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Anti-control of chaos in rigid body motion

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Abstract

Anti-control of chaos for a rigid body has been studied in the paper. For certain feedback gains, a rigid body can easily generate chaotic motion. Basic dynamical behaviors, such as symmetry, invariance, dissipativity and existence of attractor, are also discussed. The transient behaviors of the chaotic system have also been presented as the feedback gain changed. Of particular interesting is the fact that the chaotic system can generate a complex multi-scroll chaotic attractor under the appropriate feedback gains. Finally, it was shown that the system could be related to the famous Lorenz equations and Chen system. In other words, the system can easily display all the dynamical behaviors of the famous Lorenz equations and Chen system.

1. Introduction

More than 30 years of studying physical phenomena in chaotic dynamics, which has brought us much improved understanding of the world around us. Most importantly, we have learnt about chaos to a point where we are confident about the consequences of its presence under certain conditions: whether it is safe of disastrous, useful or useless, etc. During the last decade, many methods have been proposed to control chaos, i.e., to stabilize the chaotic dynamical systems to period motion, when chaos is not unwanted or undesirable. Recently, many excellent books were given by Moon [1], Chen and Dong [2], and Kapitaniak [3]. Moreover, some outstanding reports were presented by EI Naschie [4] and Kapitaniak [5]. Sometimes chaotic behavior and chaos synchronization are beneficial and desirable in many applications. For example, chaos is important in secure communication, information processing, liquid mixing, biological systems, etc. [6–8]. For this purpose, making a nonchaotic dynamical system chaotic or retaining (or enhancing) the chaos of a chaotic system is called "anti-control of chaos or chaotification [9,10]". Therefore, the anti-control of chaos is meaningful topic and worth to be investigated.

Scientists and mathematicians have been working on the problem of rigid body motion for over two centuries. Which has many practical engineering applications such as gyroscopes, satellites, spacecraft and rockets. However, an analytic solution to the general problem of a rigid body under the influence of arbitrary external torques is far from complete. In fact, most existing analytic theories were applied to highly idealized cases, such as torque-free or symmetric bodies. Solutions have been obtained for these and several other special cases by Euler, Jacobi, Poinsot and other researchers, are reported by Leimanis [11]. Unfortunately these solutions are hardly of practical importance to the complex problems encountered in spacecraft dynamics and control. Leipnik and Newton [12] found strange attractors

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in rigid body motion. Because of their works, the chaotic dynamics in rigid body motion have been intensively studied by many researchers [13–17].

In 1963, Lorenz [18] discovered chaos in a simple system of three autonomous ordinary differential equations that has only quadratic nonlinearities, in order to describe the simplified Rayleigh–Bénard problem. The Lorenz system has seven terms on the right-hand side, two of which are nonlinear (xz and xy). In 1976, Rösler [19] found a three-dimensional quadratic autonomous chaotic system, which also has seven terms on the right-hand side, but with only one quadratic nonlinearity (xz). Recently, Liu and Chen [20] created a new chaotic system. An electronic circuit was also designed to realize the new system. The system consists of three ordinary differential equations with three quadratic nonlinear terms. In 1944, Nadolschi [11] showed that the true Euler equations for the motion of a rigid body. In fact, Euler equations of a rigid body motion are a simple and important three-dimensional autonomous system in classical mechanics. In this paper, the easier method is proposed to construct a chaotic system by applying linear feedback with certain gains. It is notable that the system has six terms on the right-hand side, three of which are nonlinear (yz, xz and xy). The following problem will be investigated: For simple feedback gains, a rigid body whether can easily generate chaotic motion? Next, basic dynamical behaviors, such as symmetry, invariance, dissipativity and existence of attractor, will also be discussed. Furthermore, the transient behaviors of the chaotic system will also be studied when the feedback gains changed. Finally, the relation of the system with the famous Lorenz equations and Chen system [21] will also be discussed.

2. Equations of motion

The Euler equations for motion of a rigid body with principle axes at the center of mass are

$$\begin{cases} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + M_1, \\ I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + M_2, \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3, \end{cases} \tag{1}$$

where I_1, I_2, I_3 are the principal moment of inertias, ω_1, ω_2, ω_3 are the angular velocities about principal axes fixed at the center of mass and M_1, M_2, M_3 are applied moments. In our case, the applied moments are considered to linear feedback, so that $M = A\omega$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}. \tag{2}$$

Then the equations are represented as

$$\begin{cases} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + a_{11}\omega_1, \\ I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + a_{22}\omega_2, \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + a_{33}\omega_3. \end{cases} \tag{3}$$

Denoting $\omega_1 = x$, $\omega_2 = y$, $\omega_3 = z$, $a_{11}/I_1 = a$, $a_{22}/I_2 = b$, $a_{33}/I_3 = c$, Eq. (3) is rewritten in the form

$$\begin{cases} \dot{x} = \frac{I_2 - I_3}{I_1} yz + ax, \\ \dot{y} = \frac{I_3 - I_1}{I_2} xz + by, \\ \dot{z} = \frac{I_1 - I_2}{I_3} xy + cz. \end{cases} \tag{4}$$

3. Basic dynamical behaviors and anti-control of chaos

Yielding a possibility for chaos, the equilibrium of the system (4) must be unstable. According to the results of Liu and Chen [20], the parameters a, b, c must satisfy the following necessary condition such that the system (4) to generate chaos.

$$a > 0, \quad b < 0, \quad c < 0 \quad \text{and} \quad 0 < a < -(b + c). \tag{5}$$

It is noted that condition (5) is just one of three cases. The other two cases would be obtained similar results owing to the symmetry of system (4). It is notable that the system (4) has six terms on the right-hand side, three of which are nonlinear (yz, xz, xy). On the other hand, the parameters I_1, I_2 and I_3 need to satisfy

$$\frac{I_2 - I_3}{I_1} < 0, \quad \frac{I_3 - I_1}{I_2} > 0, \quad \frac{I_1 - I_2}{I_3} > 0 \quad \text{i.e.,} \ I_3 > I_1 > I_2 \tag{6}$$

or

$$\frac{I_2 - I_3}{I_1} > 0, \quad \frac{I_3 - I_1}{I_2} < 0, \quad \frac{I_1 - I_2}{I_3} < 0 \quad \text{i.e.,} \ I_2 > I_1 > I_3. \tag{7}$$

By suitable variables transformation, the same results could be obtained for the system (4) with conditions (6) and (7). So we can just study the case with conditions (5) and (6). For simplicity, assume that $I_3 = 3I_0$, $I_1 = 2I_0$, $I_2 = I_0(I_3 > I_1 > I_2)$, and then the system (4) could be rewritten as

$$\begin{cases} \dot{x} = -yz + ax, \\ \dot{y} = xz + by, \\ \dot{z} = (1/3)xy + cz. \end{cases} \tag{8}$$

It is note that the invariance of the system under the transforms $(x, y, z) \to (x, -y, -z)$, $(x, y, z) \to (-x, y, -z)$, and $(x, y, z) \to (-x, -y, z)$. That is, the system (4) is symmetrical about three coordinate axes x, y, z, respectively. Further, these symmetries persist for all values of the system parameters. This chaotic system is robust to various small perturbations due to its highly symmetric structure.

Furthermore, the system (8), it is noticed that

$$\nabla \cdot V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = a + b + c. \tag{9}$$

From Eq. (5), it is clear that $a + b + c < 0$, so the system (8) is dissipative, with an exponential contraction rate:

$$\frac{dV}{dt} = (a + b + c)V. \tag{10}$$

That is, a volume element V_0 is contracted by the flow into a volume element $V_0 e^{(a+b+c)t}$ in time t. This means that each volume containing the system trajectory tend to zero as $t \to \infty$ at an exponential rate, $a + b + c$. Therefore, all system orbits ultimately are confined to a specific subset of zero volume, and the asymptotic motion settles onto an attractor.

The phase portrait is the evolution of a set of trajectories emanating from various initial conditions in the state space. When the solution reaches steady state, the transient behavior disappears. By numerical integration method, the phase portrait of the system, Eq. (8), is plotted in Fig. 1 for $a = 5$, $b = -10$, $c = -3.8$. Clearly, the motion is chaotic and symmetric about an axis. With the others feedback gains, the strange attractors are shown in Figs. 2–4. So we can easily make the motion of a rigid body chaotic via choosing suitable feedback gains.

By numerical results, the system (8) exhibits both strange attractors and limit cycles for certain choices of a, b, c. As illustrated in Fig. 4, for $a = 3$, $b = -5$, $c = -1.0$, there are two strange attractors for the system (8) with the initial conditions $(0.2, 0.2, 0.2)$ and $(0.2, -0.2, -0.2)$ displayed in Fig. 5(a) and (b). There are also two limit cycles for the system (8) with the initial conditions $(0.2, -0.2, 0.2)$ and $(0.2, 0.2, -0.2)$ shown in Fig. 5(c) and (d).

4. Transient behavior analysis

In previous section, we construct a chaotic system easily and successfully. Now, attention is shifted to transient behavior analysis of this chaotic system as the feedback gain changed. A set of parameters satisfying the aforementioned conditions are: $a = 5$, $b = -10$, $c = -3.8$. The corresponding transient and steady states of the system (8) is shown in Fig. 6. The trajectory starts from initial condition $(0.2, 0.2, 0.2)$ and converges to strange attractor stepwise by several loops. Furthermore, when c is varied, some interesting phenomena can be observed, as shown in Figs. 7 and 8. For example, with $c = -0.38$, the trajectory went through three scrolls before it reaches steady state. Besides, the multi-scroll chaotic attractor is also found. As $c = -0.038$, the trajectory formed an umbrella shape is displayed in Fig. 8. It swirls into the center $(0, 0, 48)$ from initial condition $(0.2, 0.2, 0.2)$, and then converges to strange attractor directly along a straight path.

By above results, the transient behaviors of this chaotic system with the different feedback gains are conspicuously unlike. We believe that if parameters a and b are varied some interesting phenomena will also be presented. This task will leave the reader as an exercise.

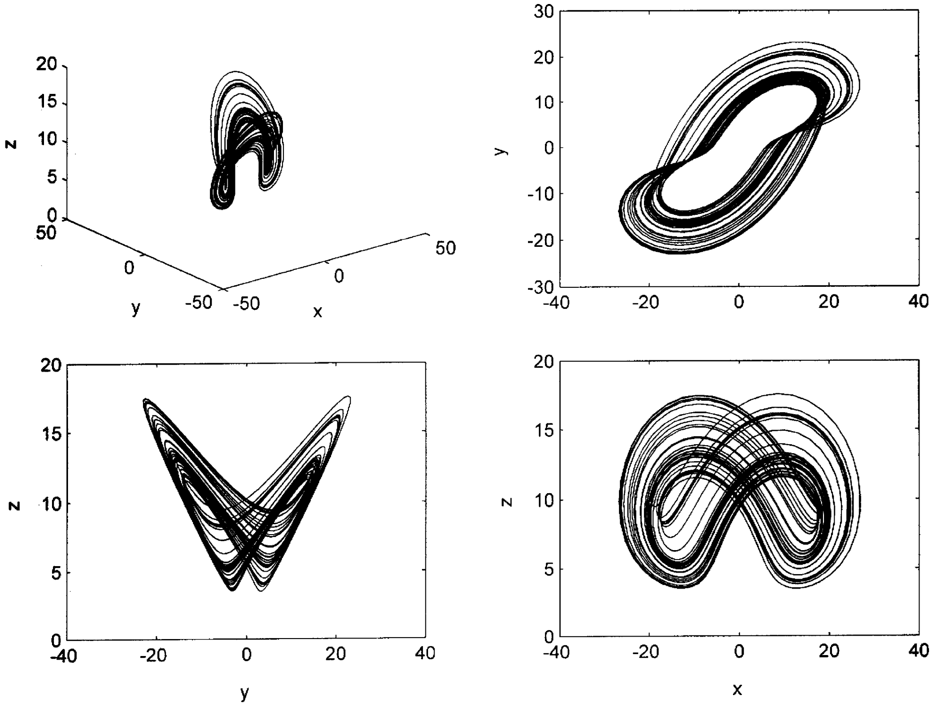

Fig. 1. The strange attractor of the system with $a = 5$, $b = -10$, $c = -3.8$.

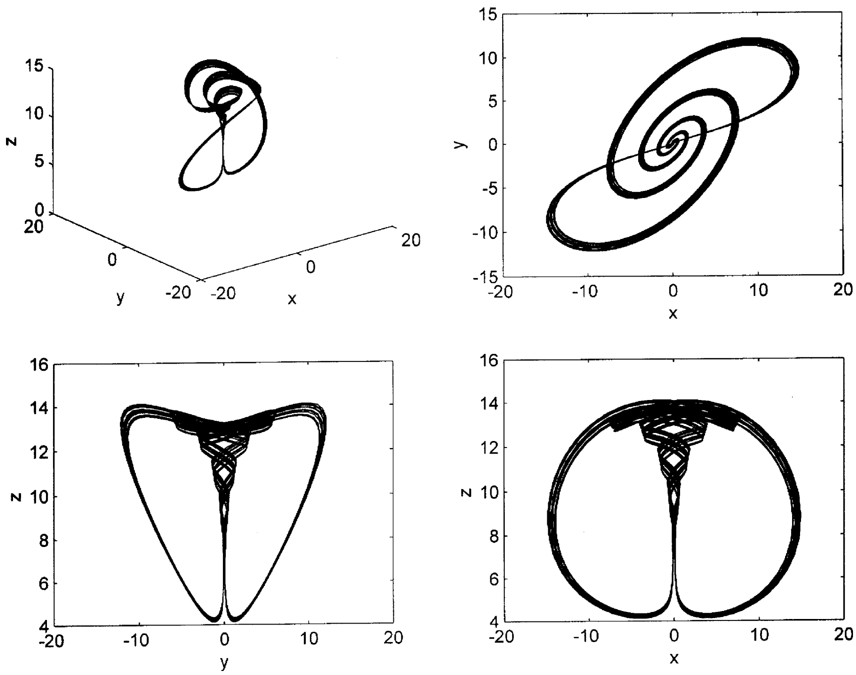

Fig. 2. The strange attractor of the system with $a = 5$, $b = -10$, $c = -0.38$.

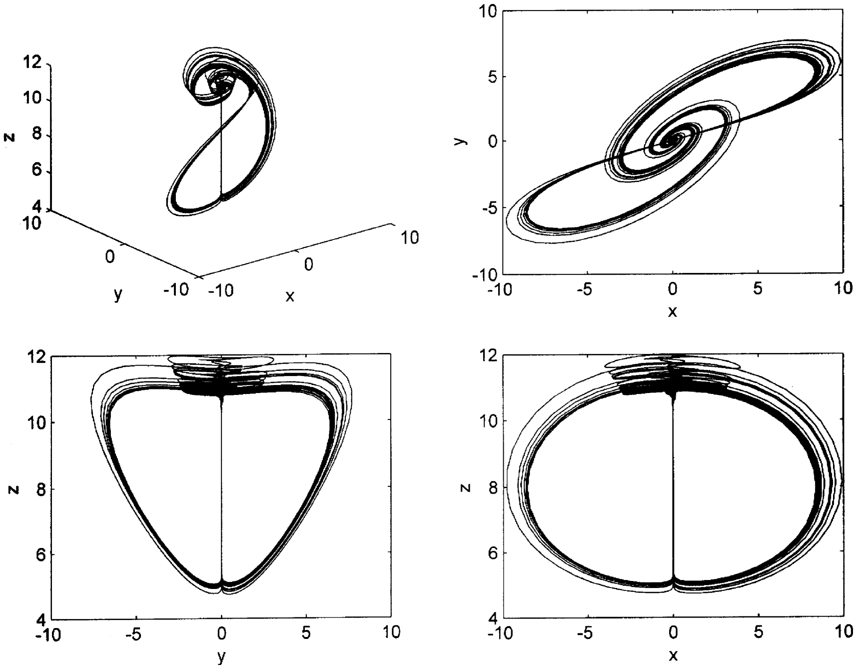

Fig. 3. The strange attractor of the system with $a = 5$, $b = -10$, $c = -0.038$.

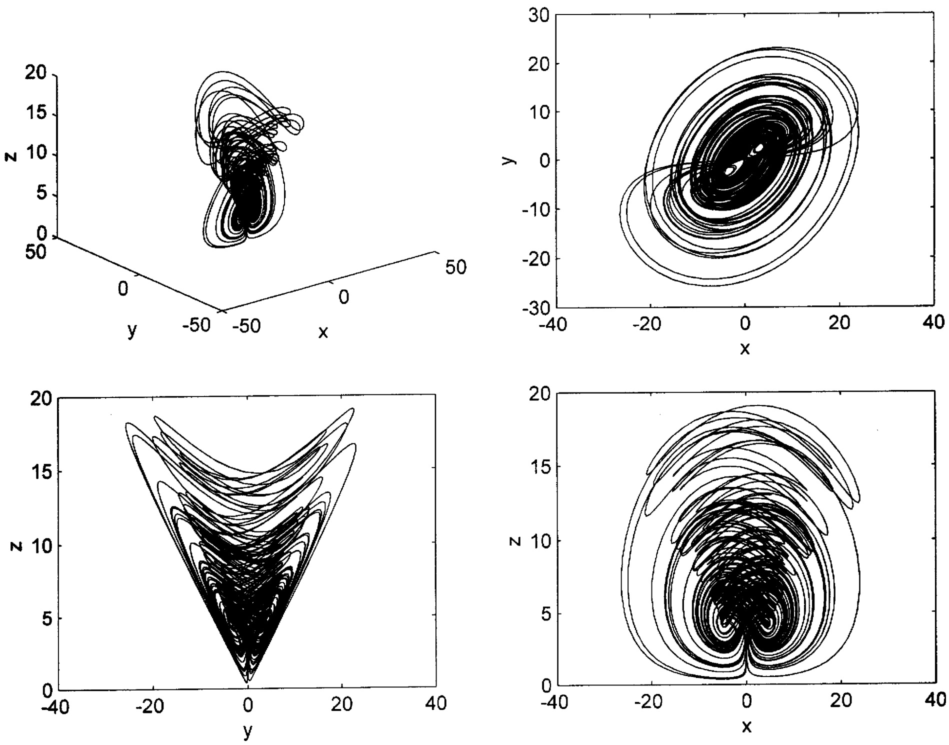

Fig. 4. The strange attractor of the system with $a = 3$, $b = -5$, $c = -1.0$.

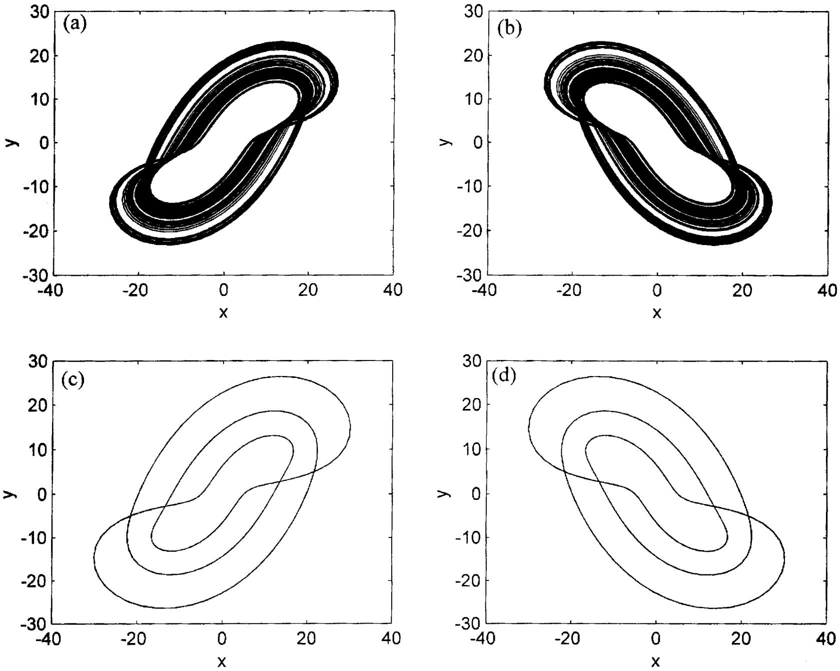

Fig. 5. For $a = 5$, $b = -10$, $c = -3.8$, the attractor for the system with initial conditions: (a) $(0.2, 0.2, 0.2)$; (b) $(0.2, -0.2, -0.2)$; (c) $(0.2, -0.2, 0.2)$; (d) $(0.2, 0.2, -0.2)$.

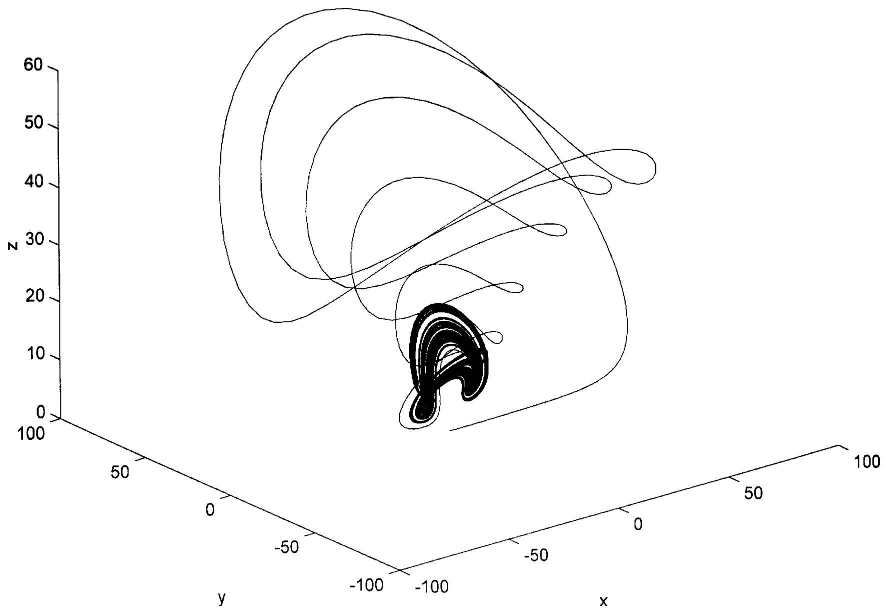

Fig. 6. The transient behavior of the system with $a = 5$, $b = -10$, $c = -3.8$.

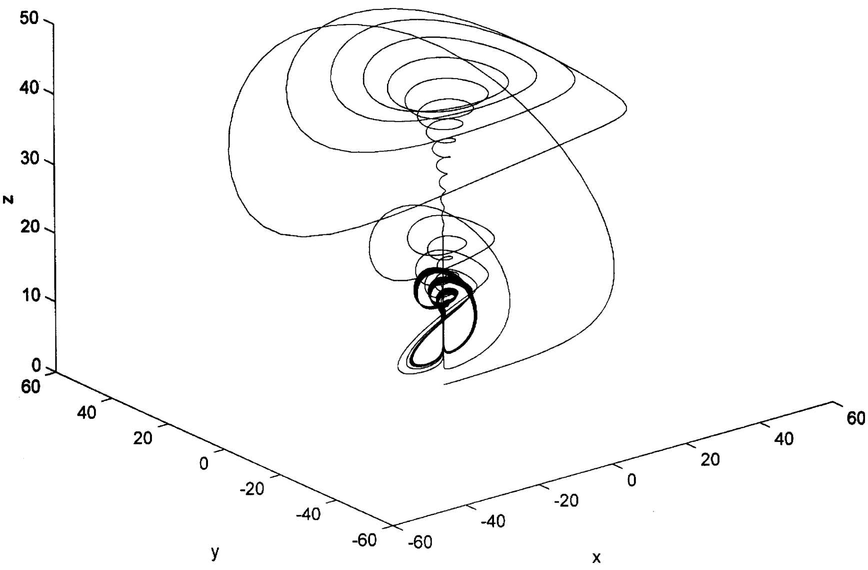

Fig. 7. The transient behavior of the system with $a = 5$, $b = -10$,$c = -0.38$.

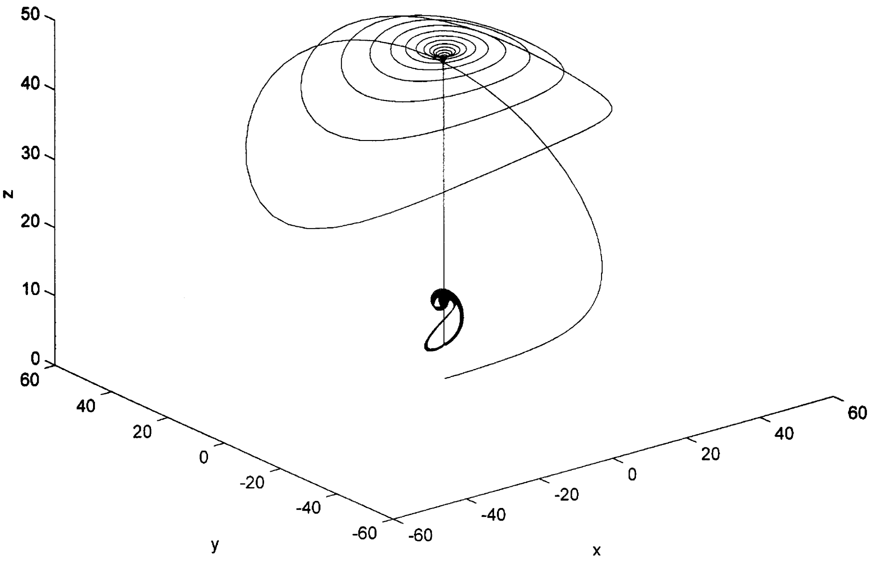

Fig. 8. The transient behavior of the system with $a = 5$, $b = -10$, $c = -0.038$.

5. Two special cases

If the applied moments of the Euler equations (1) are reconsidered to $M = A\omega$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}. \tag{11}$$

Then the system can be related to two special cases such as the Lorenz equations and Chen system.

(a) *Case 1:*
If $[I_1, I_2, I_3] = [2I_0, I_0, I_0]$ and $a_{12} = -a_{11}$, Eq. (1) can be rewritten as follows:

$$\begin{cases} \dot{x} = \frac{a_{11}}{2I_0}(y - x), \\ \dot{y} = \frac{a_{21}}{I_0}x + \frac{a_{22}}{I_0}y - xz, \\ \dot{z} = xy + \frac{a_{33}}{I_0}z. \end{cases} \tag{12}$$

The system (12) with $a_{11} = 20I_0$, $a_{21} = 28I_0$, $a_{22} = -I_0$, $a_{33} = -8I_0/3$ would become the famous Lorenz equations, i.e., the rigid body motion is chaotic.
(b) *Case 2:*
If $[I_1, I_2, I_3] = [2I_0, I_0, I_0]$, $a_{12} = -a_{11}$ and $\frac{a_{21}}{I_2} = \frac{a_{22}}{I_2} - \frac{a_{11}}{I_1}$, Eq. (1) is rewritten as follows:

$$\begin{cases} \dot{x} = \frac{a_{11}}{2I_0}(y - x), \\ \dot{y} = (\frac{a_{22}}{I_0} - \frac{a_{11}}{2I_0})x - xz + \frac{a_{22}}{I_0}y, \\ \dot{z} = xy + \frac{a_{33}}{I_0}z. \end{cases} \tag{13}$$

The system (13) with $a_{11} = 70I_0$, $a_{22} = 28I_0$ and $a_{33} = -3I_0$, is related to Chen system and it is chaotic.

From the above analysis, it has found that the Euler equations not only exhibits chaotic motions but also the system can display all the dynamical behaviors of the Lorenz equations and Chen system by easily appropriate choice the feedback gains.

6. Conclusions

A simple method has been proposed for anti-control of chaos of a rigid body motion. The chaotic motion of the system has been obtained easily by choosing suitable feedback gains. Basic dynamical behaviors, such as symmetry, invariance, dissipativity and existence of attractor, have also been issued. By applying numerical simulation, the system exhibits both strange attractors and limit cycles for certain choices of the parameters. Besides, the transient behaviors of the chaotic system depending on the feedback gains have also been studied. It has also been found that the system can generate a complex multi-scroll chaotic attractor under the appropriate feedback gains. Finally, it was shown that the system could be related to the famous Lorenz equations and Chen system. In other words, the system can easily display all the dynamical behaviors of the famous Lorenz equations and Chen system. This paper has brought us much improved understanding of a rigid body motion.

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