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Decomposing $K_n \cup P$ into triangles

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In honor of Curt Lindner on the occasion of his 65th birthday

Abstract

In this paper, we extend the work on minimum coverings of K_n with triangles. We prove that when P is any forest on n vertices the multigraph $G = K_n \cup P$ can be decomposed into triangles if and only if three trivial necessary conditions are satisfied: (i) each vertex in G has even degree, (ii) each vertex in P has odd degree, and (iii) the number of edges in G is a multiple of 3. This result is of particular interest because the corresponding packing problem where the leave is any forest is yet to be solved. We also consider some other families of packings, and provide a variation on a proof by Colbourn and Rosa which settled the packing problem when P is any 2-regular graph on at most n vertices.
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1. Introduction

A Steiner triple system of order v is a pair (S,t) where S is a v-set and t is a collection of 3-element subsets of S such that each pair of elements in S occur together in a triple exactly once. It is well known that a Steiner triple system of order v, STS(v), exists if and only if $v \equiv 1$ or 3 (mod 6) [5]. In terms of graph decompositions, an STS(v) can be viewed as a partition of the edges of K_v, each element of which induces a triangle C_3; we denote such a decomposition by $C_3 \mid K_v$.

For each value of v for which there is no STS(v), results in the literature establish how close one can come by omitting some pairs from triples. A *packing* of a graph G with triangles is a partition of the edge set of a subgraph H of G, each element of which induces a triangle; the *remainder graph* of this packing, also known as the *leave*, is the subgraph $G - H$ formed from G by removing the edges in H. If the remainder graph is minimum in size (that is, has the least number of edges among all possible leaves of G), then the packing is called a *maximum* packing. The following result is also well-known (a simple proof of this result can be found in [6], for example). A graph is said to be *odd* if each vertex has *odd* degree.

Theorem 1.1 (Hanani [4]). *The remainder graph L for the maximum packings of K_n with triangles are as follows:*

L \ n	0	1	2	3	4	5
L	F	$\emptyset$	F	$\emptyset$	F_1	C_4

(mod 6)

F is a 1-factor, F_1 is an odd spanning forest with $n/2+1$ edges (tripole), and C_4 is a cycle of length 4.

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It is natural to ask for which subgraphs H of $K_n, C_3 \mid K_n - H$. When n is odd and H is 2-regular graph, the following result has been obtained by Colbourn and Rosa [1]:

Theorem 1.2. *Let n be an odd positive integer. Let H be a 2-regular subgraph of K_n. If $n = 9$ then suppose that $H \neq C_4 \cup C_5$. Then $C_3 \mid K_n - H$ if and only if the number of edges in $K_n - H$ is a multiple of 3.*

For convenience, in what follows, we call a graph G 3-*sufficient* if each vertex of G is of even degree and $3 \mid |E(G)|$.
In the hope of determining for which H it is possible that $C_3 \mid K_n - H$, the following conjecture deserves to be mentioned:

Conjecture 1 (Nash-Williams). Let H be a subgraph of $K_n (n \neq 9)$ such that $K_n - H$ is 3-sufficient and $\Delta(H) \leqslant \frac{1}{4}(n-1)$. Then $C_3 \mid K_n - H$.

The exceptional case when $n = 9$ is required since when H is a disjoint union of a 4-cycle and a 5-cycle there is no decomposition of $K_n - H$ into triangles. Clearly, if the conjecture is proved to be true, then Theorem 1.2 would become an easy corollary. So far, this conjecture is not yet settled.

It is also worth mentioning another problem of interest which seems possible to solve, yet still remains open.

Conjecture 2. Let H be a spanning odd forest of a complete graph of even order n. Then $C_3 | K_n - H$ if and only if $K_n - H$ is 3-sufficient.

When no STS(v) exists, a second approach to approximating one has been to use some pairs more than once. A *covering* of a graph G with triangles is a collection of triangles, p, such that each edge of G occurs in at least one triangle in p. So, if $G(p)$ is the multigraph formed by joining each pair of vertices u and v with x edges if and only if p contains x triples that contain both u and v, then clearly $C_3 | G(p)$. The multigraph $G(p) - G$ is called the *excess graph* of G; it is also known as the *padding* of the covering p of G. A covering with smallest excess graph (in size) is called a *minimum* covering, and these have been found as the following result describes (a simple proof of this result can be found in [6], for example).

Theorem 1.3 (Hanani [4]). *The excess graph P for the minimum coverings of K_n with triangles are as follows:*

n	0	1	2	3	4	5	
P	F	$\emptyset$	F_1	$\emptyset$	F_1		(mod 6)

As with the packing problem, we would like to go further and know what are the possible excess graphs of K_n. For example, if $n \equiv 0 (\text{mod } 6)$ then Theorem 1.3 shows that a 1-factor can be an excess graph of K_n. In this paper we generalize such results by showing that the excess graph P can in fact be any spanning odd forest as long as the multigraph $K_n \cup P$ (formed by adding the edges of P to K_n) is 3-sufficient (see Theorem 2.5). Clearly for such a result it is necessary that n be even so that $K_n \cup P$ is 3-sufficient. This result is of particular interest in that the corresponding packing problem (see Conjecture 2) still remains unsolved. It also provides a companion result to the following theorem of Colbourn and Rosa [2]. Of course, if n is odd then one would require all vertices in P to have even degree.

Theorem 1.4 (Colbourn and Rosa [2]). *Let H be any 2-regular (not necessarily spanning) graph defined on $V(K_n)$. Then $C_3 \mid K_n \cup H$ if and only if $K_n \cup H$ is 3-sufficient.*

In this paper we provide a new proof of Theorem 1.4, presented in Section 3. Both proofs are similar in that they rely on Theorem 1.2, but are certainly different in that the original proof considers the possible congruence classes of v in turn, whereas here the argument is based on the possible lengths of the cycles in the padding.

In what follows, we assume that all the packings and coverings are done using triangles.

2. Forest paddings

Before we prove the main results, we need a couple of lemmas. The following is easy to see, but useful enough to list separately:

Lemma 2.1. *Let G be packed with remainder graph L. If there exists a graph P defined on $V(G)$ such that $C_3 \mid L \cup P$, then G can be covered with excess graph P.*

Proof. The union of the 3-cycles in the given packing of G and the 3-cycles in the decomposition of $L \cup P$ provides the result. □

The next lemma is of some interest in its own right.

Lemma 2.2. *Let H be an odd tree. Then H can be decomposed into $|V(H)|/2 - 1$ paths of length 2 and a path of length one. Moreover, this can be done so that the end vertices of these paths are all distinct, and so that the path of length one can be any edge of the tree.*

Proof. The proof is by induction on the number of vertices. Let f be the edge in the path of length one. Since all vertices have odd degree, H must have an even number $2n$ of vertices, so has an odd number $2n-1$ of edges. Clearly the result is true if H is a star, so we can assume that $n \geqslant 6$ and that if $R = (v_1, v_2, \ldots, v_x)$ is a longest path in H then it has length at least 3. Since R is a longest path, and since v_2 and v_{x-1} have odd degree, each of these 2 vertices is adjacent to at least two leaves in H. So at least one of the two, say v_2 is adjacent to two leaves, neither of which is incident with the edge f. Deleting the two leaves incident with v_2 produces an odd tree with two fewer vertices, so by induction it has a decomposition into one path of length 1 containing the edge f and the remaining edges being in paths of length 2, as described in the lemma. Adding the path of length 2 that joins the two deleted leaves in H completes the proof, since clearly these leaves cannot be the ends of any 2-path produced by induction. □

Using a similar idea, the following decomposition can also be obtained. A graph is unicyclic if it contains exactly one cycle.

Lemma 2.3. *Let H be a connected unicyclic odd graph. Then H can be decomposed into $|V(H)|/2$ paths of length two. Moreover, all the end vertices of these length two paths are distinct.*

Proof. Let $C = (v_1, v_2, \ldots, v_3)$ be the cycle in H. Since all vertices have odd degree, for $1 \leqslant i \leqslant x$ the vertex v_i is adjacent to at least one vertex w_i that is not in C. Deleting the edges in C leaves a forest with exactly x components, each of which is an odd tree. So we can apply the previous lemma to each tree with the path of length one containing the edge joining v_i to w_i for $1 \leqslant i \leqslant x$. Adding the 2-paths (w_i, v_i, v_{i+1}) for $1 \leqslant i \leqslant x$ produces the required decomposition. □

Now, we can consider the excess graphs which are either unicyclic odd graphs or spanning odd forests, beginning with the easier unicyclic graphs.

Theorem 2.4. *Let H be a spanning odd subgraph of K_n in which each component is unicyclic. Then $C_3 \mid K_n \cup H$ if and only if $K_n \cup H$ is 3-sufficient.*

Proof. The necessity is obvious so we prove the sufficiency. Since each vertex of H is of odd degree, $K_n \cup H$ being 3-sufficient means that n must be even. Since each component of H is a unicyclic odd graph, H can be decomposed into paths p_i for $1 \leqslant i \leqslant n/2$ of length two such that all the end vertices are distinct; let a_i and b_i be the end vertices of p_i. Let $L = \{\{a_i, b_i\} \mid 1 \leqslant i \leqslant n/2\}$. Thus L is a 1-factor of K_n. Furthermore, H has n edges, so since $K_n \cup H$ is 3-sufficient, it follows that $3 | \binom{n}{2} + n$ so $n \equiv 0$ or 2 (mod 6). Therefore, K_n has a maximum packing with leave a 1-factor, which we can assume is L. So adding to this maximum packing the triangles defined on the three vertices in p_i for $1 \leqslant i \leqslant n/2$ shows that $K_n \cup H$ can be decomposed into triangles. □

We now turn to the most interesting case, where H is a forest. Since every non-trivial tree has a vertex of degree 1, it is clear that if H is a forest then for each vertex in $K_n \cup H$ to have even degree it is necessary that n is even, that each vertex in H is odd, and that H spans K_n.

Theorem 2.5. *Let H be an odd spanning forest of K_n. Then $C_3 \mid K_n \cup H$ if and only if $K_n \cup H$ is 3-sufficient.*

Proof. First, we consider $n \neq 10$. Suppose that H has u components. Apply Lemma 2.2 to each component in H to decompose H into paths $p_1, p_2, \ldots, p_s$ of length two and u paths of length one. Let the ends of p_i be a_i and b_i for $1 \leqslant i \leqslant s$. Let the set of vertices in the length one paths be $\{c_i, d_i\}$ for $1 \leqslant i \leqslant u$.

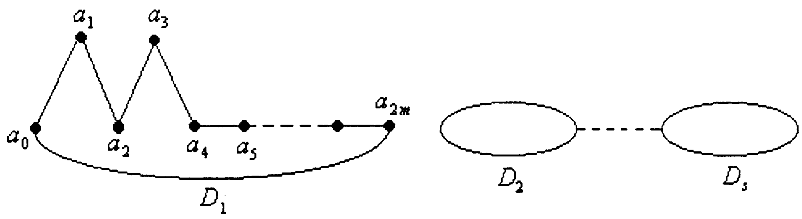

Fig. 1.

By Lemma 2.2, we know that $n=2s+2u$. Let L_1 be the set of edges $\{\{b_2,a_3\},\{b_3,a_4\},\ldots,\{b_{s-1},a_s\},\{b_s,a_2\}\}$ and L_2 be $\{\{a_i,b_i\} \mid i=2,3,\ldots,s\}$.

Note that if $s \leqslant 1$, then H is the excess graph of a minimum covering, so the result follows from Theorem 1.3. Also if $s=2$ then $|E(H)|=2+n/2$, so $|E(K_n \cup H)|=2+n^2/2$ which is not divisible by 3, so this case cannot arise since $K_n \cup H$ is 3-sufficient. So we can assume that $s \geqslant 3$, so $L_1 \cup L_2$ is an even cycle of length at least 4. Let c_1 and d_1 be in a component of size greater than 2; so we can assume that d_1 is the center of one of the paths of length 2, say the 2-path (a_1,d_1,b_1).

Let α be the collection of triangles obtained in the maximum packing of $G=(K_n-c_1)-(L_1 \cup L_2)$. Since $n \neq 10$, since $K_n - c_1$ is in fact a complete graph of order $n-1$, and since $L_1 \cup L_2$ is 2-regular, by Theorem 1.2 the above maximum packing has an empty leave as long as the number of edges in $G=(K_n-c_1)-(L_1 \cup L_2)$ is a multiple of 3. This is in fact the case since the 3-sufficient graph $K_n \cup H = G \cup L_1 \cup L_2 \cup H \cup \{c_1c_i, c_1d_j, c_1a_k, c_1b_h \mid 2 \leqslant i \leqslant u, 1 \leqslant j \leqslant u, 1 \leqslant k,h \leqslant s\}$ and since $(K_n \cup H)-G$ can be decomposed into a set of triangles $\beta=\{(c_1,d_1,a_1),(c_1,d_1,b_1),(c_1,b_2,a_3),(c_1,b_3,a_4),\ldots,(c_1,b_s,a_2),$ $\{a_2,b_2\} \cup p_2,\ldots,\{a_s,b_s\} \cup p_s,\ (c_1,c_2,d_2),(c_1,c_3,d_3),\ldots,(c_1,c_u,d_u)\}$. Now, $\alpha \cup \beta$ gives the decomposition of $K_n \cup H$ into triangles.

As for the case $n=10$, the necessary conditions imply that H can only be a spanning tree or a forest with 4 components. The latter case is a minimum covering and the decomposition is known by Theorem 1.3. In the former case we have $s=4$ and the proof in the general case works here too since $L_1 \cup L_2$ is a cycle of length 6 in this case, and by Theorem 1.2 $C_3 \mid K_9 - C_6$. □

3. 2-Regular paddings

Next, we consider the situation where the excess graph is a 2-regular graph defined on n vertices, so n must be odd. We allow a double edge (a 2-cycle) to be a part of the 2-regular graph. Recall that the join of two graphs G_1 and G_2, $G_1 \vee G_2$, is the graph defined on $V(G_1) \cup V(G_2)$ such that $E(G_1 \vee G_2)=E(G_1) \cup E(G_2) \cup E(K_{|V(G_1)|,|V(G_2)|})$. Let O_2 denote the graph with two vertices and no edges.

Theorem 3.1. *Let H be any 2-regular (not necessarily spanning) graph defined on $V(K_n)$. Then $C_3|K_n \cup H$ if and only if $K_n \cup H$ is 3-sufficient.*

Proof. The necessity is trivial, so we prove the sufficiency. Since $K_n \cup H$ is 3-sufficient, n must be odd. We prove the theorem by using a similar technique to that used to prove the last theorem. We split the proof into four cases. We assume that $n \neq 11$ in the first three cases, then deal with $n=11$ at the end of the proof.

(i) Suppose that H contains an odd cycle D_1.

Let $D_1=(a_0,a_1,a_2,\ldots,a_{2m})$, and $D_2,D_3,\ldots,D_s$ be the remaining cycles in H. Here, we consider a double edge as a 2-cycle. If there are an odd number of vertices in K_n that are in no cycle in H then let one of them be named ∞, and in any case let the remaining vertices that are in no cycle in H be $e_1,e_2,\ldots,e_{2x}$ (so possibly $x=0$). By arranging the cycles as in Fig. 1, we can convert the covering problem of K_n to the packing problem of K_{n-2} where we can use Theorem 1.2.

First, we define the triangles between $\{a_1,a_3\}$ and $\{a_0,a_2,a_i,e_j \mid 4 \leqslant i \leqslant 2m, 1 \leqslant j \leqslant 2x\}$. Let $\alpha = \{(a_1,a_0,a_{2m}),(a_1,a_2,a_3),(a_0,a_1,a_2),(a_2,a_3,a_4),(a_o,a_3,a_4),(a_1,a_{2i+2},a_{2i+3}),(a_3,a_{2i+3},a_{2i+4}) \mid i = 1,2,\ldots,m-2\}$. If $x \neq 0$ then replace the 3-cycle (a_0,a_3,a_4) in α with the 3-cycles in $\{(a_3,a_4,e_1),(a_3,e_{2i},e_{2i+1}),(a_3,e_{2x},a_0),$ $(a_1,e_{2j-1},e_{2j}) \mid 1 \leqslant i \leqslant x-1, 1 \leqslant j \leqslant x\}$. Notice that α partitions into triangles the edges in $(K_2 \vee D_1) \cup (a_0,a_2,a_4,e_1,e_2,\ldots,e_{2x})$ (the edges in the cycle $(a_0,a_2,a_4,e_1,e_2,\ldots,e_{2x})$ form part of the leave in K_{n-2} when we apply Theorem 1.2).

Now we consider the edges between $\{a_1,a_3\}$ and $V(D_i), i=2,3,\ldots,s$. If $V(D_i)=\{b_1,b_2,\ldots,b_t\}$ where t is even, then let $\beta_i=\{(a_1,b_{2i-1},b_{2i}),(a_3,b_{2i},b_{2i+1}) \mid 1\leqslant i\leqslant t/2\}$. Clearly, β_i partitions $O_2\vee D_i$ in this case. Next, pair up all except possibly one, say D_2 of the odd cycles. If $V(D_i)=\{b_1,b_2,\ldots,b_{2l+1}\}$ is paired with $D_j=\{c_1,c_2,\ldots,c_{2h+1}\}$ then define $\gamma_{i,j}=\{(a_1,b_{2i-1},b_{2i}),(a_3,b_{2i},b_{2i+1}),(a_1,c_{2j},c_{2j+1}),(a_3,c_{2j-1},c_{2j}) \mid i=1,2,\ldots,l, j=1,2,\ldots,h\}\cup\{(a_1,b_{2l+1},c_1),(a_3,c_{2h+1},b_1)\}\cup\{(b_1,b_3,b_{2l+1}),(c_1,c_3,c_{2h+1})\}$. Then $\gamma_{i,j}$ partitions the edge set of $\{O_2\vee(D_i\cup D_j)\}\cup\{(b_1,b_3,b_{2l+1},c_1,c_3,c_{2h+1})\}$. Note that as in previous case, $(b_1,b_3,b_{2l+1},c_1,c_3,c_{2h+1})$ will become a part of the leave in K_{n-2} when Theorem 1.2 is applied.

If we have an even number of odd cycles, then one, say D_2, remains unpaired. Clearly H is of even size in this case, so there exists the vertex ∞ in $V(K_n)$ which is not in $V(H)$. Let $D_2=(d_1,d_2,\ldots,d_{2k+1})$. Define $\delta=\{(a_1,d_{2i-1},d_{2i}),(a_3,d_{2i},d_{2i+1}),(a_1,d_{2k+1},\infty),(a_3,\infty,d_1),(d_1,d_3,d_{2k+1}) \mid i=1,2,\ldots,k\}$. Then δ partitions $(O_2\vee D_2)\cup\{(d_1,d_3,d_{2k+1},\infty)\}$. Again, $(d_1,d_3,d_{2k+1},\infty)$ will become part of the remainder graph in K_{n-2}.

In each subcase we are able to place the edges in the cycles D_1 to D_s into triangles, as well as the edges between $\{a_1,a_3\}$ and $V(K_n)\backslash\{a_1,a_3\}$. This implies that after taking away all the edges in triangles defined by $\alpha,\beta_i,\gamma_{i,j}$ and δ as the case may be, the remaining edges induce the graph $K_{n-2}-H'$ where H' is a 2-regular graph defined on $V(K_{n-2})$. (More precisely, the components of H' are a cycle of length $2x+3$, one 6-cycle for each of the paired odd length cycles in H, and a 4-cycle if H has an even number of odd length cycles.) Since $3\,\|\,E(K_n\cup H)|$ and since $K_{n-2}-H'$ is obtained by taking away edges in triangles from $K_n\cup H$, it follows that $K_{n-2}-H'$ is also 3-sufficient. By Theorem 1.2, the proof follows since $n\neq 11$.

(ii) Suppose that H contains only even cycles and that $|V(H)|=n-1$. Let the vertex in $V(K_n)\backslash V(H)$ be ∞, and the cycles in H be $D_i=(b_1^{(i)},b_2^{(i)},\ldots,b_{f(i)}^{(i)})$, $i=1,2,\ldots,s$. Since $f(i)$ is even for each i, by deleting the edges in the triangles in $\{(\infty,b_{2j-1}^{(i)},b_{2j}^{(i)}) \mid 1\leqslant j\leqslant f(i)/2\}$, the multi-graph that remains is the union of K_{n-1} and a 1-factor. By Theorem 1.3 the edges in this graph can be partitioned into triangles, so the proof in this case follows.

(iii) Suppose that H contains only even cycles and that $|V(H)|<n-2$. Since n is odd, $|V(H)|\leqslant n-3$ and $n-|V(H)|$ is odd.

First, if $n-|V(H)|=3$ then let the vertices in $\{a_1,a_2,a_3\}$ be the vertices in K_n that are not in H. Since $K_n\cup H$ is 3-sufficient it immediately follows that n must be equivalent to 5 (mod 6). So, by Theorem 1.1 there exists a packing T of the graph formed from K_n by deleting vertices in $\{a_1,a_2,a_3\}$ and then deleting the edges in a 1-factor F_1. Also, H has a 1-factorization (since all cycles in H have even length) into 2 1-factors F_2 and F_3. Then $R\cup\{(a_i,b,c) \mid 1\leqslant i\leqslant 3,\{b,c\}\in F_i\}$ provides the required covering.

Otherwise, since 3 does not divide $n-5+n(n-1)/2$, we can assume that $n-|V(H)|\geqslant 7$. Let $\{a_1,a_2,\ldots,a_{2t+1}\}$ be the set of vertices which are not in $V(H)$ and the set of cycles in H be $D_1,D_2,\ldots,D_s$, where $D_i=(b_1^i,b_2^i,\ldots,b_{f(i)}^i)$, $i=1,2,\ldots,s$. Now, let $T_1=\{(a_1,a_2,a_3),(a_1,a_4,a_5),(a_1,a_{2i},a_{2i+1}),(a_2,a_{2i-1},a_{2i}),(a_2,a_{2t+1},a_4) \mid 3\leqslant i\leqslant t\}$, and for $1\leqslant i\leqslant s$ let $T_2(i)=\{(a_1,b_{2j-1}^i,b_{2j}^i),(a_2,b_{2j}^i,b_{2j+1}^i) \mid 1\leqslant j\leqslant f(i)/2\}$. Removing the edges from these $s+1$ sets from $K_n\cup H$ produces the graph $K_{n-2}-(a_3,a_4,\ldots,a_{2t+1})$ which must be 3-sufficient since it is produced from a 3-sufficient graph by removing edges in triangles; so by Theorem 1.2 it can be decomposed into triangles.

(iv) Finally, we handle the case when $n=11$. Since $K_n\cup H$ is 3-sufficient, $|E(H)|=2$ (mod 3). It suffices to consider all cases where H contains no 3-cycles, since 3-cycles can always be added to the padding.

If $|E(H)|=2$, then the padding is a double edge, so this case is handled by Theorem 1.3.

The remaining cases are all handled in the same fashion: a packing with an appropriate leave L is obtained, usually using Theorem 1.1; then by appropriately naming the vertices in L and H, Lemma 2.1 is used to complete the decomposition. We describe one case in detail, then just give the ingredients for the remaining cases, it being a simple matter to check the details.

If H is a 5-cycle, say (1,2,3,4,5), then using Theorem 1.1 take a packing of K_{11} with leave the 4-cycle (1,3,5,6). Then $\{(1,2,3),(3,4,5),(1,3,5),(1,5,6)\}$ gives the required decomposition of $L\cup H$ that can be used in Lemma 2.1.

The following table gives more possible paddings H together with the leave L that can be used to obtain the result.

H	C_8	$C_4\cup C_4$	$C_2\cup C_6$	C_{11}	$C_9\cup C_2$	$C_7\cup C_4$	$C_6\cup C_5$
L	C_4	$C_5\cup C_5$	$C_4\cup C_3$	C_7	$C_6\cup C_4$	$C_5\cup C_5$	$C_3\cup C_4$

Finally, we have the cases $H\cong C_2\cup C_5\cup C_4$, $C_2\cup C_2\cup C_2\cup C_5$, and $C_2\cup C_2\cup C_7$ to consider. Each of these coverings can be obtained using the construction described in case (i). This follows since each of these paddings has exactly one odd length cycle. So when the construction requires Theorem 1.2 to be applied, in each case a packing of K_9 that has a 3-cycle leave is needed, and this clearly exists. □

4. Concluding remarks

There are other excess graphs P for which $K_n \cup P$ can be decomposed into triangles. For example, if $n = m \cdot r$ with $m, r \geqslant 2$ $C_3 \mid K_n \cup P$ provided that P is a vertex disjoint union of m K_r's and $K_n \cup P$ is 3-sufficient. This decomposition is also known as a group divisible design with index (2,1), see [3]. Unfortunately, being 3-sufficient is not a sufficient condition for $K_n \cup P$ to have a triangle decomposition. For example, it is not difficult to see that $K_{6m} \cup K_{3m,3m}$ is 3-sufficient provided that m is odd, but $K_{6m} \cup K_{3m,3m}$ cannot be decomposed into triangles. (An additional necessary condition is that for all partitions $\{S, T\}$ of the n vertices, the number of edges joining vertices in different parts must be at most half the number of edges joining vertices in the same part.) Therefore, to determine which graphs P are the excess graph for a C_3-decomposition of $K_n \cup P$ is an interesting problem, one that is not going to be easy to solve.

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