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Permutation polytopes corresponding to strongly supermodular functions

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Abstract

Throughout, let p be a positive integer and let Σ be the set of permutations over $\{1,\dots,p\}$. A real-valued function λ over subsets of $\{1,\dots,p\}$, with $\lambda(\emptyset)=0$, defines a mapping of Σ into $\mathbb{R}^p$ where $\sigma\in\Sigma$ is mapped into the vector λ_σ whose kth coordinate $(\lambda_\sigma)_k$ is the augmented λ-value obtained from adding k to the coordinates that precede it, according to the ranking induced by σ. The permutation polytope corresponding to λ is then the convex hull of the vectors corresponding to all permutations. We introduce a new class of *strongly supermodular functions* and for such functions we derive an isomorphic representation for the face-lattices of the corresponding permutation polytope.

Keywords: Polytopes; Supermodularity; Permutations; Cores of games

1. Introduction

A *permutation* (of $\{1,\dots,p\}$) is formally defined as an ordered collection of sets $\sigma=(\sigma_1,\dots,\sigma_p)$ where $\sigma_1,\dots,\sigma_p$ are singletons that partition $\{1,\dots,p\}$; given such a partition σ and $k\in\{1,\dots,p\}$ there is a unique index j with $\sigma_j=\{k\}$, which we denote $j_\sigma(k)$. Given a real-valued function λ on the subsets of $\{1,\dots,p\}$ with $\lambda(\emptyset)=0$, each permutation σ defines a vector $\lambda_\sigma\in\mathbb{R}^p$ whose kth coordinate $(\lambda_\sigma)_k$ for $k=1,\dots,p$ equals $\lambda(\bigcup_{t=1}^{j}\sigma_t)-\lambda(\bigcup_{t=1}^{j-1}\sigma_t)$ with $j\equiv j_\sigma(k)$. The *permutation polytope corresponding to* λ, denoted H^λ, is the convex hull of the vectors λ_σ with σ ranging over all permutations of $\{1,\dots,p\}$. These polytopes have been studied in the literature with different motivations.

Shapley [13] studied the core of convex p-person games, otherwise known as supermodular set function games. Such a game is a real-valued function λ on the subsets of $\{1,\dots,p\}$ that satisfies $\lambda(\emptyset)=0$ and

$$\lambda(I\cup J)+\lambda(I\cap J)\geqslant\lambda(I)+\lambda(J)\text{ for all subsets } I \text{ and } J \text{ of } \{1,\dots,p\}. \tag{1.1}$$

Shapley showed that the *core* of such games, defined as the solution set of the linear inequality system

$$\sum_{j\in I}x_j\geqslant\lambda(I)\text{ for each } I\subseteq\{1,\dots,p\}\text{ and}$$

$$\sum_{j=1}^{p}x_j=\lambda(\{1,\dots,p\}), \tag{1.2}$$

coincides with H^λ. He further examined other properties of H^λ for such games.

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Elsewhere a partition problem was studied in Gao, Hwang, Li and Rothblum (GHLR) [7] (see also [8] and references therein). The data for the problem consists of positive integers p, n, $n_1,\dots,n_p$ with $\sum_{j=1}^{p} n_j = n$ and n real numbers

$$\theta^1 \leqslant \cdots \leqslant \theta^n. \tag{1.3}$$

Given a partition $\pi=(\pi_1,\dots,\pi_p)$ of $\{1,\dots,n\}$, let θ_π be the p-vector with $(\theta_\pi)_j=\sum_{i\in\pi_j}\theta^i$ for $j=1,\dots,p$. The goal is to find a partition $(\pi_1,\dots,\pi_p)$ that maximizes an objective function $f(\theta_\pi)$ over the set of partitions π with $|\pi_j|=n_j$ for $j=1,\dots,p$. It is shown in [7] that the function λ defined on each subset I of $\{1,\dots,p\}$ by

$$\lambda(I)=\sum_{i=1}^{n(I)}\theta^i \text{ where } n(I)=\sum_{j\in I} n_j \tag{1.4}$$

satisfies (1.1); further, the convex hull of the θ_π's, referred to as the *partition polytope*, coincides with the permutation polytope H^λ corresponding to λ. So, permutation polytopes generalize partition polytopes. Also, an instance of partition polytopes with $n_j=1$ for $j=1,\dots,p$ is referred to as a *permutahedron* (see [1,7,11,12]).

Both Shapley and GHLR studied the strict version of their problems—for Shapley "strict" refers to the case where the inequalities in (1.1) are strict when S and T are not comparable by set-inclusion and for GHLR "strict" refers to strict inequalities in (1.3) which implies that the function λ defined by (1.4) satisfies Shapley's strictness condition (see [7]). Billera and Sarangarajan [1] studied this strictness for the special case of permutahedra (which form a subclass of partition polytopes). Under either of this strictness assumption, the corresponding permutation polytopes have particularly simple structure; in fact, all such polytopes are both combinatorially and normally equivalent (see [8]).

Permutation polytopes were studied extensively in the literature for functions λ satisfying (1.1) [1,2,4–6,10–13]. Further, they were extended to base polyhedra which play an important role in the analysis of combinatorial optimization and other areas of combinatorial analysis, e.g. [3,4]. Functions λ satisfying (1.1) were also studied extensively in [9].

In the current paper, we propose a new nondegeneracy condition that sharpens (1.1) and is weaker than the strict supermodularity introduced by Shapley; under this condition we derive an isomorphic representation of the face lattice of the corresponding permutation polytopes. We call the new condition *strong supermodularity* and we note that it is satisfied by set functions generated by any partitioning problems.

Preliminaries about Polytopes and Supermodularity are summarized in Sections 2 and 3, and our main results about strong supermodularity are established in Section 4.

2. Preliminaries: polytopes, permutations and supermodularity

We identify row and column vectors and use $\mathbb{R}^p$ to denote the set of either type of p-vectors. Also, we refer to the standard definitions for the *convex hulls* of subsets of $\mathbb{R}^p$ and for the *dimension* of convex sets, and use the notation conv C and dim C, respectively. A *polytope* in $\mathbb{R}^p$ is the convex hull of finitely many points in $\mathbb{R}^p$. The Main Theorem for Polytopes (see [14, Theorem 1.1, p. 29]) asserts that a subset of $\mathbb{R}^p$ is a polytope if and only if it is bounded and is the solution set of a system of linear inequalities.

Given a polytope P in $\mathbb{R}^p$, we say that a linear inequality $\sum_{j=1}^{p} c_jx_j \leqslant \gamma$ is *valid* for P if $P\subseteq\{x\in\mathbb{R}^p\colon \sum_{j=1}^{p} c_jx_j\leqslant\gamma\}$. A *face* of P is any set of the form $F=P\cap\{x\in\mathbb{R}^p: \sum_{j=1}^{p} d_jx_j=\delta\}$ where $\sum_{j=1}^{p} d_jx_j\leqslant\delta$ is a valid inequality for P. A face F of P is *proper* if $\emptyset\neq F\neq P$. Faces of dimension 0, 1 and $(\dim P)-1$ are called *vertices, edges* and *facets*, respectively. For convenience, we refer to a *vertex* not only as a face of dimension zero, but also as the single element that such a face contains. A number of results about faces of polytopes are recorded in Proposition A.1 in the Appendix.

With set inclusion as the partial order, the set of faces of a polytope P is known to be a lattice (cf., Part (b) of Proposition A.1 of the Appendix), and we refer to this lattice as the *face-lattice of* P.

Real-valued functions λ on the nonempty subsets of $\{1,\dots,p\}$ are automatically extended to the empty set with $\lambda(\emptyset)=0$; and a function λ on subsets of $\{1,\dots,p\}$ which satisfy $\lambda(\emptyset)=0$ is viewed as a function on the nonempty subsets of $\{1,\dots,p\}$. Recall the definition of supermodular functions given in the Introduction via (1.1). A real-valued function λ on the subsets of $\{1,\dots,p\}$ with $\lambda(\emptyset)=0$ is called *strictly supermodular* if strict inequality holds in (1.1) whenever the two sets I and J are not ordered by set inclusion, that is, $I\nsubseteq J$ and $J\nsubseteq I$.

Suppose λ is supermodular on subsets of $\{1,\dots,p\}$. A triplet (I,K,J) of subsets of $\{1,\dots,p\}$ is called *λ-flat* if $I\subset K\subset J$ and, with $L=I\cup(J\setminus K)$,

$$\lambda(I)+\lambda(J)=\lambda(K)+\lambda(L). \tag{2.1}$$

We observe that strict supermodularity means that there exist no λ-flat triplets.

We say that a function λ on subsets of $\{1,\dots,p\}$ is *strongly supermodular* if λ is supermodular and for every pair of subsets I,J of $\{1,\dots,p\}$ if it satisfies the following condition:

if there exists a subset K of $\{1,\dots,p\}$ such that (I,K,J) is λ-flat, then for every subset K' satisfying $I\subset K'\subset J$, (I,K',J) is λ-flat.

Of course, strict supermodularity implies strong supermodularity.

The next two examples demonstrate that supermodularity does not imply strong supermodularity and that strong supermodularity does not imply strict supermodularity.

Example 1. Let $p=3$ and λ be given by $\lambda(\{1\})=1$, $\lambda(\{2\})=2$, $\lambda(\{3\})=\lambda(\{1,2\})=3$, $\lambda(\{1,3\})=\lambda(\{2,3\})=5$, $\lambda(\{1,2,3\})=7$. Then λ is supermodular. However, λ is not strongly supermodular as $\lambda(\{2\})+\lambda(\{1,3\})=7=\lambda(\emptyset)+\lambda(1,2,3)$ assuring that $(\emptyset,\{2\},\{1,2,3\})$ is λ-flat. But, $(\emptyset,\{1\},\{1,2,3\})$ is not λ-flat because $\lambda(\{1\})+\lambda(\{2,3\})=6<7=\lambda(\emptyset)+\lambda(\{1,2,3\})$.

Example 2. Let $p=3$ and λ be given by $\lambda(I)=|I|$ if $|I|\leqslant 2$ and $\lambda(\{1,2,3\})=4$. Then λ is not strictly supermodular as equality in (1.1) occurs for all distinct sets I and J which contain a single element. As equality in (1.1) occurs only for such pairs, it immediately follows that λ is strongly supermodular.

Let λ be a supermodular function on subsets of $\{1,\dots,p\}$. We say that a pair of subsets (I,J) of $\{1,\dots,p\}$ is *λ-flat* if $|J\setminus I|\geqslant 2$ and for every $I\subset K\subset J$, the triplet (I,K,J) is λ-flat. Strong supermodularity of λ means for every λ-flat triplet (I,K,J), the pair (I,J) must be λ-flat.

We next consider an important class of strongly supermodular functions that appear in partitioning problems (e.g. [7,8]).

Throughout the end of this section, let $\theta^1,\dots,\theta^n$ be real numbers satisfying (1.3) and let $n_1,\dots,n_p$ be nonnegative integers satisfying $\sum_{j=1}^p n_j=n$. For subset I of $\{1,\dots,p\}$, let

$$n(I)\equiv\sum_{j\in I}n_j, \tag{2.2}$$

and

$$\theta_*^{(n_1,\dots,n_p)}(I)=\sum_{i=1}^{n(I)}\theta^i; \tag{2.3}$$

in particular, $n(\{1,\dots,p\})=n$ and $\theta_*^{(n_1,\dots,n_p)}(\{1,\dots,p\})=\sum_{i=1}^n\theta^i$.

We note that (1.3) implies that

$$\sum_{i=u+1}^{u+w}\theta^i\leqslant\sum_{i=v+1}^{v+w}\theta^i \text{ for nonnegative integers } u,v \text{ and } w \text{ with } u\leqslant v. \tag{2.4}$$

Further, if $u<v$ and $w>0$, equality holds in (2.4) if and only if θ^i is a constant for $u<i\leqslant v+w$. In particular, (2.4) holds strictly when the inequalities in (1.3) are strict, $u<v$ and $w>0$.

The next two lemmas and following theorem establish useful properties of $\theta_*^{(n_1,\dots,n_p)}$.

Lemma 2.1. *Let $n_1,\dots,n_p$ be nonnegative integers with $n=\sum_{j=1}^p n_j$. Then, $\theta_*^{(n_1,\dots,n_p)}$ is supermodular. Further, if the n_j's are positive, then for subsets I and J of $\{1,\dots,p\}$ where neither is a subset of the other we have that*

$$\theta_*^{(n_1,\dots,n_p)}(I\cup J)+\theta_*^{(n_1,\dots,n_p)}(I\cap J)=\theta_*^{(n_1,\dots,n_p)}(I)+\theta_*^{(n_1,\dots,n_p)}(J), \tag{2.5}$$

if and only if θ^i is constant for $n(I\cap J)<i\leqslant n(I\cup J)$.

Proof. For subsets I and J of $\{1,\dots,p\}$, $n(I\cup J)=n(I)+n(J\setminus I)$, $n(J)=n(I\cap J)+n(J\setminus I)$ and (2.4) with $u\equiv n(I\cap J)$, $v\equiv n(I)(\geqslant n(I\cap J)=u)$ and $w=n(J\setminus I)\geqslant 0$ implies that

$$\theta_*^{(n_1,\dots,n_p)}(I\cup J)-\theta_*^{(n_1,\dots,n_p)}(I)=\sum_{i=1}^{n(I\cup J)}\theta^i-\sum_{i=1}^{n(I)}\theta^i=\sum_{i=n(I)+1}^{n(I)+n(J\setminus I)}\theta^i$$

$$\geqslant \sum_{i=n(I\cap J)+1}^{n(I\cap J)+n(J\setminus I)} \theta^i = \sum_{i=n(I\cap J)+1}^{n(J)} \theta^i = \sum_{i=1}^{n(J)} \theta^i - \sum_{i=1}^{n(I\cap J)} \theta^i$$

$$= \theta_*^{(n_1,\dots,n_p)}(J) - \theta_*^{(n_1,\dots,n_p)}(I\cap J). \tag{2.6}$$

Next, assume that the n_j's are positive, that $I \nsubseteq J$ and that $J \nsubseteq I$. Then $n(I\cap J) < n(I)$ and $n(J\setminus I) > 0$; hence, the comment following (2.4) implies that the left-most and right-most expressions of (2.6) are equal if and only if θ^i is a constant for $n(I\cap J) < i \leqslant n(I\cup J)$. □

Lemma 2.2. *Let $n_1,\dots,n_p$ be positive integers with $n=\sum_{j=1}^p n_j$ and let K and L be subsets of $\{1,\dots,p\}$ with $K\subseteq L$. Then the following are equivalent*:

(a) (K,I,L) is $\theta_*^{(n_1,\dots,n_p)}$*-flat for some subset I of* $\{1,\dots,p\}$,
(b) $|L\setminus K|\geqslant 2$ *and (K,L) is $\theta_*^{(n_1,\dots,n_p)}$-flat, and*
(c) $|L\setminus K|\geqslant 2$ *and θ^i is constant for $n(K)<i\leqslant n(L)$.*

Proof. (b) ⇒ (a): Assume (b) holds. The assertion $|L\setminus K|\geqslant 2$ assures that there is a set I with $K\subset I\subset L$; for such a set, (b) implies that (K,I,L) is $\theta_*^{(n_1,\dots,n_p)}$-flat.

(c) ⇒ (b): Suppose $|L\setminus K|\geqslant 2$ and θ^i is constant for $n(K)<i\leqslant n(L)$. As $|L\setminus K|\geqslant 2$, there exists a set I satisfying $K\subset I\subset L$. Let $J\equiv K\cup(L\setminus I)$. Then, $I\cap J=K$, $I\cup J=L$ and I and J are not ordered by set inclusion. Lemma 2.1 then implies that (2.5) must be satisfied, assuring that (K,I,L) is $\theta_*^{(n_1,\dots,n_p)}$-flat.

(a) ⇒ (c): Suppose (K,I,L) is $\theta_*^{(n_1,\dots,n_p)}$-flat for some subset I of $\{1,\dots,p\}$. Then $K\subset I\subset L$ assuring that $|L\setminus K|\geqslant 2$. Also, let $J\equiv K\cup(L\setminus I)$. Then, $I\cap J=K$, $I\cup J=L$ and I and J are not ordered by set inclusion; further, the $\theta_*^{(n_1,\dots,n_p)}$-flatness of (K,I,L) assures that (2.5) is satisfied. Lemma 2.1 then implies that θ^i is constant for $n(K)=n(I\cap J)<i\leqslant n(I\cup J)=n(L)$. □

Theorem 2.3. *Let $n_1,\dots,n_p$ be positive integers with $n=\sum_{j=1}^p n_j$. Then $\theta_*^{(n_1,\dots,n_p)}$ is strongly supermodular; further, if the θ^i's are distinct, then $\theta_*^{(n_1,\dots,n_p)}$ is strictly supermodular.*

Proof. Lemma 2.1 shows that $\theta_*^{(n_1,\dots,n_p)}$ is supermodular, and the implication (a) ⇒ (b) in Lemma 2.2 proves that $\theta_*^{(n_1,\dots,n_p)}$ is strongly supermodular.

Next assume that θ^i's are distinct and let I and J be subsets of $\{1,\dots,p\}$ where neither is a subset of the other. Then, $n(I\cup J)-n(I\cap J)\geqslant 2$; thus, the second conclusion of Lemma 2.1 implies that (2.5) cannot hold, that is necessarily

$$\theta_*^{(n_1,\dots,n_p)}(I\cup J)+\theta_*^{(n_1,\dots,n_p)}(I\cap J) > \theta_*^{(n_1,\dots,n_p)}(I)+\theta_*^{(n_1,\dots,n_p)}(J). \tag{2.7}$$

Thus, $\theta_*^{(n_1,\dots,n_p)}$ is strictly supermodular. □

Let $\Pi^{(n_1,\dots,n_p)}$ be the set of partitions $\pi=(\pi_1,\dots,\pi_p)$ of $\{1,\dots,n\}$ satisfying $|\pi_j|=n_j$ for $j=1,\dots,p$. Then for each $J\subseteq\{1,\dots,p\}$,

$$\theta_*^{(n_1,\dots,n_p)}(J)=\min\left\{\sum_{j\in J}(\theta_\pi)_j\colon\ \pi\in\Pi^{(n_1,\dots,n_p)}\right\}.$$

Also, it is shown in [8] that the permutation polytope $H^{\theta_*^{(n_1,\dots,n_p)}}$ equals the (partition) polytope

$$P^{(n_1,\dots,n_p)}\equiv \operatorname{conv}\left[\left(\sum_{i\in\pi_1}\theta^i,\dots,\sum_{i\in\pi_p}\theta^i\right)\colon\ \pi=(\pi_1,\dots,\pi_p)\in\Pi^{(n_1,\dots,n_p)}\right\}.$$

3. Permutation polytopes corresponding to supermodular functions

In this section, we record some results on permutation polytopes corresponding to supermodular functions to facilitate proofs on permutation polytopes corresponding to strongly supermodular functions to be presented in Section 4. These results have been established in a more general context (see [4]). We include some elementary proofs for the sake of completeness.

Throughout we assume that λ is a real-valued function on the nonempty subsets of $\{1,\dots,p\}$. For such λ, the permutation polytope H^λ is defined in the first paragraph of Section 1. Also, let C^λ be the solution set of the system of linear inequalities given by (1.2).

For each $I\subseteq\{1,\dots,p\}$, let F_I be the subset of C^λ obtained by tightening the inequality corresponding to I in (1.2), that is,

$$F_I \equiv \left\{x\in C^\lambda : \sum_{j\in I} x_j = \lambda(I)\right\}. \tag{3.1}$$

We note that the faces of C^λ are precisely intersections of F_I's (see Proposition A.1, parts (b), (c) and (g)).

Lemma 3.1. *Suppose λ is supermodular and I and J are subsets of $\{1,\dots,p\}$.*

(a) *If $(I\cap J, I, I\cup J)$ is λ-flat, then $F_{I\cap J}\cap F_{I\cup J}\subseteq F_I\cap F_J$.*

(b) *If $F_I\cap F_J\neq\emptyset$, then $F_{I\cap J}\cap F_{I\cup J}=F_I\cap F_J$ and either I and J are ordered by set-inclusion or $(I\cap J, I, I\cup J)$ is λ-flat.*

Proof. (a) Suppose $(I\cap J,I,I\cup J)$ is λ-flat and $y\in F_{I\cap J}\cap F_{I\cup J}$. Then

$$\lambda(I)+\lambda(J)\leqslant\sum_{j\in I}y_j+\sum_{j\in J}y_j=\sum_{j\in I\cap J}y_j+\sum_{j\in I\cup J}y_j=\lambda(I\cap J)+\lambda(I\cup J)=\lambda(I)+\lambda(J);$$

it follows that all of the above inequalities hold as equalities. Thus, $\sum_{j\in I} y_j=\lambda(I)$ and $\sum_{j\in J} y_j=\lambda(J)$, that is, $y\in F_I\cap F_J$. So, the inclusion $F_{I\cap J}\cap F_{I\cup J}\subseteq F_I\cap F_J$ has been established.

(b) Assume that $F_I\cap F_J\neq\emptyset$ and $x\in F_I\cap F_J$. Then

$$\lambda(I)+\lambda(J)=\sum_{j\in I}x_j+\sum_{j\in J}x_j=\sum_{j\in I\cap J}x_j+\sum_{j\in I\cup J}x_j\geqslant\lambda(I\cap J)+\lambda(I\cup J)\geqslant\lambda(I)+\lambda(J).$$

It follows that all of the above inequalities hold as equalities. Thus, $\sum_{i\in I\cup J}x_i=\lambda(I\cup J)$, $\sum_{i\in I\cap J}x_i=\lambda(I\cap J)$ and $\lambda(I\cap J)+\lambda(I\cup J)=\lambda(I)+\lambda(J)$. In particular, $x\in F_{I\cap J}\cap F_{I\cup J}$ and either I and J are ordered by set-inclusion or $(I\cap J,I,I\cup J)$ is λ-flat. As $x\in F_I\cap F_J$ was selected arbitrarily, we conclude that $F_I\cap F_J\subseteq F_{I\cap J}\cap F_{I\cup J}$. Next, the reverse inclusion follows from part (a). If $(I\cap J,I,I\cup J)$ is λ-flat and is trivial otherwise (when I and J are ordered by set-inclusion).

A (possibly empty) sequence $I_1,I_2,\dots,I_k$ of subsets of $\{1,\dots,p\}$ is called a *chain* if $\emptyset\subset I_1\subset I_2\subset\cdots\subset I_k\subset\{1,\dots,p\}$, in which case we refer to k as the *length* of the chain. Such a chain is usually augmented with $I_0\equiv\emptyset$ and $I_{k+1}\equiv\{1,\dots,p\}$. We say that a chain $I_1,I_2,\dots,I_k$ is a *representing chain* of a subset F of $\mathbb{R}^p$, if $F=\bigcap_{t=1}^k F_{I_t}$. In this case, F is a face of H^λ (as an intersection of F_I's). We say that chain $I'_1,I'_2,\dots,I'_{k'}$ is a *subchain* of $I_1,I_2,\dots,I_k$ and that $I_1,I_2,\dots,I_k$ is a *superchain* of $I'_1,I'_2,\dots,I'_{k'}$, if $\{I'_1,I'_2,\dots,I'_{k'}\}\subseteq\{I_1,I_2,\dots,I_k\}$; we say that $I'_1,I'_2,\dots,I'_{k'}$ is a *proper subchain* of $I_1,I_2,\dots,I_k$ and that $I_1,I_2,\dots,I_k$ is a *proper superchain* of $I'_1,I'_2,\dots,I'_{k'}$ when the above inclusion is strict. The maximal length of a chain is $p-1$ and every chain has a superchain of length $p-1$. A chain $I_1,\dots,I_k$ is called *maximal* (*minimal*) if it has no proper superchain (subchain) which is a representing chain of $\bigcap_{t=1}^k F_{I_t}$.

For a chain $I_1,I_2,\dots,I_k$, we have that $\{I_t\setminus I_{t-1}\colon t=1,\dots,k+1\}$ is a partition of $\{1,\dots,p\}$. In particular, if the length of the chain is $p-1$, each of the sets $I_t\setminus I_{t-1}$ is a singleton and $\{I_t\setminus I_{t-1}\colon t=1,\dots,p\}=\{\{j\}:1\leqslant j\leqslant p\}$. Thus, a chain of length $p-1$ defines a permutation σ of $\{1,\dots,p\}$ with $\sigma_t=I_t\setminus I_{t-1}$ for $t=1,\dots,p$. We note that the correspondence of chains of length $p-1$ into permutations is one-to-one and onto, with the pre-image of permutations σ being the chain $I_1,\dots,I_{p-1}$ with $I_t=\bigcup_{s=1}^t\sigma_s$. We say that a permutation σ *is consistent with* a chain $I_1,\dots,I_k$ if $I_1,\dots,I_k$ is a subchain of the unique chain of length $p-1$ corresponding to σ.

The following result is due to Shapley [13].

Theorem 3.2. *Suppose λ is supermodular. Then*:

(a) $H^\lambda=C^\lambda$,

(b) *the vertices of H^λ are precisely the λ_σ's where σ ranges over the permutations of $\{1,\dots,p\}$,*

(c) *the nonempty faces of H^λ are precisely the sets represented by chains*, *and*
(d) *a chain of length* $p-1$ *is a representing chain of* $\{\lambda_\sigma\}$ *where* σ *is the corresponding permutation.*

The next result records a relationship between face-inclusion and representing chains. It has been established in the more general framework of base polytopes (see [4]). We include an elementary proof using the following lemma.

Lemma 3.3. *Suppose* λ *is supermodular. Let* $I_1,\dots,I_k$ *be a chain and let* $I'_1,\dots,I'_{k'}$ *be nonempty proper subsets of* $\{1,\dots,p\}$ *with* $F'=(\bigcap_{t=1}^k F_{I_t})\cap(\bigcap_{t=1}^{k'} F_{I'_t})\neq\emptyset$. *Then there exists a superchain of* $I_1,\dots,I_k$ *which is a representing chain of* F'.

Proof. It suffices to consider the case with $k'=1$, in which case we let I stand for I'_1. So, $F'\equiv\left(\bigcap_{t=1}^k F_{I_t}\right)\cap F_I\neq\emptyset$. For $t=1,\dots,k+1$, let $J_t\equiv I_{t-1}\cup[(I_t\setminus I_{t-1})\cap I]$. We next prove by induction that for $s=0,1,\dots,k$, $F'=\left(\bigcap_{t=1}^s F_{J_t}\right)\cap F_{I_s\cup I}\cap\left(\bigcap_{t=s+1}^k F_{I_t}\right)$. As $I_0=\emptyset$, the case where $s=0$ follows from the representation $F'=F_I\cap\left(\bigcap_{t=1}^k F_{I_t}\right)$. Next assume that the asserted representation holds for $0\leqslant s<k$. As $I_s\subseteq I_{s+1}$, we have that $(I_s\cup I)\cap I_{s+1}=I_s\cup[(I_{s+1}\setminus I_s)\cap I]=J_{s+1}$ and $(I_s\cup I)\cup I_{s+1}=I_{s+1}\cup I$; by the induction assumption $F_{I_s\cup I}\cap F_{I_{s+1}}\supseteq F'\neq\emptyset$. Hence, part (b) of Lemma 3.1 implies that $F_{I_s\cup I}\cap F_{I_{s+1}}=F_{(I_s\cup I)\cap I_{s+1}}\cap F_{(I_s\cup I)\cup I_{s+1}}=F_{J_{s+1}}\cap F_{I_{s+1}\cup I}$ and therefore $F'=(\bigcap_{t=1}^s F_{J_t})\cap F_{I_s\cup I}\cap(\bigcap_{t=s+1}^k F_{I_t})=(\bigcap_{t=1}^{s+1} F_{J_t})\cap F_{I_{s+1}\cup I}\cap(\bigcap_{t=s+2}^k F_{I_t})$. Thus, the induction hypothesis has been established with $s+1$ replacing s. As $I_{k+1}=\{1,\dots,p\}$, we have that $I_k\cup I=I_k\cup[(I_{k+1}\setminus I_k)\cap I]=J_{k+1}$ and the verified inductive hypothesis with $s=k$ proves that $F'=\bigcap_{t=1}^{k+1} F_{J_t}$. We next observe that

$$\emptyset=I_0\subseteq J_1\subseteq I_1\subseteq J_2\subseteq\cdots\subseteq J_k\subseteq I_k\subseteq J_{k+1}\subseteq I_{k+1}=\{1,\dots,p\}. \tag{3.2}$$

As $\bigcap_{t=1}^{k+1} F_{J_t}=F'\subseteq\bigcap_{t=1}^k F_{I_t}$, we have that $(\bigcap_{t=1}^{k+1} F_{J_t})\cap(\bigcap_{t=1}^k F_{I_t})=F'$; further, by dropping J_t's which coincide with either I_{t-1} or with I_t, we get a superchain of $I_1,\dots,I_k$ which is a representing chain of F'. □

Theorem 3.4. *Suppose* λ *is supermodular and* F *and* F' *are nonempty faces of* H^λ. *Then the following are equivalent*:

(i) $F'\subseteq F$,
(ii) *each representing chain of* F *has a superchain which is a representing chain of* F', *and*
(iii) *some representing chain of* F *has a superchain which is a representing chain of* F',

Proof. The implication (i) ⇒ (ii) follows directly from Lemma 3.3, and the implication (ii) ⇒ (iii) is immediate from the existence of representing chains of faces (Theorem 3.2). Finally, if $I_1,\dots,I_k$ is a representing chain of F and $I'_1,\dots,I'_{k'}$ is a superchain of $I_1,\dots,I_k$ which is a representing chain of F', then $F=\bigcap_{t=1}^k F_{I_t}\supseteq\bigcap_{t=1}^{k'} F_{I'_t}=F'$. □

The following corollary of Theorem 3.4 demonstrates that representing chains of a face of H^λ provide characterization of the vertices in that face.

Corollary 3.5. *Suppose* λ *is supermodular. Let* F *be a nonempty face of* H^λ *with representing chain* $I_1,\dots,I_k$. *Then a vertex* v *of* H^λ *is in* F *if and only if there is a permutation* σ *which is consistent with* $I_1,\dots,I_k$ *and has* $v=\lambda_\sigma$, *in particular*, $F=conv\{\lambda_\sigma:\sigma$ *is a permutation which is consistent with* $I_1,\dots,I_k\}$.

4. Permutation polytopes corresponding to strongly supermodular functions

In this section we study permutation polytopes corresponding to strongly supermodular functions. For such polytopes, we show that minimal chain representation of faces is unique and use the minimal chain representation of faces to derive an isomorphic representation of the corresponding face lattice.

Lemma 4.1. *Suppose* λ *is strongly supermodular and* (I,J) *is* λ*-flat. Then every triplet* (I',K',J') *with* $I\subseteq I'\subset K'\subset J'\subseteq J$ *is* λ*-flat.*

Proof. Let $L' \equiv I' \cup (J' \setminus K')$. By the λ-flatness of (I,J) we then have that

$$\begin{aligned}\lambda(I) + \lambda(J) &= \lambda(I') + \lambda[I \cup (J \setminus I')],\\ &= \lambda(J') + \lambda[I \cup (J \setminus J')],\\ &= \lambda(K') + \lambda[I \cup (J \setminus K')],\\ &= \lambda(L') + \lambda[I \cup (J \setminus L')].\end{aligned} \tag{4.1}$$

As $K' \cap L' = I'$ and $K' \cup L' = J'$, the supermodularity of λ implies that

$$\lambda(K') + \lambda(L') \leqslant \lambda(I') + \lambda(J'). \tag{4.2}$$

Also, as

$$[I \cup (J \setminus K')] \cap [I \cup (J \setminus L')] = I \cup [J \setminus (K' \cup L')] = I \cup (J \setminus J')$$

and

$$[I \cup (J \setminus K')] \cup [I \cup (J \setminus L')] = I \cup [J \setminus (K' \cap L')] = I \cup (J \setminus I'),$$

we have that

$$\lambda[I \cup (J \setminus K')] + \lambda[I \cup (J \setminus L')] \leqslant \lambda[I \cup (J \setminus I')] + \lambda[I \cup (J \setminus J')]. \tag{4.3}$$

We conclude from (4.1)–(4.3) that

$$\begin{aligned}0 &= \lambda(I') + \lambda(J') - \lambda(K') - \lambda(L') + \lambda[I \cup (J \setminus I')]\\ &\quad + \lambda[I \cup (J \setminus J')] - \lambda[I \cup (J \setminus K')] - \lambda[I \cup (J \setminus L')] \geqslant 0.\end{aligned}$$

It follows that equality holds in (4.2) (and (4.3)), assuring that (I', K', J') is λ-flat. □

Corollary 4.2. *Suppose λ is strongly supermodular and (I,J) is λ-flat. Then every pair of subsets (I',J') with $I \subseteq I' \subset J' \subseteq J$ and $|J' \setminus I'| \geqslant 2$ is λ-flat.*

Lemma 4.3. *Suppose λ is strongly supermodular and I', I, J' and J are subsets of $\{1,\ldots,p\}$ such that $I' \subseteq I \subset J' \subseteq J$ where both (I',J') and (I,J) are λ-flat. Then (I',J) is λ-flat.*

Proof. The result is trivial when either $I' = I$ or $J' = J$. In the remaining case $I' \subset I \subset J' \subset J$ and the assumptions of the lemma imply that (I', I, J') and (I, J', J) are λ-flat. Let $K \equiv I' \cup (J' \setminus I)$ and $L \equiv I \cup (J \setminus J') = I' \cup (J \setminus K)$. As (I', I, J') is λ-flat, we have that

$$\lambda(I) + \lambda(K) = \lambda(I') + \lambda(J'),$$

and as (I, J', J) is λ-flat, we have that

$$\lambda(J') + \lambda(L) = \lambda(I) + \lambda(J).$$

Summing the above equalities and canceling identical terms, we see that

$$\lambda(K) + \lambda(L) = \lambda(I') + \lambda(J);$$

as $L = I' \cup (J \setminus K)$, we conclude that (I', K, J) is λ-flat; hence, by the strong supermodularity of λ, (I',J) is λ-flat. □

Lemma 4.4. *Suppose λ is strongly supermodular; $I_1,\ldots,I_k$ is a chain and $s \in \{1,\ldots,k\}$. Then the following are equivalent:*

(a) $\bigcap_{t=1}^{k} F_{I_t} = \bigcap_{t=1,\ t\neq s}^{k} F_{I_t}$,
(b) $F_{I_{s-1}} \cap F_{I_{s+1}} \subseteq F_{I_s}$, *and*
(c) (I_{s-1}, I_{s+1}) *is λ-flat.*

Proof. (b) $\Rightarrow$ (a): This implication is trite.

(c) $\Rightarrow$ (b): Suppose (I_{s-1}, I_{s+1}) is λ-flat. Then, (I_{s-1}, I_s, I_{s+1}) is λ-flat. Let $J_s \equiv I_{s-1} \cup (I_{s+1} \setminus I_s)$. As $I_s \cap J_s = I_{s-1}$ and $I_s \cup J_s = I_{s+1}$, part (a) of Lemma 3.1 next implies that $F_{I_{s-1}}, \cap F_{I_{s+1}} \subseteq F_{I_s} \cap F_{J_s} \subseteq F_{I_s}$.

(a) $\Rightarrow$ (c): Suppose (a) holds. Again, let $J_s \equiv I_{s-1} \cup (I_{s+1} \setminus I_s)$. As $I_1, \ldots, I_{s-1}, J_s, I_{s+1}, \ldots, I_k$ is a chain, it represents a nonempty face F (Theorem 3.2) which is contained in the face $F' \equiv \bigcap_{t=1,\ t\neq s}^k F_{I_t} = \bigcap_{t=1}^k F_{I_t}$. It follows that $\emptyset \neq F' \subseteq F_{J_s} \cap F_{I_s}$. As I_s and J_s are not ordered by set inclusion, $I_s \cap J_s = I_{s-1}$ and $I_s \cup J_s = I_{s+1}$, we conclude from part (b) of Lemma 3.1 that (I_{s-1}, I_s, I_{s+1}) is λ-flat; hence, by the strong supermodularity of λ, (I_{s-1}, I_{s+1}) is λ-flat. □

Corollary 4.5. *Suppose λ is strongly supermodular; $I_1, \ldots, I_k$ is a minimal chain and $F = \bigcap_{t=1}^k F_{I_t}$, and $s \in \{0, \ldots, k\}$. If there exists a set I satisfying $I_s \subset I \subset I_{s+1}$ and $F_I \supseteq F$, then (I_s, I_{s+1}) is λ-flat.*

Proof. The insertion of I into the chain $I_1, \ldots, I_k$, between I_s and I_{s+1}, yields a superchain which is another representing chain of F. As the removal of I from this superchain is the original chain which represents F, the equivalence of (a) and (c) in Lemma 4.4 implies that (I_s, I_{s+1}) is λ-flat. □

The next result establishes uniqueness of minimal representing chains of faces.

Lemma 4.6. *Suppose λ is strongly supermodular. If $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are minimal chains with $\bigcap_{t=1}^k F_{I_t} = \bigcap_{t=1}^{k'} F_{I'_t} \neq \emptyset$, then $k = k'$ and $I_t = I'_t$ for $t = 1, \ldots, k = k'$.*

Proof. Let $F \equiv \bigcap_{t=1}^k F_{I_t} = \bigcap_{t=1}^{k'} F_{I'_t} \neq \emptyset$. We proceed with an inductive argument and prove that for each positive integer $s \leqslant \min\{k, k'\} + 1$, $I_s = I'_s$, in particular, if $s = \min\{k, k'\} + 1$, then $I_s = I'_s = \{1, \ldots, p\}$ and $s = k + 1 = k' + 1$. As $I_0 = I'_0 = \emptyset$, the assertion is trite for $s = 0$. Assume the assertion holds for integer $s < \min\{k, k'\} + 1$ and we will establish it with $s + 1$ replacing s.

We first observe that (I_s, I_{s+2}) is not λ-flat, for otherwise Lemma 4.4 implies that $I_1, \ldots, I_s, I_{s+2}, \ldots, I_k$ is also a representing chain of F, contradicting the minimality of $I_1, \ldots, I_k$.

We next argue that I_{s+1} and I'_{s+1} are ordered by set inclusion. Aiming to establish a contradiction we assume that this conclusion is false. In particular, neither I_{s+1} nor I'_{s+1} equals $\{1, \ldots, p\}$ assuring that $s + 1 < \min\{k, k'\} + 1$. With $J \equiv I_{s+1} \cap I'_{s+1}$ and $K \equiv I_{s+1} \cup I'_{s+1}$, we have that $I_s \subseteq J \subset I_{s+1} \subset K$. As $F_{I_{s+1}} \cap F_{I'_{s+1}} \supseteq F \neq \emptyset$, Lemma 3.1 implies that $F_J \cap F_K = F_{I_{s+1}} \cap F_{I'_{s+1}} \supseteq F$ and $\lambda(J) + \lambda(K) = \lambda(I_{s+1}) + \lambda(I'_{s+1})$; in particular, the strong supermodularity of λ implies that (J, K) is λ-flat. Now, if $I_s \neq J$, then $I_s \subset J \subset I_{s+1} \subset K$; as $F_J \supseteq F_J \cap F_K \supseteq F$, Corollary 4.5 implies that (I_s, I_{s+1}), is λ-flat. So both (I_s, I_{s+1}) and (J, K) are λ-flat and Lemma 4.3 assures that (I_s, K) is λ-flat. When $I_s = J$, we have that (I_s, K) is λ-flat from the established λ-flatness of (J, K). Thus, we conclude that regardless of whether or not $I_s = J$, (I_s, K) is λ-flat. As it was shown that (I_s, I_{s+2}) is not λ-flat and (I_s, K) is λ-flat, it now follows from Corollary 4.2 that $I_{s+2} \nsubseteq K$.

We next argue that necessarily $K \subset I_{s+2}$. Indeed, suppose this is not the case. As $I_{s+2} \nsubseteq K$, we then have that K and I_{s+2} are not ordered by set-inclusion. As $F_{I_{s+2}} \cap F_K \supseteq F \neq \emptyset$, Lemma 3.1 and the strong supermodularity of λ imply that $(I_{s+2} \cap K,\ I_{s+2} \cup K)$ is λ-flat. As $I_s \subseteq K \cap I_{s+2} \subset K \subset K \cup I_{s+2}$ and (I_s, K) was shown to be λ-flat, we conclude from Lemma 4.3 that $(I_s, K \cup I_{s+2})$ is λ-flat; as $I_s \subset I_{s+2} \subseteq K \cup I_{s+2}$, it then follows from Corollary 4.2 that (I_s, I_{s+2}) is λ-flat, a contradiction. Thus, indeed, $K \subset I_{s+2}$. So, $I_s \subset I_{s+1} \subset K \subset I_{s+2}$. As $F_K \supseteq F_K \cap F_J \supseteq F$, Corollary 4.5 implies that (I_{s+1}, I_{s+2}) is λ-flat and therefore the λ-flatness of (I_s, K) and another application of Lemma 4.3 imply that (I_s, I_{s+2}) is λ-flat, a contradiction. This contradiction establishes that I_{s+1} and I'_{s+1} are ordered by set inclusion.

Without loss of generality, we proceed under the assumption that $I_{s+1} \subseteq I'_{s+1}$. So, $K = I_{s+1} \cup I'_{s+1} = I'_{s+1}$. Now, suppose $I_{s+1} \neq I'_{s+1}$, that is, $I_{s+1} \subset I'_{s+1}$. As $I'_s = I_s \subset I_{s+1} \subset I'_{s+1}$ and $F_{I_{s+1}} \supseteq F = \bigcap_{t=1}^{k'} F_{I'_t}$, we conclude from Corollary 4.5 that $(I'_s = I_s, I'_{s+1} = K)$ is λ-flat. The arguments of the above paragraph then imply that (I_s, I_{s+2}) is λ-flat, a contradiction. This contradiction proves that $I_{s+1} = I'_{s+1}$. □

We observe that subchains of minimal chains are minimal, and every chain $I_1, \ldots, I_k$ has a minimal subchain $I'_1, \ldots, I'_{k'}$ with $\bigcap_{t=1}^{k'} F_{I'_t} = \bigcap_{t=1}^k F_{I_t}$. We say that minimal chain $I'_1, \ldots, I'_{k'}$ *refines* minimal chain $I_1, \ldots, I_k$ if $I'_1, \ldots, I'_{k'}$ can be constructed from $I_1, \ldots, I_k$ by augmenting this chain with additional sets and then dropping sets which become superfluous; formally $I'_1, \ldots, I'_{k'}$ refines $I_1, \ldots, I_k$ if there exists a chain $I''_1, \ldots, I''_{k''}$ which is a superchain of both $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ and $\bigcap_{t=1}^{k''} F_{I''_t} = \bigcap_{t=1}^{k'} F_{I'_t}$. We observe that the refining relationship is a partial order on the set of minimal chains.

Example 3. Let $p = 3$, $\lambda(I) = 2|I| - 1$ for each $\emptyset \subset I \subseteq \{1, 2, 3\}$. Then λ is supermodular, in fact, strongly supermodular, and (by Theorem 3.2) $C^\lambda = H^\lambda$ is the convex hull of $\{(1, 2, 2),\ (2, 1, 2)$ and $(2, 2, 1)\}$. It is easy to verify that the chains $I_1 = \{1, 2\}$ and $I'_1 = \{1\}$ are minimal chains representing the faces $\{x \in \mathbb{R}^3\colon\ x_1 + x_2 = 3,\ x_3 = 2\}$ and $\{(1, 2, 2)\}$, respectively. Now, the chain $I''_1 = I'_1$, $I''_2 = I_1$ is a superchain of the above two minimal chains and $F_{I''_1} \cap F_{I''_2} = F_{I'_1}$. So, I'_1 refines I_1.

Theorem 4.7. *Suppose λ is strongly supermodular. A subset F of $\mathbb{R}^p$ is a nonempty face of H^λ if and only if there is a minimal chain $I_1,\ldots,I_k$ with $F=\bigcap_{t=1}^k F_{I_t}$, and the correspondence of nonempty faces of H^λ onto minimal chains is one-to-one. Further, if F is a nonempty face of H^λ corresponding to minimal chain $I_1,\ldots,I_k$, then*:

(a) *if $I'_1,\ldots,I'_{k'}$ is a chain with $F=\bigcap_{t=1}^{k'} F_{I'_t}$, then $I_1,\ldots,I_k$ is a subchain of $I'_1,\ldots,I'_{k'}$*,
(b) *if F' is a nonempty face of H^λ, then the following are equivalent*:
  (i) $F'\subseteq F$,
  (ii) *F' has a representing chain which is a superchain of $I_1,\ldots,I_k$*,
  (iii) *the minimal chain representing F' refines $I_1,\ldots,I_k$*,
(c) $\dim F\leqslant \dim H^\lambda - k$.

Proof. By Theorem 3.2, a subset F of $\mathbb{R}^p$ is a nonempty face of H^λ if and only if there is a chain $I_1,\ldots,I_k$ with $F=\bigcap_{t=1}^k F_{I_t}$; each such chain has a minimal subchain $I'_1,\ldots,I'_{k'}$ with $\bigcap_{t=1}^{k'} F_{I'_t}=\bigcap_{t=1}^k F_{I_t}=F$. We conclude that a set $F\subseteq\mathbb{R}^p$ is a nonempty face of H^λ if and only if it has a representing chain which is minimal. By Lemma 4.6, a minimal chain representing a given face is unique; also, trivially, a minimal chain uniquely defines the corresponding face. So the correspondence of nonempty faces of H^λ to minimal chains is one-to-one and onto.

Next, let F be a nonempty face of H^λ with representing minimal chain $I_1,\ldots,I_k$.

(a): If $I'_1,\ldots,I'_{k'}$ is a representing chain of F, then it has a minimal subchain $I''_1,\ldots,I''_{k''}$ with $\bigcap_{t=1}^{k''} F_{I''_t}=\bigcap_{t=1}^{k'} F_{I'_t}=F=\bigcap_{t=1}^k F_{I_t}$. As $\bigcap_{t=1}^{k''} F_{I''_t}=\bigcap_{t=1}^k F_{I_t}$, Lemma 4.6 assures that the minimal chains $I_1,\ldots,I_k$ and $I''_1,\ldots,I''_{k''}$ coincide, thus $I_1,\ldots,I_k$ is a subchain of $I'_1,\ldots,I'_{k'}$.

(b): Suppose F' is a nonempty face of H^λ. The implication (i) $\Rightarrow$ (ii) follows from Theorem 3.4. To see that (ii) $\Rightarrow$ (iii) assume that F' has a representing chain $I''_1,\ldots,I''_{k''}$ which is a superchain of $I_1,\ldots,I_k$. It then follows from the established part (a) that the minimal chain representing F', say $I'_1,\ldots,I'_{k'}$, is a subchain of $I''_1,\ldots,I''_{k''}$. As minimal chains $I_1,\ldots,I_k$ and $I'_1,\ldots,I'_{k'}$ are both subchains of $I''_1,\ldots,I''_{k''}$ and $\bigcap_{t=1}^{k''} F_{I''_t}=F'=\bigcap_{t=1}^{k'} F_{I'_t}$, we have that $I'_1,\ldots,I'_{k'}$ refines $I_1,\ldots,I_k$. Finally, to see that (iii) $\Rightarrow$ (i) assume the minimal chain representing F', say $I'_1,\ldots,I'_{k'}$, refines $I_1,\ldots,I_k$. Then there exists a chain $I''_1,\ldots,I''_{k''}$ which is a superchain of both $I_1,\ldots,I_k$ and $I'_1,\ldots,I'_{k'}$ with $\bigcap_{t=1}^{k''} F_{I''_t}=\bigcap_{t=1}^{k'} F_{I'_t}$; in particular, $F'=\bigcap_{t=1}^{k''} F_{I''_t}\subseteq\bigcap_{t=1}^k F_{I_t}=F$.

(c): The minimality of the chain $I_1,\ldots,I_k$ implies that the sets $F_0\equiv H^\lambda$ and $F_s\equiv\bigcap_{t=1}^s F_{I_t}$ for $s=1,\ldots,k$ are distinct. As these sets are faces of H^λ and $F_0=H^\lambda\supset F_1\supset\cdots\supset F_{k-1}\supset F_k=F$, it follows (cf., part (f) of Proposition A.1) that $\dim H^\lambda=\dim F_0>\dim F_1>\cdots>\dim F_{k-1}>\dim F_k=\dim F$; it follows that $\dim F\leqslant\dim H^\lambda-k$. □

Property (a) of the minimal chain corresponding to a face F of H^λ characterizes that chain as the common subchain of all representing chains of F, namely as the unique *minimal* representing chain for F. Property (b) shows that the correspondence of faces to minimal representing chains is an isomorphism of the face-lattice with set inclusion as the partial order onto the set of minimal chains with the "refining" partial order; in particular, we obtain a lattice structure for the minimal chains. Finally, property (c) shows the length of the minimal chain corresponding to a face of H^λ yields an upper bound on the dimension of that face. □

The next example demonstrates that strong supermodularity does not suffice for the unique representation of a face via minimal chains.

Example 1 (continued)**.** We observe that in Example 1, C^λ is the polytope defined by the linear system

$$x_1\geqslant 1,\ \ x_2\geqslant 2,\ \ x_3\geqslant 3,$$

$$x_1+x_2\geqslant 3,\ \ x_1+x_3\geqslant 5,\ \ x_2+x_3\geqslant 5,$$

$$x_1+x_2+x_3=7.$$

By Theorem 3.2, the vertices of $H^\lambda=C^\lambda$ are the λ_σ's with σ ranging over the permutations over $\{1,2,3\}$; these permutations together with the corresponding λ_σ's are listed below in Table 1.

In particular, $v=(1,2,4)^{\mathrm{T}}$ is a vertex which lies in F_I for $I\in\{\{1\},\{2\},\{1,2\},\{1,3\}\}$ and $F_I=\{v\}$ for $I\in\{\{1\},\{1,2\}\}$. Hence, $\{1\}$ and $\{1,2\}$ are two distinct minimal representing chains (each of length 1). There are 3 maximal representing chains of $\{v\}$, namely: (i) $\{1\},\{1,2\}$, (ii) $\{1\},\{1,3\}$, and (iii) $\{2\},\{1,2\}$. Each of these maximal chains has the length $3-1=2$.

Finally, the next example demonstrates that the bound in part (c) of Theorem 4.7 need not be tight.

Table 1

Permutation σ	λ_σ
$(\{1\}, \{2\}, \{3\})$	$(1,2,4)^{\mathrm{T}}$
$(\{1\}, \{3\}, \{2\})$	$(1,2,4)^{\mathrm{T}}$
$(\{2\}, \{1\}, \{3\})$	$(1,2,4)^{\mathrm{T}}$
$(\{2\}, \{3\}, \{1\})$	$(2,2,3)^{\mathrm{T}}$
$(\{3\}, \{1\}, \{2\})$	$(2,2,3)^{\mathrm{T}}$
$(\{3\}, \{2\}, \{1\})$	$(2,2,3)^{\mathrm{T}}$

Table 2

Permutation σ	λ_σ
$(\{1\}, \{2\}, \{3\})$	(3, 3, 3)
$(\{1\}, \{3\}, \{2\})$	(3, 3, 3)
$(\{2\}, \{1\}, \{3\})$	(5, 1, 3)
$(\{2\}, \{3\}, \{1\})$	(6, 1, 2)
$(\{3\}, \{1\}, \{2\})$	(5, 3, 1)
$(\{3\}, \{2\}, \{1\})$	(6, 2, 1)

Example 4. Let $p=3$, $\lambda(\{1\})=\lambda(\{2,3\})=3$, $\lambda(\{2\})=\lambda(\{3\})=1$, $\lambda(\{1,2\})=\lambda(\{1,3\})=6$ and $\lambda(\{1,2,3\})=9$. Then λ is not strictly supermodular, as $\lambda(\{1,2\})+\lambda(\{1,3\})=6+6=9+3=\lambda(\{1,2,3\})+\lambda(\{1\})$; in fact, $\{1,2\}$ and $\{1,3\}$ is the only pair of subsets of $\{1,2,3\}$ which is not ordered and satisfies (2.1) with equality; consequently, it follows that λ is strongly supermodular. By Theorem 3.2, the vertices of $H^\lambda=C^\lambda$ are the λ_σ's with σ ranging over the permutations of $\{1,2,3\}$; these permutations along with the λ_σ's are listed below in Table 2.

It is easy to verify that H^λ, namely, the convex hull of the λ_σ's, has dimension 2. The chain $\{1\}$ of length $k=1$ is a minimal chain representing the vertex $F=\{(3,3,3)\}$ and $(\dim H^\lambda)-k=2-1=1>0=\dim F$.

Appendix

In this Appendix, we summarize results about polytopes and their faces that are used in our paper. For a proof of the first result, see [14, Propositions 2.2 and 2.3, pp. 52–53, and Theorem 2.7 and followwing discussion, pp. 57–58].

Proposition A.1. *Let P be a polytope in $\mathbb{R}^p$. Then*:

(a) *P is the convex hull of its vertices,*
(b) *intersections of faces of P are faces of P,*
(c) *each face of P is the intersection of facets of P,*
(d) *each proper face F of P is a facet of a face F' of P,*
(e) *the faces of a face F of P are exactly the faces of P that are contained in F, in particular, the vertices of F are the vertices of P that are contained in F,*
(f) *a face F' of P is strictly included in a face F of P if and only if $F'\subseteq F$ and $\dim F'<\dim F$,*
(g) *if P is a polytope with representation*

$$\sum_{j=1}^{n} B_{kj}x_j \leqslant b_k \quad \text{for all } k\in\beta, \tag{A.1}$$

where β is a finite index set, then each facet F of P has a representation of the form $F=\{x\in P: \sum_{j=1}^{n} B_{rj}x_j=b_r\}$ for some $r\in\beta$,
(h) *if P is the convex hull of a finite subset B, then B contains all the vertices of P, in particular, maximizers of a linear function over B is a maximizer of that function over P,*
(i) *a linear function on P attains a maximum at a vertex of P,*
(j) *if $\dim P=1$, then P has exactly two vertices,*
(k) *a polytope of dimension m has a face of dimension k for each $0\leqslant k<m$, and*
(l) *F is a face of P if and only if F is a convex extreme set of P.*

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