_{k-2}, n - (i_k - i_{k-1})) \sim \cdots \sim (n - \sum_{p=1}^{k} (-1)^{k-p} i_p)$ and proceed with similar arguments, or by observing that (N) and $(n-N)$ correspond to the Lie algebras $sp(N, n-N) \cong sp(n-N, N)$ [4, p. 355]. This proves Table 1 for C_n.

For D_n, the following proposition considers the various situations based on the colors of $n-1$ and n.

Proposition 2.5. *In D_n:*

(a) *If $i_k \leqslant n-2$, then $(i_1, \ldots, i_k) \sim (\sum_{p=1}^{k} (-1)^{k-p} i_p)$.*
(b) $(i_1, \ldots, i_k, n-1, n) \sim (1 + \sum_{p=1}^{k} (-1)^{k-p} i_p)$.
(c) $(i_1, \ldots, i_k, n-1) \sim (n-1)$.

Proof. The argument for (a) is similar to A_n; we simply move pairs of painted vertices to the left by Lemma 2.1(a). We perform this operation in (b), and get

$$(i_1, \ldots, i_k, n-1, n) \sim \left(\sum_{p=1}^{k} (-1)^{k-p} i_p, n-1, n \right).$$

By $F[n-1]$ followed by $F[n-2]$, we get

$$\left(\sum_{p=1}^{k} (-1)^{k-p} i_p, n-1, n \right) \sim \left(\sum_{p=1}^{k} (-1)^{k-p} i_p, n-3, n-2 \right).$$

This reduces to (a), and simple operations show that the last expression is equivalent to $(1+\sum_{p=1}^{k}(-1)^{k-p}i_p)$. This proves (b).

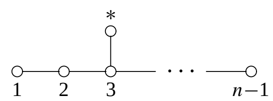
Now consider $(i_1,\ldots,i_k,n-1)$ in (c). The first k painted vertices can be dealt with as before, leaving $(i_1,\ldots,i_k,n-1)\sim(I,n-1)$, where $I=\sum_{p=1}^{k}(-1)^{k-p}i_p$. By performing $F[n-1], F[n-2],\ldots,F[I+1]$ to $(I,n-1)$, we get $(I,n-1)\sim(I+1,n)\sim(I+1,n-1)$. Repeating this method gives $(I,n-1)\sim(I+1,n-1)\sim\cdots\sim(n-2,n-1)$. Then $F[n-1](n-2,n-1)=(n-1)$ and we are done. □

Most equivalence classes of type D in Table 1 are covered by Proposition 2.5. The remaining cases follow from two simple observations. Firstly, $(N)\sim(n-N)$ because they correspond to Lie algebras $so(2N,2n-2N)\cong so(2n-2N,2N)$. Secondly, if exactly one of $n-1$, n is painted, obviously it does not matter which of them is painted due to symmetry of the diagram.

We have checked the equivalence classes of Vogan diagrams of types A, B, C, D given in Table 1. The next section considers the diagrams of types E, F, G.

3. Exceptional diagrams

In this section, we consider the Vogan diagrams of types E, F, G in Table 1 with $\theta=1$. We first treat the diagrams of E_n. Label the vertices as follows:



Lemma 3.1.

(a) *For* $q\geqslant 4$ *and* $p=2,3$, *we get* $(p,q)\sim(p-1,q-1,*)$ *and* $(p,q,*)\sim(p-1,q-1)$.
(b) *For* $q\geqslant 4$, $(1,q)\sim(q-1,*)$ *and* $(1,q,*)\sim(q-1)$.

Proof. For (p,q) or $(p,q,*)$, where $q\geqslant 4$, apply $F[p],\ldots,F[q-1]$ to it and we get the desired results. □

The next proposition simplifies a Vogan diagram to one of the form (α) or $(\alpha,*)$. However, it excludes the special cases in (1.1) because they are not valid in argument (3.7) below. We will deal with them separately in Proposition 3.5. Although argument (3.7) also cannot be applied to (1.1) of E_8, Propositions 3.3 and 3.5 show that they all happen to be equivalent to (7) in E_8, which coincides with the formulae in Proposition 3.2. Therefore, we need not exclude (1.1) of E_8 in Proposition 3.2.

As in (1.2), the Vogan diagrams are denoted by $(i_1,\ldots,i_k,j_1,\ldots,j_l,s)$, where $1\leqslant i_1<\cdots<i_k\leqslant 3<j_1<\cdots<j_l\leqslant n-1$ and s is $*$ or empty. Throughout this section, let I, J be defined as in (1.3), and let

$$\alpha=\begin{cases} J-I & \text{if } J\geqslant 4,\\ n-J-I & \text{if } J<4.\end{cases} \tag{3.1}$$

The next proposition simplifies $(i_1,\ldots,i_k,j_1,\ldots,j_l,s)$ to (α,t), where t is $*$ or empty.

Proposition 3.2. *Consider* $(i_1,\ldots,i_k,j_1,\ldots,j_l)$ *or* $(i_1,\ldots,i_k,j_1,\ldots,j_l,*)$ *other than in* (1.1). *Then* $(i_1,\ldots,i_k,j_1,\ldots,j_l,s)\sim(\alpha,t)$, *where* $s=t$ *if* I *is even, and* $s\neq t$ *if* I *is odd.*

Proof. For the case $(i_1,j_1)=(3,4)$, by $F[3]$, $F[2]$, $F[1]$, we get $(1,*)$. Now consider the case $(i_1,\ldots,i_k,j_1,\ldots,j_l)$, we may regard $(i_1,\ldots,i_k)$ and $(j_1,\ldots,j_l)$ as painted diagrams of A_3 and A_{n-4}, respectively. By Proposition 2.3, we have

$$(i_1,\ldots,i_k)\sim\left(\sum_{p=1}^{k}(-1)^{k-p}i_p\right)=(I)\quad\text{in } A_3\quad\text{and}\tag{3.2}$$

$$(j_1,\ldots,j_l)\sim\left(\sum_{p=1}^{l}(-1)^{l-p}j_p\right)=(J)\quad\text{in } A_{n-4}.\tag{3.3}$$

Notice that $J\geqslant 4$ if and only if $l=1$, this implies that there is a single painted vertex in $\{j_1,\ldots,j_l\}$; and if $J<4$, then the corresponding single painted vertex of E_n is $n-J$. Let β denote the single painted vertex of E_n reduced from the painted vertices $\{j_1,\ldots,j_l\}$, then

$$\beta=\begin{cases} n-J & \text{if } J<4,\\ J & \text{if } J\geqslant 4,\end{cases}\quad\text{and}\quad \alpha=\beta-I.\tag{3.4}$$

In reducing the diagrams (3.2) and (3.3), we did not use the operation $F[3]$. So $*$ does not occur and

$$(i_1,\ldots,i_k,j_1,\ldots,j_l)\sim(I,\beta).\tag{3.5}$$

Since $\beta\geqslant 4$ and by Lemma 3.1(a), we see that

$$\begin{aligned}(I,\beta)&\sim(I-1,\beta-1,*)\\ &\sim(I-2,\beta-2)\\ &\;\vdots\\ &\sim\begin{cases}(1,\beta-I+1,*) & \text{if } I-1 \text{ is odd},\\ (1,\beta-I+1) & \text{if } I-1 \text{ is even}.\end{cases}\end{aligned}\tag{3.6}$$

Hence we have

$$\begin{aligned}(i_1,\ldots,i_k,j_1,\ldots,j_l)&\sim(I,\beta) && \text{by (3.5)}\\ &\sim\begin{cases}(1,\beta-I+1) & \text{if } I-1 \text{ is even},\\ (1,\beta-I+1,*) & \text{if } I-1 \text{ is odd},\end{cases} && \text{by (3.6)}\\ &\sim\begin{cases}(\beta-I,*) & \text{if } I \text{ is odd},\\ (\beta-I) & \text{if } I \text{ is even},\end{cases} && \text{by Lemma 3.1(b)}\\ &\sim\begin{cases}(\alpha,*) & \text{if } I \text{ is odd},\\ (\alpha) & \text{if } I \text{ is even},\end{cases} && \text{by (3.4).}\end{aligned}\tag{3.7}$$

The use of Lemma 3.1(b) in (3.7) requires $\beta - I + 1 \geqslant 4$, which is not valid for the diagrams in (1.1). This is the reason which excludes them from this proposition.

By (3.7), we solve the case $(i_1, \dots, i_k, j_1, \dots, j_l)$. The case of $(i_1, \dots, i_k, j_1, \dots, j_l, *)$ follows from similar argument. This completes the proof. □

The above proposition shows how a Vogan diagram is equivalent to one of the form (α) or $(\alpha, *)$. The next two propositions deal with $(\alpha, *)$ and (α), respectively.

Table 3

$(\alpha, *)$	E_6	E_7	E_8
$(1, *)$	(5)	(6)	(7)
$(2, *)$	(5)	(6)	(7)
$(3, *)$	$(*)$	$(*)$	$(*)$
$(4, *)$	(1)	(1)	(1)
$(5, *)$	(5)	(6)	(7)
$(6, *)$	–	(1)	(7)
$(7, *)$	–	–	(1)

Proposition 3.3. *The Vogan diagrams of the form $(\alpha, *)$ are equivalent to diagrams with single painted vertex in Table* 3.

Proof. For $(1, *)$, we apply $F[1], F[2], \dots, F[n-1]$ consecutively and get

$$(1, *) \sim (1, 2, *) \sim (2, 3, *) \sim (3, 4) \sim \cdots \sim (n-1). \tag{3.8}$$

For $(2, *)$, we apply $F[*]$ to it and get $(2, 3, *)$, then proceed as in (3.8). Clearly, $(3, *) \sim (3)$. For $(4, *)$, we apply $F[*]$, $F[3]$, $F[2]$, $F[1]$ to it and get $(4, *) \sim (1)$.

We next show that $(5, *) \sim (2, *)$, so that we can proceed with $(2, *)$ as above. By $F[*]$, $F[3]$, $F[4]$, we get $(5, *) \sim (2, 4)$. By Lemma 3.1(a), $(2, 4) \sim (1, 3, *)$. Then apply $F[1]$, $F[2]$ to $(1, 3, *)$, we get $(2, *)$. This solves $(5, *)$.

We now consider $(6, *)$ in E_7 and E_8. In E_7, apply $F[6], \dots, F[1]$ consecutively to $(6, *)$ and we get (1). In E_8, by Lemma 3.1(b), $(6, *) \sim (1, 7)$. Apply $F[7], F[6], \dots, F[2]$ to $(1, 7)$ and we get $(2, *)$. This solves $(6, *)$.

Finally, for $(7, *)$ in E_8, we apply $F[7], \dots, F[1]$ to it and get (1). This proves the proposition. □

By Propositions 3.2 and 3.3, we have simplified all type E diagrams to single painted vertex diagrams. We consider these single painted vertex diagrams in the following proposition.

Proposition 3.4.

(a) E_6 *has two equivalence classes* $(1) \sim (5)$ *and* $(2) \sim (3) \sim (4) \sim (*)$.
(b) E_7 *has three equivalence classes* (6), $(1) \sim (2) \sim (3) \sim (5)$, *and* $(4) \sim (*)$.
(c) E_8 *has two equivalence classes* $(1) \sim (4) \sim (5) \sim (*)$ *and* $(2) \sim (3) \sim (6) \sim (7)$.

Proof. We only have to prove the equivalence claimed in this proposition. The inequivalence of different groupings follows from [4]. For example, [4, pp. 533–534] says that (1) and $(*)$ in E_6 are not equivalent.

We first claim that $(*) \sim (4)$ in all E_n:

$$\begin{aligned} (*) &\sim (3,*) && \text{apply } F[*] \\ &\sim (2,5,*) && \text{by Proposition 3.2} \\ &\sim (4) && \text{apply } F[3], F[4]. \end{aligned} \tag{3.9}$$

Hence $(*) \sim (4)$ as claimed. In the following we consider E_6, E_7, E_8 separately.

In E_6, clearly $(1) \sim (5)$ and $(2) \sim (4)$ by symmetry of the diagram. So by (3.9) it suffices to show that $(3) \sim (*)$. By applying $F[3]$, $F[4]$, $F[5]$ to (3), we get $(2,5,*)$, and by Proposition 3.2, $(2,5,*) \sim (3,*)$. Clearly $(3,*) \sim (*)$. This proves (a).

We next consider E_7 in (b):

$$\begin{aligned} (3) &\sim (3,6,*) && \text{by Proposition 3.2} \\ &\sim (4,*) && \text{apply } F[6], F[5], F[4] \\ &\sim (1) && \text{by Proposition 3.3.} \end{aligned}$$

We conclude that $(3) \sim (1)$. Next we claim that $(2) \sim (3)$:

$$\begin{aligned} (2) &\sim (1,3) && \text{apply } F[2], F[1] \\ &\sim (2,4,*) && \text{by Lemma 3.1(a)} \\ &\sim (3) && \text{apply } F[*], F[3]. \end{aligned}$$

Hence $(2) \sim (3)$ as claimed. We next prove that $(5) \sim (2)$:

$$\begin{aligned} (5) &\sim (1,6,*) && \text{by Proposition 3.2} \\ &\sim (2) && \text{apply } F[6], F[5], \ldots, F[2]. \end{aligned}$$

Together with (3.9), this proves (b).

Finally, we consider E_8 in (c):

$$\begin{aligned} (5) &\sim (2,7) && \text{by Proposition 3.2} \\ &\sim (3,*) && \text{apply } F[7], F[6], \ldots, F[3] \\ &\sim (*) && \text{apply } F[*]. \end{aligned}$$

On the other hand,

$$\begin{aligned} (4) &\sim (3,7,*) && \text{by Proposition 3.2} \\ &\sim (4,*) && \text{apply } F[7], F[6], \ldots, F[4] \\ &\sim (1) && \text{by Proposition 3.3.} \end{aligned}$$

Recall that $(4) \sim (*)$ by (3.9), so we conclude that $(1) \sim (4) \sim (5) \sim (*)$. We next check the other equivalence class $(2) \sim (3) \sim (6) \sim (7)$:

$$
\begin{aligned}
(6) &\sim (1,7,*) && \text{by Proposition 3.2}\\
&\sim (2) && \text{apply } F[7], F[6], \ldots, F[2]\\
&\sim (1,4,*) && \text{apply } F[2], F[3], F[*]\\
&\sim (3) && \text{by Proposition 3.2}\\
&\sim (3,6,*) && \text{by Proposition 3.2}\\
&\sim (6,*) && \text{apply } F[*]\\
&\sim (7) && \text{by Proposition 3.3.}
\end{aligned}
$$

That is, $(6) \sim (2) \sim (3) \sim (7)$. This completes the proof. □

The next proposition deals with the Vogan diagrams in (1.1). They have been excluded by Proposition 3.2.

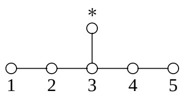
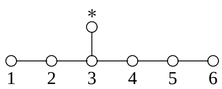
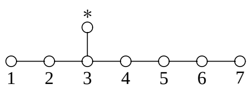
Proposition 3.5. *The equivalence classes of the Vogan diagrams in* (1.1) *are given in Table* 4. *In particular, each of them is equivalent to* $(n-2,*)$ *or* $(n-2)$.

Proof. In all E_n,

$$
\begin{aligned}
(3,5,*) &\sim (2,4) && \text{by Lemma 3.1(a)}\\
&\sim (1,3,4) && \text{apply } F[2], F[1]\\
&\sim (1,n-1) && \text{apply } F[4], \ldots, F[n-1]\\
&\sim (n-2,*) && \text{by Lemma 3.1(b).}
\end{aligned}
$$

The equivalence class of $(n-2,*)$ is given by Proposition 3.3. By similar arguments, we have $(3,5) \sim (2,4,*) \sim (1,3,4,*) \sim (n-2)$. The equivalence class of $(n-2)$ is given by Proposition 3.4. And clearly, in E_7, $(3,4,6) \sim (3,4,5,6) \sim (3,5)$ and $(3,4,6,*) \sim (3,4,5,6,*) \sim (3,5,*)$. This completes the proof. □

Table 4

Dynkin diagram	Single painted vertex	Equivalent diagrams
E_6 	(1)	(5), (2, 4), (1, 3, 4) (3, 5, ∗), (α, ∗), α = 1, 2, 4, 5
	(∗)	(2), (3), (4), (3, 5) (3, ∗), (2, 4, ∗), (1, 3, 4, ∗)
E_7 	(1)	(2), (3), (5), (3, 5), (3, 4, 6), (3, 4, 5, 6) (2, 4, ∗), (1, 3, 4, ∗), (α, ∗), α = 4, 6
	(6)	(2, 4), (1, 3, 4) (3, 5, ∗), (3, 4, 6, ∗), (3, 4, 5, 6, ∗), (α, ∗), α = 1, 2, 5
	(∗)	(4), (3, ∗)
E_8 	(7)	(2), (3), (6) (α, ∗), α = 1, 2, 5, 6
	(∗)	(1), (4), (5) (α, ∗), α = 3, 4, 7

By Propositions 3.2–3.5, we have completely characterized all the equivalence classes of Vogan diagrams of type E. We summarize these results in Table 4. Recall that α is defined in (3.1).

Table 4 summarizes the following method to determine the equivalence class of a Vogan diagram of E_n:

(1) Diagrams belong to the special cases (1.1).
Use Proposition 3.5 to reduce it to the form $(n-2,*)$ or $(n-2)$, then use Proposition 3.3 or 3.4 to find the equivalence class. The result is in Table 4.

(2) Diagrams not in (1.1).
Write it as $(i_1,\dots,i_k,j_1,\dots,j_l,s)$, then use (1.3) and (3.1) to compute I, J, α. Use Proposition 3.2 to reduce it to $(\alpha,*)$ or (α). The equivalence classes of $(\alpha,*)$ and (α) are given in Propositions 3.3 and 3.4, and are summarized in Table 4.

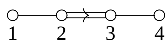
Methods (1) and (2) here correspond to methods (1) and (2) for Table 1. The methods for Table 4 have been simplified to the various cases of Table 1.

For example, consider $(1,2,3,5,*)$ in E_7. It does not belong to (1.1), so we compute

$$I = i_3 - i_2 + i_1 = 3-2+1 = 2 \quad \text{and} \quad J = j_1 = 5 > 4,$$

hence $\alpha = J - I = 3$. Since I is even, by Proposition 3.2, $(1,2,3,5,*) \sim (3,*)$. By Proposition 3.3, $(3,*) \sim (*)$. So Table 4 shows that $(1,2,3,5,*) \sim (*)$ in E_7. Alternatively, from $j_1 = 3 + I$ and $I + s = 2 + *$ is odd, we find $(1,2,3,5,*) \sim (*)$ in E_7 of Table 1.

We next consider the Vogan diagrams of F_4. We label the vertices as follows:



Proposition 3.6. *In* F_4, $(i_1,\dots,i_k) \sim (1)$ *if and only if* $\{i_1,\dots,i_k\} \cap \{1,2\} \neq \emptyset$.

Proof. Suppose that $(i_1,\dots,i_k)$ does not contain 1 or 2. That is $(3,4) \sim (4)$ or (3). It follows either from [4, pp. 541–542] or Theorem 5.1 later that $(3) \not\sim (2)$ and $(3) \not\sim (1)$. By applying $F[4]$, $F[3]$ on (4), we get (3). Conversely,

$$\begin{aligned} (1) &\sim (1,2) && \text{by } F[1] \\ &\sim (2,3) && \text{by } F[2] \\ &\sim (2,3,4) && \text{by } F[3] \\ &\sim (2,4) && \text{by } F[4] \\ &\sim (1,2,3,4) && \text{by } F[2] \\ &\sim (1,2,3) && \text{by } F[3] \\ &\sim (2) && \text{by } F[2]. \end{aligned} \tag{3.10}$$

And clearly,

$$(1,2,3,4) \sim (1,3,4) \sim (1,4) \sim (1,2,4) \quad \text{and} \quad (1,2,3) \sim (1,3). \tag{3.11}$$

All cases of $\{i_1, \ldots, i_k\} \cap \{1, 2\} \neq 0$ are considered in (3.10) and (3.11), this completes the proof. □

Proposition 3.6 shows that there are only two equivalence classes of Vogan diagrams of F_4 as listed in Table 1.

It is clear that all paintings on G_2 (unless we keep all vertices unpainted) are equivalent to one another. This can be checked by the performing various $F[i]$, or by looking at its painted root system.

4. Nontrivial involutions

In this section we study the equivalence classes of the Vogan diagrams with nontrivial involutions, and prove the informations in Table 2.

The condition $\theta \neq 1$ restricts the underlying Dynkin diagrams to A_n, D_n, and E_6. We also ignore the diagrams without painted vertex, since they cannot be equivalent to one with painted vertices. So the possibilities for $\theta \neq 1$ and with painted vertices are limited to A_n (n odd), D_n, and E_6. We may not paint vertices that are not fixed by θ (since compactness of roots makes sense only on the imaginary ones). We label the vertices as in Table 2. The only way to paint A_n (n odd) is by painting the vertex $(n+1)/2$, so it is not equivalent to any other diagram.

Next we consider D_n with vertex N painted, where $N \leqslant n-2$. In the previous case where $\theta = 1$, we have shown in Proposition 2.5(a) that

$$(N) \sim (i_1, \ldots, i_k) \quad \text{for } N = \sum_{p=1}^{k} (-1)^{k-p} i_p \text{ and } i_k \leqslant n-2. \tag{4.1}$$

This argument uses $F[i]$ for $i \leqslant n-3$. In general, $F[i]$ differs in the cases $\theta = 1$ and $\theta \neq 1$ only if a vertex adjacent to i is not fixed by θ. Therefore, since $n-1$ and n are the only vertices not fixed by θ here, the arguments in Proposition 2.5(a) are still valid in our present situation. Namely, we also have (4.1) for $\theta \neq 1$. Also, $(N) \sim (n-N-1)$ because they represent the Lie algebras $so(2N+1, 2n-2N-1) \cong so(2n-2N-1, 2N+1)$. This proves D_n in Table 2.

Finally in E_6, there is only one equivalence class with $\theta \neq 1$ and with some vertices painted [4, pp. 532–535]. Therefore, all such cases are equivalent to one another. This completely verifies Table 2.

5. Graph paintings

This section develops an idea in the opposite direction: The Vogan diagrams can classify almost all the simply-laced Dynkin diagrams. Since we are interested in the underlying Dynkin diagrams, we may consider only the Vogan diagrams with $\theta = 1$ in this section.

Recall that the algorithm $F[i]$ in (2.1) is used to reduce the number of painted vertices within an equivalence class of Vogan diagrams until we end up with a single painted vertex. This is not so surprising, by the following theorem.

Theorem 5.1. *Two Vogan diagrams with $\theta = 1$ are equivalent if and only if one can be transformed to the other by a sequence of $F[i]$ operations.*

Proof. The "if" part of the theorem is obvious, since $F[i]$ preserves equivalence classes. The converse has been verified explicitly for each Dynkin diagram in Sections 2, 3 when we check Table 1. We now give a more intrinsic argument which does not take into account the shapes of the Dynkin diagrams. Recall that two equivalent Vogan diagrams correspond to the same Lie algebra under different choices of Weyl chambers. The Weyl group W acts transitively on the chambers, and so it acts transitively on each equivalence class of Vogan diagrams. Since $\theta = 1$, all roots are imaginary, and they are either compact or noncompact. Let W_c and W_n denote the subgroups generated by reflections about the compact and noncompact simple roots, respectively. Clearly, W is generated by W_c and W_n. Further, since W_c acts trivially on the Vogan diagrams, it follows that W_n acts transitively on each equivalence class of Vogan diagrams. Since $F[i]$ corresponds to reflection about the noncompact simple root labelled i, this proves the theorem. □

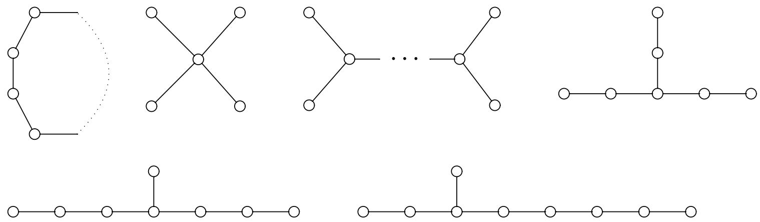
The proof of this theorem does not make use of knowledge on the shapes of the Dynkin diagrams. Therefore, if we accept the Borel–de Siebenthal theorem, then it gives a necessary condition for a connected graph to be a Dynkin diagram.

Corollary 5.2. *If a connected graph Γ is a Dynkin diagram, then*

(a) *every painting on Γ can be simplified via a sequence of $F[i]$ to a painting with single painted vertex*;
(b) *every connected subgraph of Γ satisfies property* (a).

Proof. To prove (a), let Γ be a Dynkin diagram. Suppose that p is a painting on Γ. By the Borel–de Siebenthal theorem, $(\Gamma, p) \sim (\Gamma, s)$, where s paints just a single vertex of Γ. By Theorem 5.1, (Γ, p) can be transformed to (Γ, s) with some sequence of $F[i]$ operations. This proves (a). Since connected subgraphs of a Dynkin diagram correspond to simple subalgebras, condition (b) is trivial. The corollary follows. □

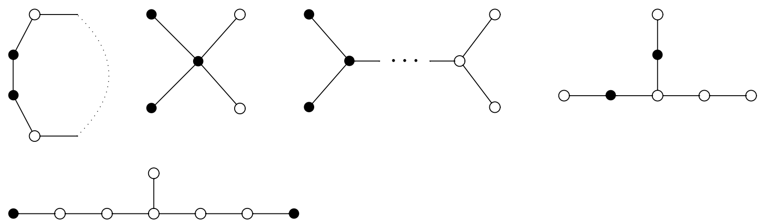
The corollary provides an obstruction for a graph to be Dynkin via conditions (a) and (b). We shall see that they come close to being sufficient conditions. The simply-laced Dynkin diagrams are classified by showing that they cannot contain the following subgraphs:



(5.1)

In the top row of (5.1), the first two diagrams say that a Dynkin diagram has no loop and no node (branch point) with more than three edges. The third diagram says that there is at most one node. In this case we can topologically think of the node as being joined to three "lines" l_1, l_2, l_3 whose lengths are defined in the obvious manner. The fourth diagram of the top row says that one of the l_i, say l_1, is of length 1. Then the remaining diagrams put some restrictions based on the lengths of l_2 and l_3.

Corollary 5.2(b) says that a connected subgraph of a Dynkin diagram is again Dynkin; so it suffices to show that the six graphs in (5.1) are not Dynkin. We attempt to use Corollary 5.2(a) to achieve this; namely, we find a painting which cannot be simplified to a graph with single painted vertex via the algorithms $F[i]$. Such attempt is successful for all but one of them:



(5.2)

For instance, in the loop in (5.2), no matter how we apply $F[i]$, we always end up with a loop with two painted vertices. So we conclude that every Dynkin diagram cannot contain any loop. Unfortunately, in the last graph of (5.1), any painting can be reduced to a diagram with a single painted vertex. This "fake E_9" is the only structure which does not exist in Dynkin diagrams but cannot be dismissed by the algorithms $F[i]$.

Acknowledgments

The authors thank David Vogan and the referee for many helpful suggestions.

References

[1] P. Batra, Invariants of real forms of affine Kac–Moody Lie algebras, J. Algebra 223 (2000) 208–236.

[2] P. Batra, Vogan diagrams of real forms of affine Kac–Moody Lie algebras, J. Algebra 251 (2002) 80–97.

[3] A. Borel, J. de Siebenthal, Les sous-groupes fermés de rang maximum des groupes de Lie clos, Comment. Math. Helv. 23 (1949) 200–221.

[4] A.W. Knapp, Lie Groups beyond an Introduction, in: Progr. Math., vol. 140, Birkhäuser, 1996.