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Zero-norm states and high-energy symmetries of string theory

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Abstract

We derive stringy Ward identities from the decoupling of two types of zero-norm states in the old covariant first quantized (OCFQ) spectrum of open bosonic string. These Ward identities are valid to all energy α' and all loop orders χ in string perturbation theory. The high-energy limit $\alpha' \to \infty$ of these stringy Ward identities can then be used to fix the proportionality constants between scattering amplitudes of different string states algebraically *without* referring to Gross and Mende's saddle point calculation of high-energy string-loop amplitudes. As examples, all Ward identities for the mass level $M^2 = 4, 6$ are derived, their high-energy limits are calculated and the proportionality constants between scattering amplitudes of different string states are determined. In addition to those identified before, we discover some *new* nonzero components of high-energy amplitudes not found previously by Gross and Manes. These components are essential to preserve massive gauge invariances or decouple massive zero-norm states of string theory. A set of massive scattering amplitudes and their high-energy limits are calculated explicitly for each mass level $M^2 = 4, 6$ to justify our results.

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1. Introduction

It is often of fundamental importance to study the high-energy behavior of a local quantum field theory. In the quantum chromodynamics, for example, the renormalization group and the discovery of asymptotic freedom [1] turned out to be one of the most important properties of Yang–Mills theories. On the other hand, the spontaneously broken

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symmetries are often hidden at low energy, but become evident in the high-energy behavior of the theory. In string theory, one expects even more rich fundamental structures at high-energy since only then will an infinite number of particles be excited. Being a consistent quantum theory with no free parameter, it is conceivable that an huge symmetry group or Ward identities get restored at high-energy, which are responsible for the ultraviolet finiteness of string theory.

Recently it was discovered that [2] the high-energy limits $\alpha' \to \infty$ of stringy Ward identities can be used to fix the proportionality constants between scattering amplitudes of different string states algebraically *without* referring to Gross and Mende's [3] saddle point calculation of high-energy string-loop amplitudes. These proportionality constants are, as conjectured by Gross [4], independent of the scattering angle ϕ_{CM} and the order χ of string perturbation theory. As a result, all high-energy string scattering amplitudes can be expressed in terms of those of tachyons. These Ward identities, which are valid to all energy α' and all loop orders χ in string perturbation theory, are derived from the decoupling of two types of zero-norm states in the old covariant first quantized (OCFQ) spectrum of open bosonic string. A prescription to explicitly calculate zero-norm states for arbitrary mass levels, or stringy symmetry charges with arbitrarily high spins, was given in [5]. The importance of zero-norm states and their implication on stringy symmetries were first pointed out in the context of massive σ-model approach of string theory [6]. These stringy symmetries were also demonstrated recently in Witten's string field theory (WSFT), and the background ghost fields in the off-shell BRST spectrum were identified, in a one to one manner, to the lifting of the on-shell conditions of zero-norm states in the OCFQ approach [7]. On the other hand, zero-norm states were also shown [8] to carry the spacetime ω_∞ symmetry charges of toy 2D string theory, and the corresponding ω_∞ Ward identities were powerful enough to determine the tachyon scattering amplitudes algebraically *without* any integration [9].

In this paper, all Ward identities for the mass level $M^2 = 4, 6$ will be derived, their high-energy limits are calculated and the proportionality constants between scattering amplitudes of different string states are determined directly from these Ward identities. General formula of high-energy amplitudes for arbitrary mass levels will be given in terms of those of tachyons. In addition to those identified before, we discover some new nonzero components of high-energy amplitudes at each mass level not found previously by Gross and Manes [10]. These components are essential to preserve massive gauge invariances or decouple massive zero-norm states of string theory. A set of massive scattering amplitudes and their high energy limits are calculated explicitly for each mass level $M^2 = 4, 6$ to justify our results. This paper is organized as following. In Section 2, we derive stringy Ward identities for the mass level $M^2 = 4$ [11], and then take high-energy limits of them to determine the proportionality constants between scattering amplitudes of different string states algebraically. At the subleading order energy, one finds 6 unknown amplitudes and 4 equations. Presumably, they are not proportional to each other or the proportional coefficients do depend on the scattering angle ϕ_{CM}. This result will be confirmed at Section 3. In Section 3, the high energy limits of a set of string-tree level amplitudes with one tensor at mass level $M^2 = 4$ and three tachyons are explicitly calculated to justify the results of Section 2. The whole program is then generalized to mass level $M^2 = 6$ in

Section 4. We make a comparison of our results with those of Gross and Manes [10] in Section 5. Finally a brief conclusion is given in Section 6.

2. High-energy stringy Ward identities of mass level $M^2 = 4$

In the OCFQ spectrum of open bosonic string theory, the solutions of physical states conditions include positive-norm propagating states and two types of zero-norm states which were neglected in the most literature. They are [12]

$$\text{Type I:} \quad L_{-1}|x\rangle, \quad \text{where } L_1|x\rangle = L_2|x\rangle = 0,\ L_0|x\rangle = 0; \tag{2.1}$$

$$\text{Type II:} \quad \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\tilde{x}\rangle, \quad \text{where } L_1|\tilde{x}\rangle = L_2|\tilde{x}\rangle = 0,\ (L_0+1)|\tilde{x}\rangle = 0. \tag{2.2}$$

Eqs. (2.1) and (2.2) can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at $D = 26$. The existence of type II zero-norm states signals the importance of zero-norm states in the structure of the theory of string. In the first quantized approach of string theory, the stringy *on-shell* Ward identities are proposed to be (for our purpose we choose four-point amplitudes in this paper)

$$\mathcal{T}_\chi(k_i) = g_c^{2-\chi} \int \frac{Dg_{\alpha\beta}}{\mathcal{N}} DX^\mu \exp\left(-\frac{\alpha'}{2\pi}\int d^2\xi \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right) \times \prod_{i=1}^{4} v_i(k_i) = 0, \tag{2.3}$$

where at least one of the 4 vertex operators corresponds to the zero-norm state solution of Eqs. (2.1) or (2.2). In Eq. (2.3) g_c is the closed string coupling constant, $\mathcal{N}$ is the volume of the group of diffeomorphisms and Weyl rescalings of the worldsheet metric, and $v_i(k_i)$ are the on-shell vertex operators with momenta k_i. The integral is over orientable open surfaces of Euler number χ parametrized by moduli $\vec{m}$ with punctures at ξ_i. The simplest zero-norm state $k \cdot \alpha_{-1}|0, k\rangle$, $k^2 = 0$ with polarization k is the massless solution of Eq. (2.1), which reproduces the Ward identity of string QED when substituting into Eq. (2.3). A simple prescription to systematically solve Eqs. (2.1) and (2.2) for an infinite number of zero-norm states was given in [5]. A more thorough understanding of the solution of these equations and their relation to spacetime ω_∞ symmetry of toy $D = 2$ string was discussed in [8]. For our purpose here, there are four zero-norm states at mass level $M^2 = 4$, the corresponding Ward identities were calculated to be [11]

$$k_\mu \theta_{\nu\lambda} \mathcal{T}_\chi^{(\mu\nu\lambda)} + 2\theta_{\mu\nu} \mathcal{T}_\chi^{(\mu\nu)} = 0, \tag{2.4}$$

$$\left(\frac{5}{2}k_\mu k_\nu \theta'_\lambda + \eta_{\mu\nu}\theta'_\lambda\right)\mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu\theta'_\nu \mathcal{T}_\chi^{(\mu\nu)} + 6\theta'_\mu \mathcal{T}_\chi^{\mu} = 0, \tag{2.5}$$

$$\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu}\theta_\lambda\right)\mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu\theta_\nu \mathcal{T}_\chi^{[\mu\nu]} - 6\theta_\mu \mathcal{T}_\chi^{\mu} = 0, \tag{2.6}$$

$$\left(\frac{17}{4}k_\mu k_\nu k_\lambda + \frac{9}{2}\eta_{\mu\nu}k_\lambda\right)\mathcal{T}_\chi^{(\mu\nu\lambda)} + (9\eta_{\mu\nu} + 21k_\mu k_\nu)\mathcal{T}_\chi^{(\mu\nu)} + 25k_\mu\mathcal{T}_\chi^\mu = 0, \tag{2.7}$$

where $\theta_{\mu\nu}$ is transverse and traceless, and θ'_λ and θ_λ are transverse vectors. In each equation, we have chosen, say, $v_2(k_2)$ to be the vertex operators constructed from zero-norm states and $k_\mu \equiv k_{2\mu}$. Note that Eq. (2.6) is the inter-particle Ward identity corresponding to D_2 vector zero-norm state obtained by antisymmetrizing those terms which contain $\alpha^\mu_{-1}\alpha^\nu_{-2}$ in the original type I and type II vector zero-norm states. We will use 1 and 2 for the incoming particles and 3 and 4 for the scattered particles. In Eqs. (2.4)–(2.7), 1, 3 and 4 can be any string states (including zero-norm states) and we have omitted their tensor indices for the cases of excited string states. For example, one can choose $v_1(k_1)$ to be the vertex operator constructed from another zero-norm state which generates an inter-particle Ward identity of the third massive level. The resulting Ward-identity of Eq. (2.6) then relates scattering amplitudes of particles at different mass level. $\mathcal{T}_\chi$'s in Eqs. (2.4)–(2.7) are the mass level $M^2 = 4$, χth order string-loop amplitudes. At this point, $\{\mathcal{T}_\chi^{(\mu\nu\lambda)}, \mathcal{T}_\chi^{(\mu\nu)}, \mathcal{T}_\chi^\mu\}$ is identified to be the *amplitude triplet* of the spin-three state. $\mathcal{T}_\chi^{[\mu\nu]}$ is obviously identified to be the scattering amplitude of the antisymmetric spin-two state with the same momenta as $\mathcal{T}_\chi^{(\mu\nu\lambda)}$. Eq. (2.6) thus relates the scattering amplitudes of two different string states at mass level $M^2 = 4$. Note that Eqs. (2.4)–(2.7) are valid order by order and are *automatically* of the identical form in string perturbation theory. This is consistent with Gross's argument through the calculation of high-energy scattering amplitudes. However, it is important to note that Eqs. (2.4)–(2.7) are, in contrast to the high-energy $\alpha' \to \infty$ result of Gross, valid to *all* energy α' and their coefficients do depend on the center of mass scattering angle ϕ_{CM}, which is defined to be the angle between $\vec{k}_1$ and $-\vec{k}_3$, through the dependence of momentum k.

We will calculate high energy limit of Eqs. (2.4)–(2.7) without referring to the saddle point calculation in [3,4,10]. Let us define the normalized polarization vectors

$$e_P = \frac{1}{m_2}(E_2, k_2, 0) = \frac{k_2}{m_2}, \tag{2.8}$$

$$e_L = \frac{1}{m_2}(k_2, E_2, 0), \tag{2.9}$$

$$e_T = (0, 0, 1) \tag{2.10}$$

in the CM frame contained in the plane of scattering. They satisfy the completeness relation

$$\eta^{\mu\nu} = \sum_{\alpha,\beta} e^\mu_\alpha e^\nu_\beta \eta^{\alpha\beta}, \tag{2.11}$$

where $\mu, \nu = 0, 1, 2$ and $\alpha, \beta = P, L, T$. Diag $\eta^{\mu\nu} = (-1, 1, 1)$. One can now transform all μ, ν coordinates in Eqs. (2.4)–(2.7) to coordinates α, β. For Eq. (2.4), we have $\theta^{\mu\nu} = e^\mu_L e^\nu_L - e^\mu_T e^\nu_T$ or $\theta^{\mu\nu} = e^\mu_L e^\nu_T + e^\mu_T e^\nu_L$. In the high energy $E \to \infty$, fixed angle ϕ_{CM} limit, one identifies $e_P = e_L$ and Eq. (2.4) gives (we drop loop order χ here to simplify the notation)

$$\mathcal{T}^{6\to 4}_{LLL} - \mathcal{T}^4_{LTT} + \mathcal{T}^4_{(LL)} - \mathcal{T}^2_{(TT)} = 0, \tag{2.12}$$

$$\mathcal{T}^{5\to 3}_{LLT} + \mathcal{T}^3_{(LT)} = 0. \tag{2.13}$$

In Eqs. (2.12) and (2.13), we have assigned a relative energy power for each amplitude. For each longitudinal L component, the order is E^2 and for each transverse T component, the order is E. This is due to the definitions of e_L and e_T in Eqs. (2.9) and (2.10), where e_L got one energy power more than e_T. By Eq. (2.12), the E^6 term of the energy expansion for $\mathcal{T}_{LLL}$ is forced to be zero. As a result, the possible leading order term is E^4. Similar rule applies to $\mathcal{T}_{LLT}$ in Eq. (2.13). For Eq. (2.5), we have $\theta'^{\mu} = e_L^{\mu}$ or $\theta'^{\mu} = e_T^{\mu}$ and one gets, in the high energy limit,

$$10\mathcal{T}_{LLL}^{6\to 4} + \mathcal{T}_{LTT}^{4} + 18\mathcal{T}_{(LL)}^{4} + 6\mathcal{T}_{L}^{2} = 0, \tag{2.14}$$

$$10\mathcal{T}_{LLT}^{5\to 3} + \mathcal{T}_{TTT}^{3} + 18\mathcal{T}_{(LT)}^{3} + 6\mathcal{T}_{T}^{1} = 0. \tag{2.15}$$

For the D_2 Ward identity, Eq. (2.6), we have $\theta^{\mu} = e_L^{\mu}$ or $\theta^{\mu} = e_T^{\mu}$ and one gets, in the high energy limit,

$$\mathcal{T}_{LLL}^{6\to 4} + \mathcal{T}_{LTT}^{4} + 9\mathcal{T}_{[LL]}^{4\to 2} - 3\mathcal{T}_{L}^{2} = 0, \tag{2.16}$$

$$\mathcal{T}_{LLT}^{5\to 3} + \mathcal{T}_{TTT}^{3} + 9\mathcal{T}_{[LT]}^{3} - 3\mathcal{T}_{T}^{1} = 0. \tag{2.17}$$

It is important to note that $\mathcal{T}_{[LL]}$ in Eq. (2.16) originate from the high energy limit of $\mathcal{T}_{[PL]}$, and the antisymmetric property of the tensor forces the leading E^4 term to be zero. Finally the singlet zero norm state Ward identity, Eq. (2.7), implies, in the high energy limit,

$$34\mathcal{T}_{LLL}^{6\to 4} + 9\mathcal{T}_{LTT}^{4} + 84\mathcal{T}_{(LL)}^{4} + 9\mathcal{T}_{(TT)}^{2} + 50\mathcal{T}_{L}^{2} = 0. \tag{2.18}$$

One notes that all components of high energy amplitudes of symmetric spin three and antisymmetric spin two states appear at least once in Eqs. (2.12)–(2.18). It is now easy to see that the naive leading order amplitudes corresponding to E^4 appear in Eqs. (2.12), (2.14), (2.16) and (2.18). However, a simple calculation shows that $\mathcal{T}_{LLL}^{4} = \mathcal{T}_{LTT}^{4} = \mathcal{T}_{(LL)}^{4} = 0$. So the real leading order amplitudes correspond to E^3, which appear in Eqs. (2.13), (2.15) and (2.17). A simple calculation shows that

$$\mathcal{T}_{TTT}^{3} : \mathcal{T}_{LLT}^{3} : \mathcal{T}_{(LT)}^{3} : \mathcal{T}_{[LT]}^{3} = 8 : 1 : -1 : -1. \tag{2.19}$$

Note that these proportionality constants are, as conjectured by Gross [4], independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory. They are also independent of particles chosen for vertex $v_{1,3,4}$. *Most importantly, we now understand that they originate from zero-norm states in the OCFQ spectrum of the string*! The subleading order amplitudes corresponding to E^2 appear in Eqs. (2.12), (2.14), (2.16) and (2.18). One has 6 unknown amplitudes and 4 equations. Presumably, they are not proportional to each other or the proportional coefficients do depend on the scattering angle ϕ_{CM}. We will justify this point later in our sample calculation in Section 3. Our calculation here is purely algebraic *without any integration* and is independent of saddle point calculation in [3,4,10]. It is important to note that our result in Eq. (2.19) is gauge invariant as it should be since we derive it from Ward identities (2.4)–(2.7). On the other hand, the result obtained in [10] with $\mathcal{T}_{TTT}^{3} \propto \mathcal{T}_{[LT]}^{3}$, and $\mathcal{T}_{LLT}^{3} = 0$ in the leading order energy at this mass level is, on the contrary, *not* gauge invariant. In fact, with $\mathcal{T}_{LLT}^{3} = 0$, an inconsistency arises, for example, between Eqs. (2.13) and (2.15). We give one example here to illustrate the meaning of the massive gauge invariant amplitude. To be more specific, we will use two different gauge choices to calculate the high-energy scattering amplitude of symmetric spin three state. The first gauge choice is

$$\begin{aligned}&\left(\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}+\epsilon_{(\mu\nu)}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu}\right)|0,k\rangle,\\&\epsilon_{(\mu\nu)}=-\frac{3}{2}k^{\lambda}\epsilon_{\mu\nu\lambda},\qquad k^{\mu}k^{\nu}\epsilon_{\mu\nu\lambda}=0,\qquad \eta^{\mu\nu}\epsilon_{\mu\nu\lambda}=0.\end{aligned}\tag{2.20}$$

In the high-energy limit, using the helicity, decomposition and writing $\epsilon_{\mu\nu\lambda}=\sum_{\alpha,\beta,\delta}e_{\mu}^{\alpha}\times e_{\nu}^{\beta}e_{\lambda}^{\delta}u_{\alpha\beta\delta}$; $\alpha,\beta,\delta=P,L,T$, we get

$$\begin{aligned}&\left(\epsilon_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}+\epsilon_{(\mu\nu)}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu}\right)|0,k\rangle\\&\quad=\left[u_{PLT}\left(6\alpha_{-1}^{PLT}+6\alpha_{-1}^{(L}\alpha_{-2}^{T)}\right)\right.\\&\qquad+u_{TTP}\left(3\alpha_{-1}^{TTP}-3\alpha_{-1}^{LLP}+3\alpha_{-1}^{(T}\alpha_{-2}^{T)}-3\alpha_{-1}^{(L}\alpha_{-2}^{L)}\right)\\&\qquad\left.+u_{TTL}\left(3\alpha_{-1}^{TTL}-\alpha_{-1}^{LLL}\right)+u_{TTT}\left(\alpha_{-1}^{TTT}-3\alpha_{-1}^{LLT}\right)\right]|0,k\rangle.\end{aligned}\tag{2.21}$$

The second gauge choice is

$$\tilde{\varepsilon}_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle,\qquad k^{\mu}\tilde{\varepsilon}_{\mu\nu\lambda}=0,\qquad \eta^{\mu\nu}\tilde{\varepsilon}_{\mu\nu\lambda}=0.\tag{2.22}$$

In the high-energy limit, similar calculation gives

$$\tilde{\varepsilon}_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle=\left[\tilde{u}_{TTL}\left(3\alpha_{-1}^{TTL}-\alpha_{-1}^{LLL}\right)+\tilde{u}_{TTT}\left(\alpha_{-1}^{TTT}-3\alpha_{-1}^{LLT}\right)\right]|0,k\rangle.\tag{2.23}$$

It is now easy to see that the first and second terms of Eq. (2.21) will not contribute to the high-energy scattering amplitudes, of the symmetric spin three state due to the spin two Ward identities Eqs. (2.13) and (2.12) if we identify $e_P=e_L$. Thus the two different gauge choices Eqs. (2.20) and (2.22) give the same high-energy scattering amplitude. It can be shown that this massive gauge symmetry is valid to all energy and is the result of the decoupling of massive spin two zero-norm state at mass level $M^2=4$. Note that the α_{-1}^{LLT} term of Eq. (2.23), which corresponds to the amplitude $\mathcal{T}_{LLT}^{3}$, was missing in the calculation of Ref. [10]. We will discuss this issue in Section 5.

To further justify our result, we give a sample calculation in Section 3.

3. A sample calculation of mass level $M^2=4$

In this section, we give a detailed calculation of a set of sample scattering amplitudes to explicitly justify our results presented in Section 2. Since the proportionality constants in Eq. (2.19) are independent of particles chosen for vertex $v_{1,3,4}$, for simplicity, we will choose them to be tachyons. For the string-tree level $\chi=1$, with one tensor v_2 and three tachyons $v_{1,3,4}$, all scattering amplitudes of mass level $M^2=4$ were calculated in [11]. They are

$$\mathcal{T}^{\mu\nu\lambda}=\int\prod_{i=1}^{4}dx_i\left\langle e^{ik_1X}\partial X^{\mu}\partial X^{\nu}\partial X^{\lambda}e^{ik_2X}e^{ik_3X}e^{ik_4X}\right\rangle$$

$$
\begin{aligned}
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\big[-t/2\big(t^2/4-1\big)k_1^\mu k_1^\nu k_1^\lambda \\
&\quad + 3(s/2+1)t/2(t/2+1)k_1^\mu k_1^\nu k_3^\lambda - 3s/2(s/2+1)(t/2+1)k_1^\mu k_3^\nu k_3^\lambda \\
&\quad + s/2\big(s^2/4-1\big)k_3^\mu k_3^\nu k_3^\lambda\big],
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
\mathcal{T}^{(\mu\nu)} &= \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X}\partial^2 X^{(\mu}\partial X^{\nu)} e^{ik_2X} e^{ik_3X} e^{ik_4X}\right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\big[t/2\big(t^2/4-1\big)k_1^\mu k_1^\nu \\
&\quad - (s/2+1)t/2(t/2+1)k_1^{(\mu}k_3^{\nu)} + s/2(s/2+1)(t/2+1)k_3^{(\mu}k_1^{\nu)} \\
&\quad - s/2\big(s^2/4-1\big)k_3^\mu k_3^\nu\big],
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
\mathcal{T}^{\mu} &= \frac{1}{2}\int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X}\partial^3 X^{\mu} e^{ik_2X} e^{ik_3X} e^{ik_4X}\right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\big[s/2\big(s^2/4-1\big)k_3^\mu - t/2\big(t^2/4-1\big)k_1^\mu\big],
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
\mathcal{T}^{[\mu\nu]} &= \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X}\partial^2 X^{[\mu}\partial X^{\nu]} e^{ik_2X} e^{ik_3X} e^{ik_4X}\right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\left[\left(\frac{s+t}{2}\right)(s/2+1)(t/2+1)k_3^{[\mu}k_1^{\nu]}\right],
\end{aligned} \tag{3.4}
$$

where $s=-(k_1+k_2)^2$, $t=-(k_1+k_3)^2$ and $u=-(k_1+k_4)^2$ are the Mandelstam variables. In deriving Eqs. (3.1)–(3.4), we have made the $SL(2,R)$ gauge fixing by choosing $x_1=0$, $0 \leqq x_2 \leqq 1$, $x_3=1$, $x_4=\infty$. To calculate the high energy expansions ($s,t\to\infty$, $\frac{s}{t}=$ fixed) of these scattering amplitudes, one needs the following energy expansion formulas

$$
e_P\cdot k_1 = \left(\frac{-2E^2}{m_2}\right)\left[1-\left(\frac{m_2^2-2}{4}\right)\frac{1}{E^2}\right], \tag{3.5}
$$

$$
\begin{aligned}
e_L\cdot k_1 &= \left(\frac{-2E^2}{m_2}\right)\left[1-\left(\frac{m_2^2-2}{4}\right)\frac{1}{E^2}+\left(\frac{m_2^2}{4}\right)\frac{1}{E^4}+\left(\frac{m_2^4-2m_2^2}{16}\right)\frac{1}{E^6}\right. \\
&\quad \left. + O\left(\frac{1}{E^8}\right)\right],
\end{aligned} \tag{3.6}
$$

$$
e_T\cdot k_1 = 0, \tag{3.7}
$$

$$
\begin{aligned}
e_P\cdot k_3 &= \left(\frac{E^2}{m_2}\right)\left\{2\xi^2+\left[\frac{m_2^2}{2}\eta^2+\big(3\xi^2-1\big)\right]\frac{1}{E^2}\right. \\
&\quad \left. + \big(2\xi^2-1\big)\left(\frac{m_2^2+2}{4}\right)^2\frac{1}{E^6}+O\left(\frac{1}{E^8}\right)\right\},
\end{aligned} \tag{3.8}
$$

$$e_L \cdot k_3 = \left(\frac{E^2}{m_2}\right)\left\{2\xi^2 + \left[-\frac{m_2^2}{2}\eta^2 + \left(3\xi^2 - 1\right)\right]\frac{1}{E^2} + \left(\frac{m_2^2}{2}\xi^2\right)\frac{1}{E^4} + \left(\frac{m_2^4 - 4m_2^2\xi^2 + 8\xi^2 - 4}{16}\right)\frac{1}{E^6} + O\left(\frac{1}{E^8}\right)\right\}, \tag{3.9}$$

$$e_T \cdot k_3 = (-2\xi\eta)E - \left(\frac{2\xi\eta}{E}\right) + \left(\frac{\xi\eta}{E^3}\right) - \left(\frac{\xi\eta}{E^5}\right) + O\left(\frac{1}{E^7}\right), \tag{3.10}$$

where $\xi = \sin\frac{\phi_{\mathrm{CM}}}{2}$ and $\eta = \cos\frac{\phi_{\mathrm{CM}}}{2}$. The high-energy expansions of Mandelstam variables are given by

$$s = (E_1 + E_2)^2 = 4E^2, \tag{3.11}$$

$$t = \left(-4\xi^2\right)E^2 + \left(m_2^2 - 6\right)\xi^2 + \frac{1}{8}\left(m_2^2 + 2\right)^2\left(1 - 2\xi^2\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right). \tag{3.12}$$

We can now explicitly calculate all amplitudes in Eq. (2.19). After some algebra, we get

$$\mathcal{T}_{TTT} = -8E^9 \exp\left(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2}\right) \times \sin^3\phi_{\mathrm{CM}}\left[1 + \frac{3}{E^2} + \frac{5}{4E^4} - \frac{5}{4E^6} + O\left(\frac{1}{E^8}\right)\right], \tag{3.13}$$

$$\mathcal{T}_{LLT} = -E^9 \exp\left(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2}\right) \times \left[\sin^3\phi_{\mathrm{CM}} + \left(6\sin\phi_{\mathrm{CM}}\cos^2\phi_{\mathrm{CM}}\right)\frac{1}{E^2} - \sin\phi_{\mathrm{CM}}\left(\frac{11}{2}\sin^2\phi_{\mathrm{CM}} - 6\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right], \tag{3.14}$$

$$\mathcal{T}_{[LT]} = E^9 \exp\left(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2}\right) \times \left[\sin^3\phi_{\mathrm{CM}} - \left(2\sin\phi_{\mathrm{CM}}\cos^2\phi_{\mathrm{CM}}\right)\frac{1}{E^2} + \sin\phi_{\mathrm{CM}}\left(\frac{3}{2}\sin^2\phi_{\mathrm{CM}} - 2\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right], \tag{3.15}$$

$$\mathcal{T}_{(LT)} = E^9 \exp\left(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2}\right) \times \left[\sin^3\phi_{\mathrm{CM}} + \sin\phi_{\mathrm{CM}}\left(\frac{3}{2} - 10\cos\phi_{\mathrm{CM}} - \frac{3}{2}\cos^2\phi_{\mathrm{CM}}\right)\frac{1}{E^2} - \sin\phi_{\mathrm{CM}}\left(\frac{1}{4} + 10\cos\phi_{\mathrm{CM}} + \frac{3}{4}\cos^2\phi_{\mathrm{CM}}\right)\frac{1}{E^4} + O\left(\frac{1}{E^6}\right)\right]. \tag{3.16}$$

We thus have justified Eq. (2.19) with $\mathcal{T}^3_{TTT} = -8E^9\sin^3\phi_{\mathrm{CM}}\exp(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2})$ and $\mathcal{T}^5_{LLT} = 0$. We have also checked that $\mathcal{T}^6_{LLL} = \mathcal{T}^4_{LLL} = \mathcal{T}^4_{LTT} = \mathcal{T}^4_{(LL)} = 0$ as claimed in Section 2. Note that, unlike the leading E^9 order, the angular dependences of E^7 order are different for each amplitudes. The subleading order amplitudes corresponding to $\mathcal{T}^2$ (E^8 order) appear in Eqs. (2.12), (2.14), (2.16) and (2.18). One has 6 unknown amplitudes. An explicit sample calculation gives

$$\mathcal{T}^2_{LLL} = -4E^8 \sin\phi_{\mathrm{CM}} \cos\phi_{\mathrm{CM}} \exp\left(-\alpha' \frac{s\ln s + t\ln t + u\ln u}{2}\right), \tag{3.17}$$

$$\mathcal{T}^2_{LTT} = -8E^8 \sin^2\phi_{\mathrm{CM}} \cos\phi_{\mathrm{CM}} \exp\left(-\alpha' \frac{s\ln s + t\ln t + u\ln u}{2}\right), \tag{3.18}$$

which show that their angular dependences are indeed different or the proportional coefficients do depend on the scattering angle ϕ_{CM}.

4. The calculation of mass level $M^2 = 6$

In this section we generalize the calculation of Sections 2 and 3 to mass level $M^2 = 6$. There are four positive-norm physical propagating states at this mass level [13], a totally symmetric spin four state, a mixed symmetric spin three state, a symmetric spin two state and a scalar state. There are nine zero-norm states at this mass level. One can use the simplified method [5] to calculate all of them. The spin three and spin two zero-norm states are (from now on, unless otherwise stated, each spin polarization is assumed to be transverse, traceless and is symmetric with respect to each group of indices)

$$L_{-1}|x\rangle = \theta_{\mu\nu\lambda}\left(k_\beta \alpha^{\mu\nu\lambda\beta}_{-1} + 3\alpha^{\mu\nu}_{-1}\alpha^{\lambda}_{-2}\right)|0,k\rangle, \quad |x\rangle = \theta_{\mu\nu\lambda}\alpha^{\mu\nu\lambda}_{-1}|0,k\rangle, \tag{4.1}$$

$$\begin{aligned} &L_{-1}|x\rangle = \left[k_\lambda \theta_{\mu\nu}\alpha^{\mu_\lambda}_{-1}\alpha^{\nu}_{-2} + 2\theta_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-3}\right]|0,k\rangle, \\ &|x\rangle = \theta_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-2}|0,k\rangle, \quad \text{where } \theta_{\mu\nu} = -\theta_{\nu\mu}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} L_{-1}|x\rangle = \Big[&2\theta_{\mu\nu}\alpha^{\mu\nu}_{-2} + 4\theta_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-3} + 2(k_\lambda\theta_{\mu\nu} + k_{(\lambda}\theta_{\mu\nu)})\alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2} \\ &+ \frac{2}{3}k_\lambda k_\beta \theta_{\mu\nu}\alpha^{\mu\nu\lambda\beta}_{-1}\Big]|0,k\rangle; \\ |x\rangle = \Big[&2\theta_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-2} + \frac{2}{3}k_\lambda\theta_{\mu\nu}\alpha^{\mu\nu\lambda}_{-1}\Big]|0,k\rangle, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \left(L_{-2} + \frac{3}{2}L^2_{-1}\right)|\tilde{x}\rangle = \Big[&3\theta_{\mu\nu}\alpha^{\mu\nu}_{-2} + 8\theta_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-3} + \left(k_\lambda\theta_{\mu\nu} + \frac{15}{2}k_{(\lambda}\theta_{\mu\nu)}\right)\alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2} \\ &+ \left(\frac{1}{2}\eta_{\lambda\beta}\theta_{\mu\nu} + \frac{3}{2}k_\lambda k_\beta\theta_{\mu\nu}\right)\alpha^{\mu\nu\lambda\beta}_{-1}\Big]|0,k\rangle, \\ |\tilde{x}\rangle = \theta_{\mu\nu}\alpha^{\mu\nu}_{-1}|0,k\rangle, \end{aligned} \tag{4.4}$$

where $\alpha_{-1}^{\mu\nu}=\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}$, etc. There are two type I degenerate vector zero-norm states which can be calculated as following:

$$\text{Ansatz:}\quad |x\rangle=\big[a(\theta\cdot\alpha_{-3})+b(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-1})+c(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-2}) + d(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})+f(k\cdot\alpha_{-1})^2(\theta\cdot\alpha_{-1})\big]|0,k\rangle. \tag{4.5}$$

The L_1 and L_2 constraints of Eq. (2.1) give

$$a-2c=0,\qquad b+c+d-6f=0,\qquad 3a-12b+28d-6f=0, \tag{4.6}$$

which can be easily used to determine, for example, $a:b:c:d:f=26:5:13:0:3$ or $0:81:0:39:20$. This gives two type I vector zero-norm states

$$\begin{aligned}L_{-1}|x\rangle=\big[&3a(\theta\cdot\alpha_{-4})+2b(k\cdot\alpha_{-3})(\theta\cdot\alpha_{-1})+(2c+a)(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-3})\\ &+(b+c)(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-2})+(b+2f)(k\cdot\alpha_{-1})(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-1})\\ &+2d(\alpha_{-2}\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})+(c+f)(k\cdot\alpha_{-1})^2(\theta\cdot\alpha_{-2})\\ &+d(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-2})+d(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})\\ &+f(k\cdot\alpha_{-1})^3(\theta\cdot\alpha_{-1})\big]|0,k\rangle.\end{aligned} \tag{4.7}$$

The type II vector zero-norm state is

$$\begin{aligned}&\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\tilde{x}\rangle\\ &\quad=\bigg[33(\theta\cdot\alpha_{-4})+4(k\cdot\alpha_{-3})(\theta\cdot\alpha_{-1})+22(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-3})\\ &\qquad+\frac{21}{2}(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-2})+\frac{11}{2}(k\cdot\alpha_{-1})(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-1})\\ &\qquad+\frac{15}{2}(k\cdot\alpha_{-1})^2(\theta\cdot\alpha_{-2})+\frac{3}{2}(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-2})\\ &\qquad+\frac{1}{2}(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})+\frac{3}{2}(k\cdot\alpha_{-1})^3(\theta\cdot\alpha_{-1})\bigg]|0,k\rangle,\\ &|\tilde{x}\rangle=\big[3(\theta\cdot\alpha_{-2})+(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})\big]|0,k\rangle.\end{aligned} \tag{4.8}$$

The type I singlet zero-norm state was calculated to be the following [5]:

$$\text{Ansatz:}\quad |x\rangle=\big[a(k\cdot\alpha_{-1})^3+b(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-1})+c(k\cdot\alpha_{-1})(k\cdot\alpha_{-2}) + d(\alpha_{-1}\cdot\alpha_{-2})+f(k\cdot\alpha_{-3})\big]|0,k\rangle. \tag{4.9}$$

The L_1 and L_2 constraints of Eq. (2.1) can be easily used to determine $a:b:c:d:f=37:72:261:216:450$. This gives the type I singlet zero-norm state

$$\begin{aligned}L_{-1}|x\rangle=\big[&a(k\cdot\alpha_{-1})^4+b(k\cdot\alpha_{-1})^2(\alpha_{-1}\cdot\alpha_{-1})+(2b+d)(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-2})\\ &+(c+3a)(k\cdot\alpha_{-1})^2(k\cdot\alpha_{-2})+c(k\cdot\alpha_{-2})^2+d(\alpha_{-2}\cdot\alpha_{-2})\\ &+b(k\cdot\alpha_{-2})(\alpha_{-1}\cdot\alpha_{-1})+(2c+f)(k\cdot\alpha_{-3})(k\cdot\alpha_{-1})\\ &+2d(\alpha_{-1}\cdot\alpha_{-3})+3f(k\cdot\alpha_{-4})\big]|0,k\rangle.\end{aligned} \tag{4.10}$$

Finally, the type II singlet zero-norm state can be calculated to be

$$\begin{aligned}
&\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\tilde{x}\rangle \\
&\quad=\Big[11a(k\cdot\alpha_{-4})+(6a+8c)(k\cdot\alpha_{-3})(k\cdot\alpha_{-1})+8b(\alpha_{-1}\cdot\alpha_{-3}) \\
&\qquad+\left(\frac{5}{2}a+3c\right)(k\cdot\alpha_{-2})^2+\left(\frac{3}{2}a+\frac{17}{2}c\right)(k\cdot\alpha_{-1})^2(k\cdot\alpha_{-2}) \\
&\qquad+3b(\alpha_{-2}\cdot\alpha_{-2})+\left(\frac{5}{2}b+\frac{1}{2}a\right)(\alpha_{-1}\cdot\alpha_{-1})(k\cdot\alpha_{-2}) \\
&\qquad+6b(k\cdot\alpha_{-1})(\alpha_{-2}\cdot\alpha_{-1})+\left(\frac{3}{2}b+\frac{1}{2}c\right)(k\cdot\alpha_{-1})^2(\alpha_{-1}\cdot\alpha_{-1}) \\
&\qquad+\frac{3}{2}c(k\cdot\alpha_{-1})^4+\frac{1}{2}b(\alpha_{-1}\cdot\alpha_{-1})^2\Big]|0,k\rangle, \\
&|\tilde{x}\rangle=\big[a(k\cdot\alpha_{-2})+b(\alpha_{-1}\cdot\alpha_{-1})+c(k\cdot\alpha_{-1})^2\big]|0,k\rangle \qquad (4.11)
\end{aligned}$$

where $a:b:c=75:39:19$. We are now ready to calculate the high-energy Ward identities. The high-energy limit of stringy Ward identity corresponding to Eq. (4.1) are

$$\sqrt{6}\big(-\mathcal{T}^{8\to 6}_{LLLL}+3\mathcal{T}^{6}_{LLTT}\big)+3\big(-\mathcal{T}^{6}_{LLL}+3\mathcal{T}^{4}_{LTT}\big)=0, \qquad (4.12)$$

$$\sqrt{6}\big(-3\mathcal{T}^{7\to 5}_{LLLT}+\mathcal{T}^{5}_{LTTT}\big)+3\big(-3\mathcal{T}^{5}_{LLT}+\mathcal{T}^{3}_{TTT}\big)=0, \qquad (4.13)$$

where $\mathcal{T}_{\mu\nu\lambda}$ is the amplitude corresponding to $\alpha^{(\mu\nu}_{-1}\alpha^{\lambda)}_{-2}$. Eqs. (4.12) and (4.13) correspond to $\theta^{\mu\nu\lambda}=-e^\mu_L e^\nu_L e^\lambda_L+3e^\mu_{(L}e^\nu_T e^\lambda_{T)}$ and $\theta^{\mu\nu\lambda}=-3e^\mu_{(L}e^\nu_L e^\lambda_{T)}+e^\mu_T e^\nu_T e^\lambda_T$ respectively. Similarly, Eq. (4.2) gives

$$\tilde{\mathcal{T}}^{5\to 3}_{LL,T}+\sqrt{6}\tilde{\mathcal{T}}^{3}_{[LT]}=0, \qquad (4.14)$$

where $\tilde{\mathcal{T}}_{\mu\nu}$ is the amplitude corresponding to $\alpha^\mu_{-1}\alpha^\nu_{-3}$ and $\tilde{\mathcal{T}}_{\mu\nu,\lambda}$ is the amplitude corresponding to mixed symmetric part of $\alpha^{\mu\nu}_{-1}\alpha^{\lambda}_{-2}$, that is, first symmetrizing w.r.t. $\mu\nu$ and then antisymmetrizing w.r.t. $\mu\lambda$. This is exactly the amplitude for the positive-norm mixed symmetric spin three state. The type I symmetric spin two zero-norm state Eq. (4.3) gives, in the high-energy limit,

$$\begin{aligned}
&2\big(\mathcal{T}^{8\to 6}_{LLLL}-\mathcal{T}^{6}_{LLTT}\big)+2\sqrt{6}\Big[\big(\mathcal{T}^{6}_{LLL}-\mathcal{T}^{4}_{LTT}\big)+\frac{1}{3}\big(\tilde{\mathcal{T}}^{6\to 4}_{LL,P}+\tilde{\mathcal{T}}^{4}_{LT,T}\big)\Big] \\
&\quad+2\big(\tilde{\mathcal{T}}^{4}_{(LL)}-\tilde{\mathcal{T}}^{2}_{(TT)}\big)+\big(\mathcal{T}^{4}_{LL}-\mathcal{T}^{2}_{TT}\big)=0, \qquad (4.15)
\end{aligned}$$

$$2\mathcal{T}^{7\to 5}_{LLLT}+\sqrt{6}\left[2\mathcal{T}^{5}_{LLT}+\frac{1}{3}\tilde{\mathcal{T}}^{5}_{LL,T}\right]+2\tilde{\mathcal{T}}^{3}_{(LT)}+\mathcal{T}^{3}_{LT}=0, \qquad (4.16)$$

where $\mathcal{T}_{\mu\nu}$ is the amplitude corresponding to $\alpha^{\mu\nu}_{-2}$. The E^6 order of $\tilde{\mathcal{T}}^{6\to 4}_{PL,L}$ in Eq. (4.15) is forced to be zero in the high-energy limit $(e_P=e_L)$ due to the antisymmetric property of the tensor $\tilde{\mathcal{T}}_{\mu\nu,\lambda}$ w.r.t. $\mu\lambda$. It is important to note that in deriving Eqs. (4.15) and (4.16), we have made the following irreducible decomposition of the term

$$k_\lambda\theta_{\mu\nu}\alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2}=\left[\frac{1}{3}(k_\lambda\theta_{\mu\nu}+k_\mu\theta_{\nu\lambda}+k_\nu\theta_{\lambda\mu})+\frac{1}{3}(k_\lambda\theta_{\mu\nu}-k_\nu\theta_{\mu\lambda})\right]\alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2} \qquad (4.17)$$

in Eq. (4.3). The first term with totally symmetric spin three index corresponds to the gauge artifact of the positive-norm spin four state, and the mixed symmetric tensor structure of the second term is exactly the same as that of the positive-norm spin three state. In general, there are three other possible mixed symmetric spin three terms, which do not appear in Eq. (4.17). This is a nontrivial consistent check of zero-norm states spectrum in the OCFQ string. We shall see another similar mechanism happens in our later calculations. The type II symmetric spin two zero-norm state Eq. (4.4) gives, in the high-energy limit,

$$9\mathcal{T}^{8\to 6}_{LLLL} - \frac{17}{2}\mathcal{T}^{6}_{LLTT} - \frac{1}{2}\mathcal{T}^{4}_{TTTT} + \frac{17}{2}\sqrt{6}\biggl[\bigl(\mathcal{T}^{6}_{LLL} - \mathcal{T}^{4}_{LTT}\bigr) + \frac{2\sqrt{6}}{3}\bigl(\tilde{\mathcal{T}}^{6\to 4}_{LL,P} + \tilde{\mathcal{T}}^{4}_{LT,T}\bigr)\biggr] + 8\bigl(\tilde{\mathcal{T}}^{4}_{(LL)} - \tilde{\mathcal{T}}^{2}_{(TT)}\bigr) + 3\bigl(\mathcal{T}^{4}_{LL} - \mathcal{T}^{2}_{TT}\bigr) = 0, \tag{4.18}$$

$$18\mathcal{T}^{7\to 5}_{LLLT} + \mathcal{T}^{5}_{LTTT} + \frac{4\sqrt{6}}{3}\tilde{\mathcal{T}}^{5}_{LL,T} + 17\sqrt{6}\mathcal{T}^{5}_{LLT} + 16\tilde{\mathcal{T}}^{3}_{(LT)} + 6\mathcal{T}^{3}_{LT} = 0. \tag{4.19}$$

Two type I vector zero-norm states Eq. (4.7) give, in the high-energy limit,

$$6\sqrt{6}f\mathcal{T}^{8\to 6}_{LLLL} + \sqrt{6}d\mathcal{T}^{6}_{LLTT} + 6(b+c+3f)\mathcal{T}^{6}_{LLL} + 3d\mathcal{T}^{4}_{LTT} + (4b-8c)\tilde{\mathcal{T}}^{6\to 4}_{LP,P} + \sqrt{6}(2b+2c+a)\tilde{\mathcal{T}}^{4}_{(LL)} + (2b-2c-a)\tilde{\mathcal{T}}^{4\to 2}_{[LP]} + \sqrt{6}(b+c)\mathcal{T}^{4}_{LL} + 3a\mathcal{T}^{2}_{L} = 0, \tag{4.20}$$

$$6\sqrt{6}f\mathcal{T}^{7\to 5}_{LLLT} + \sqrt{6}d\mathcal{T}^{5}_{LTTT} + 6(b+c+3f)\mathcal{T}^{5}_{LLT} + 3d\mathcal{T}^{3}_{TTT} - (4b-8c)\tilde{\mathcal{T}}^{5\to 3}_{PP,T} + \sqrt{6}(2b+2c+a)\tilde{\mathcal{T}}^{3}_{(LT)} + \sqrt{6}(2b-2c-a)\tilde{\mathcal{T}}^{3}_{[TL]} + \sqrt{6}(b+c)\mathcal{T}^{3}_{LT} + 3a\mathcal{T}^{1}_{T} = 0, \tag{4.21}$$

where $\mathcal{T}_{\mu}$ is the amplitude corresponding to α^{μ}_{-4}. Note that $\tilde{\mathcal{T}}^{6\to 4}_{LP,P}$ in Eq. (4.20) is identical to $\tilde{\mathcal{T}}^{6\to 4}_{LL,P}$ in Eqs. (4.15) and (4.18) in the high-energy limit. However, $\tilde{\mathcal{T}}^{2}_{LP,P}$ and $\tilde{\mathcal{T}}^{2}_{LL,P}$ can be different. Also $\tilde{\mathcal{T}}^{5}_{PP,T}$ in Eq. (4.21) is zero since it equals to $\tilde{\mathcal{T}}^{5}_{LL,T}$ in Eq. (4.14), which is zero, in the high-energy limit. However, $\tilde{\mathcal{T}}^{3}_{PP,T}$ and $\tilde{\mathcal{T}}^{3}_{LL,T}$ can be different. In deriving Eqs. (4.20) and (4.21), in addition to (4.17), one needs another projection formula

$$k_\lambda k_\mu \theta_\nu \alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2} = \biggl[\frac{1}{3}(k_\lambda k_\mu \theta_\nu + k_\mu k_\nu \theta_\lambda + k_\nu k_\lambda \theta_\mu) + \frac{2}{3}(k_\lambda k_\mu \theta_\nu - k_\nu k_\mu \theta_\lambda)\biggr]\alpha^{\lambda\mu}_{-1}\alpha^{\nu}_{-2}. \tag{4.22}$$

Again, the first term of Eq. (4.22) with totally symmetric spin three index corresponds to the gauge artifact of the positive-norm spin four state, and the mixed symmetric tensor structure of the second term is exactly the same as that of the positive-norm spin three state. This is another consistent check of zero-norm states spectrum in the OCFQ string. In the following, we will use Eqs. (4.17) and (4.22) whenever they are needed. Type II vector zero-norm state Eq. (4.6) gives, in the high-energy limit,

$$9\sqrt{6}\mathcal{T}^{8\to 6}_{LLLL}+\frac{\sqrt{6}}{2}\mathcal{T}^{6}_{LLTT}+78\mathcal{T}^{6}_{LLL}+\frac{3}{2}\mathcal{T}^{4}_{LTT}+2\tilde{\mathcal{T}}^{4}_{LT,T}+38\tilde{\mathcal{T}}^{6\to 4}_{LP,P}+26\sqrt{6}\tilde{\mathcal{T}}^{4}_{(LL)}-18\tilde{\mathcal{T}}^{4\to 2}_{[LP]}+\frac{21}{2}\sqrt{6}\mathcal{T}^{4}_{LL}+33\mathcal{T}^{2}_{L}=0, \tag{4.23}$$

$$9\sqrt{6}\mathcal{T}^{7\to 5}_{LLLT}+\frac{\sqrt{6}}{2}\mathcal{T}^{5}_{LTTT}+78\mathcal{T}^{5}_{LLT}+\frac{3}{2}\mathcal{T}^{3}_{TTT}+38\tilde{\mathcal{T}}^{5}_{TL,L}+26\sqrt{6}\tilde{\mathcal{T}}^{3}_{(LT)}-18\tilde{\mathcal{T}}^{3}_{[TL]}+\frac{21}{2}\sqrt{6}\mathcal{T}^{3}_{LT}+33\mathcal{T}^{1}_{T}=0. \tag{4.24}$$

Note that $\tilde{\mathcal{T}}^{5}_{TL,L}$ in Eq. (4.23) is identical to $\tilde{\mathcal{T}}^{5}_{TP,P}$ in Eq. (4.21) in the high-energy limit. Finally, type I and type II singlet zero-norm states give, in the high-energy limit,

$$74\mathcal{T}^{8\to 6}_{LLLL}+24\mathcal{T}^{6}_{LLTT}+124\sqrt{6}\mathcal{T}^{6}_{LLL}+24\sqrt{6}\mathcal{T}^{4}_{LTT}-8\sqrt{6}\tilde{\mathcal{T}}^{4}_{LT,T}+324\tilde{\mathcal{T}}^{4}_{(LL)}+87\mathcal{T}^{4}_{LL}=0, \tag{4.25}$$

$$342\mathcal{T}^{8\to 6}_{LLLL}+136\mathcal{T}^{6}_{LLTT}+\frac{13}{2}\mathcal{T}^{4}_{TTTT}+548\sqrt{6}\mathcal{T}^{6}_{LLL}+123\sqrt{6}\mathcal{T}^{4}_{LTT}+8\sqrt{6}\tilde{\mathcal{T}}^{4}_{LT,T}+1204\tilde{\mathcal{T}}^{4}_{(LL)}+489\mathcal{T}^{4}_{LL}=0. \tag{4.26}$$

This completes the calculation of high-energy Ward identities. It is easy to count the high-energy amplitudes for each tensor. For $\mathcal{T}_{\mu\nu\lambda\gamma}$, one has $\mathcal{T}_{LLLL}, \mathcal{T}_{LLLT}, \mathcal{T}_{LLTT}, \mathcal{T}_{LTTT}$ and $\mathcal{T}_{TTTT}$. For $\mathcal{T}_{\mu\nu\lambda}$, one has $\mathcal{T}_{LLL}, \mathcal{T}_{LLT}, \mathcal{T}_{LTT}$ and $\mathcal{T}_{TTT}$. For $\tilde{\mathcal{T}}_{\mu\nu,\lambda}$, one has $\tilde{\mathcal{T}}_{LL,T}$ and $\tilde{\mathcal{T}}_{LT,T}$. For $\mathcal{T}_{\mu\nu}$, one has $\mathcal{T}_{LL}, \mathcal{T}_{LT}$ and $\mathcal{T}_{TT}$. For $\tilde{\mathcal{T}}_{\mu\nu,}$ one has $\tilde{\mathcal{T}}_{LL,}\tilde{\mathcal{T}}_{(LT),}\tilde{\mathcal{T}}_{[LT]}$ and $\tilde{\mathcal{T}}_{TT}$. For $\mathcal{T}_{\mu}$, one has $\mathcal{T}_{L}$ and $\mathcal{T}_{T}$. It is very important to note that in the E^4 order, one gets one more amplitude $\tilde{\mathcal{T}}^{4}_{LP,P}$, and in the E^3 order, one gets another amplitude $\tilde{\mathcal{T}}^{3}_{PP,T}$ described after Eq. (4.21). It can be checked by Eqs. (4.12)–(4.26) that all the amplitudes of orders E^8 E^7 E^6 and E^5 are zero. So the real leading order amplitudes correspond to E^4, which appear in Eqs. (4.12), (4.15), (4.18), (4.20), (4.23), (4.25) and (4.26). Note that there are two equations for (4.20). We thus end up with 8 equations and 9 amplitudes. A calculation by Gauss elimination shows that

$$\mathcal{T}^{4}_{TTTT}:\mathcal{T}^{4}_{TTLL}:\mathcal{T}^{4}_{LLLL}:\mathcal{T}^{4}_{TTL}:\mathcal{T}^{4}_{LLL}:\tilde{\mathcal{T}}^{4}_{LT,T}:\tilde{\mathcal{T}}^{4}_{LP,P}:\mathcal{T}^{4}_{LL}:\tilde{\mathcal{T}}^{4}_{LL}=16:\frac{4}{3}:\frac{1}{3}:-\frac{4\sqrt{6}}{9}:-\frac{\sqrt{6}}{9}:-\frac{2\sqrt{6}}{3}:0:\frac{2}{3}:0. \tag{4.27}$$

Note that these proportionality constants are again, as conjectured by Gross, independent of the scattering angle ϕ_{CM} and the loop order χ of string perturbation theory. They are also independent of particles chosen for vertex $v_{1,3,4}$. The subleading order amplitudes corresponding to E^3 appear in Eqs. (4.13), (4.14), (4.16), (4.19), (4.21) and (4.24). Note that there are two equations for (4.21). One has 7 equations with 9 amplitudes, $\mathcal{T}^{3}_{TTTL}, \mathcal{T}^{3}_{TLLL}, \mathcal{T}^{3}_{TLL}, \mathcal{T}^{3}_{TTT}, \tilde{\mathcal{T}}^{3}_{TL,L}, \tilde{\mathcal{T}}^{3}_{PP,T}, \mathcal{T}^{3}_{LT}, \tilde{\mathcal{T}}^{3}_{(LT)}$ and $\tilde{\mathcal{T}}^{3}_{[LT]}$. Presumably, they are not proportional to each other or the proportional coefficients do depend on the scattering angle ϕ_{CM}. Our calculation here is again purely algebraic *without any integration* and is independent of saddle point calculation in [3,4,10]. It is important to note that our result in Eq. (4.27) is gauge invariant. On the other hand, the result obtained in [10] with

$\mathcal{T}^4_{TTTT} \propto \tilde{\mathcal{T}}^4_{LT,T} \propto \mathcal{T}^4_{LL}$, and $\mathcal{T}^4_{TTLL} = \mathcal{T}^4_{LLLL} = \mathcal{T}^4_{TTL} = \mathcal{T}^4_{LLL} = \tilde{\mathcal{T}}^4_{LP,P} = \tilde{\mathcal{T}}^4_{LL} = 0$ in the leading order energy is, on the contrary, *not* gauge invariant. In fact, with only three non-zero amplitudes, it would be very difficult to satisfy all 8 equations. The situation gets even worse if one goes to higher mass level where number of zero-norm states, or constraint equations, increases much faster than that of positive-norm states [5]. To further justify our result, we give a sample calculation in the following.

Since the proportionality constants in Eq. (4.27) are independent of particles chosen for vertex $v_{1,3,4}$, for simplicity, we will choose them to be tachyons. For the string-tree level $\chi = 1$, with one tensor v_2 and three tachyons $v_{1,3,4}$, all scattering amplitudes for mass level $M^2 = 6$ were explicitly calculated in [14]. They are

$$\begin{aligned}
\mathcal{T}^{\mu\nu\alpha\beta} &= \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X} \partial X^\mu \partial X^\nu \partial X^\alpha \partial X^\beta e^{ik_2X} e^{ik_3X} e^{ik_4X} \right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} \left[\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right) k_3^\mu k_3^\nu k_3^\alpha k_3^\beta \right. \\
&\quad - t\left(\frac{t^2}{4}-1\right)(s+2) k_1^{(\mu} k_1^\nu k_1^\alpha k_3^{\beta)} + \frac{3st}{2}\left(\frac{s}{2}+1\right)\left(\frac{t}{2}+1\right) k_1^{(\mu} k_1^\nu k_3^\alpha k_3^{\beta)} \\
&\quad \left. - s\left(\frac{s^2}{4}-1\right)(t+2) k_1^{(\mu} k_3^\nu k_3^\alpha k_3^{\beta)} + \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right) k_1^\mu k_1^\nu k_1^\alpha k_1^\beta \right],
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
\mathcal{T}^{\mu\nu\lambda} &= \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X} \partial X^\mu \partial X^\nu \partial^2 X^\lambda e^{ik_2X} e^{ik_3X} e^{ik_4X} \right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} \left[-\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right) k_3^\mu k_3^\nu k_3^\lambda \right. \\
&\quad + t\left(\frac{t^2}{4}-1\right)\left(\frac{s}{2}+1\right) k_1^\lambda k_1^{(\mu} k_3^{\nu)} \\
&\quad - \frac{st}{4}\left(\frac{s}{2}+1\right)\left(\frac{t}{2}+1\right)\left(k_1^\mu k_1^\nu k_3^\lambda + k_3^\mu k_3^\nu k_1^\lambda\right) \\
&\quad \left. + s\left(\frac{s^2}{4}-1\right)\left(\frac{t}{2}+1\right) k_3^\lambda k_1^{(\mu} k_3^{\nu)} - \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right) k_1^\mu k_1^\nu k_1^\lambda \right],
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\mathcal{T}^{\mu\nu} &= \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X} \partial^2 X^\mu \partial^2 X^\nu e^{ik_2X} e^{ik_3X} e^{ik_4X} \right\rangle \\
&= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)} \left[\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right) k_3^\mu k_3^\nu \right. \\
&\quad \left. + \frac{st}{2}\left(\frac{s}{2}+1\right)\left(\frac{t}{2}+1\right) k_1^{(\mu} k_3^{\nu)} + \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right) k_1^\mu k_1^\nu \right],
\end{aligned} \tag{4.30}$$

$$\tilde{\mathcal{T}}^{\mu\nu} = \frac{1}{2}\int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X}\partial X^\mu \partial^3 X^\nu e^{ik_2X} e^{ik_3X} e^{ik_4X}\right\rangle$$
$$= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\left[\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right)k_3^\mu k_3^\nu \right.$$
$$- \frac{s}{2}\left(\frac{s^2}{4}-1\right)\left(\frac{t}{2}+1\right)k_1^\mu k_3^\nu - \frac{t}{2}\left(\frac{t^2}{4}-1\right)\left(\frac{s}{2}+1\right)k_3^\mu k_1^\nu$$
$$\left. + \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right)k_1^\mu k_1^\nu\right], \tag{4.31}$$

$$\mathcal{T}^{\mu} = \frac{1}{6}\int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1X}\partial^4 X^\mu e^{ik_2X} e^{ik_3X} e^{ik_4X}\right\rangle$$
$$= \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\left[-\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right)k_3^\mu \right.$$
$$\left. - \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right)k_1^\mu\right]. \tag{4.32}$$

We can now explicitly calculate all amplitudes in Eq. (4.27). After a lengthy algebra, we have justified Eq. (4.27) with $\mathcal{T}^4_{TTTT} = 16E^{12}\sin^4\phi_{\mathrm{CM}}\exp(-\alpha'\frac{s\ln s+t\ln t+u\ln u}{2})$ in the high-energy limit. We have also checked that $\mathcal{T}^8_{LLLL} = \mathcal{T}^6_{LLLL} = \mathcal{T}^6_{LLL} = \mathcal{T}^6_{TTLL} = \tilde{\mathcal{T}}^6_{LP,P} = \tilde{\mathcal{T}}^6_{LL,P} = 0$ and $\tilde{\mathcal{T}}^4_{LP,P} = \tilde{\mathcal{T}}^4_{LL,P}$ as claimed above. The calculation of $\mathcal{T}_{LLLL}$, for example, gives

$$\mathcal{T}_{LLLL} = \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\left[\left(\frac{s^2}{4}-s\right)\left(\frac{s^2}{4}-1\right)(e_Lk_3)^4\right.$$
$$- s\left(\frac{s^2}{4}-1\right)(t+2)(e_Lk_3)^3(e_Lk_1)$$
$$+ \frac{3st}{2}\left(\frac{s}{2}+1\right)\left(\frac{t}{2}+1\right)(e_Lk_3)^2(e_Lk_1)^2$$
$$\left. - t\left(\frac{t^2}{4}-1\right)(s+2)(e_Lk_3)(e_Lk_1)^3 + \left(\frac{t^2}{4}-t\right)\left(\frac{t^2}{4}-1\right)(e_Lk_1)^4\right]. \tag{4.33}$$

By using Eqs. (3.6), (3.9), (3.11) and (3.12) and after a lengthy algebra, we find that the contributions of orders E^{16} and E^{14} of $\mathcal{T}_{LLLL}$ are zero. The leading order E^{12} term gives $\frac{1}{3}\sin^4\phi_{\mathrm{CM}}\exp(-\alpha'\frac{s\ln s+t\ln t+u\ln u}{2})$ as expected from Eq. (4.27). Similar calculations apply to other 8 amplitudes. Finally, by Eqs. (3.1), (3.7), (3.10) and (4.28), it is easy to deduce in general that

$$\mathcal{T}_n^{TT\cdots} = \left[(-2)^n E^{3n}\sin^n\phi_{\mathrm{CM}}\right]\mathcal{T}, \tag{4.34}$$

where n is the number of T and $\mathcal{T} = \exp(-\alpha'\frac{s\ln s+t\ln t+u\ln u}{2})$ is the high energy four tachyons amplitude. As a result, all high-energy string scattering amplitudes can be expressed in terms of those of tachyons.

5. A comparison with saddle point calculation

To compare our results with Ref. [10], we briefly review the works in [3,4,10]. In Ref. [10], it was shown that the high-energy, fixed angle scattering amplitudes of oriented open strings can be obtained from those of closed strings calculated by Gross and Mende [3] by using the reflection principle. First, from Eq. (2.3), one notes that the high-energy limit $\alpha' \to \infty$ is equivalent to the semi-classical limit of first-quantized string theory. In this limit, the closed string G-loop scattering amplitudes is dominated by a saddle point in the moduli space $\vec{m}$. For the oriented open string amplitudes, the saddle point configuration can be constructed from an associated configuration of the closed string via reflection principle. It was also found that the Euler number χ of the oriented open string saddle is always $\chi = 1 - G$, where G is the genus of the associated closed string saddle. Thus the integral in Eq. (2.3) is dominated in the $\alpha' \to \infty$ limit by an associated G-loop closed string saddle point in X^μ, $\hat{\vec{m}}_i$ and $\hat{\xi}_i$. The closed string classical trajectory at G-loop order was found to behave at the saddle point as [3]

$$X^{\mu}_{c1}(z) = \frac{i}{1+G}\sum_{i=1}^{4} k_i \ln|z-a_i| + O\left(\frac{1}{\alpha'}\right), \tag{5.1}$$

which leads to the χth order open string four-tachyon amplitude

$$\mathcal{T}_\chi \approx g_c^{2-\chi}\exp\left(-\alpha'\frac{s\ln s + t\ln t + u\ln u}{2(2-\chi)}\right). \tag{5.2}$$

Eq. (5.2) reproduces the very soft exponential decay $e^{-\alpha' s}$ of the well-known string-tree $\chi = 1$ amplitude. The exponent of Eq. (5.2) can be thought of as the electrostatic energy E_G of two-dimensional Minkowski charges k_i placed at a_i on a Riemann surface of genus G. One can use the $SL(2,C)$ invariance of the saddle to fix 3 of the 4 points a_i, then the only modulus is the cross ratio $\lambda = \frac{(a_1-a_3)(a_2-a_4)}{(a_1-a_2)(a_3-a_4)}$, which takes the value $\lambda = \hat{\lambda} \approx -\frac{t}{s} \approx \sin^2\frac{\phi_{\mathrm{CM}}}{2}$ to extremize E_G if we neglect the mass of the tachyons in the high-energy limit. For excited string states, it was found that only polarizations in the plane of scattering will contribute to the amplitude at high energy. To leading order in the energy E, the products of e_T and e_L with $\partial^n X$ are given by [10]

$$e_T\cdot\partial^n X \sim i(-)^n\frac{(n-1)!}{\lambda^n}E\sin\phi_{\mathrm{CM}}, \quad n>0, \tag{5.3}$$

$$e_L\cdot\partial^n X \sim i(-)^{(n-1)}\frac{(n-1)!}{\lambda^n}\frac{E^2\sin^2\phi_{\mathrm{CM}}}{2m_2}\sum_{l=0}^{n-2}\lambda^l, \quad n>1, \tag{5.4}$$

$$e_L\cdot\partial^n X \sim 0, \quad n=1, \tag{5.5}$$

where m_2 is the mass of the particle. Now, we would like to point out that naive uses of Eqs. (5.3)–(5.5) will miss some high-energy amplitudes and will give, for example, a wrong result $\mathcal{T}^3_{LLT} = 0$ [10] since $e_L\cdot\partial X \sim 0$. This is inconsistent with our result Eq. (2.19) or Eq. (3.14). The missing terms can be seen as following. We will use the $M^2 = 4$ string-tree

$\chi = 1$ amplitude $\mathcal{T}_{LLT}$ to illustrate our point. Let us first use the path integral calculation

$$\mathcal{T}_{LLT} = \int \prod_{i=1}^{4} dx_i \left\langle e^{ik_1 X} e_L \cdot \partial X e_L \cdot \partial X e_T \cdot \partial X e^{ik_2 X} e^{ik_3 X} e^{ik_4 X} \right\rangle, \tag{5.6}$$

which is similar to the calculation of moments of the Gaussian integral

$$\sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} dx\, x^n e^{-\frac{a}{2}x^2 + bx} = \frac{\partial^n}{\partial b^n} \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + bx}. \tag{5.7}$$

For $n = 1$, the value obtained by Eq. (5.7) is

$$\frac{b}{a} e^{\frac{b^2}{2a}} = x e^{-\frac{a}{2}x^2 + bx} \big|_{x = \frac{b}{a}}$$

where $\frac{b}{a}$ is exactly the saddle point of the Gaussian integrand. For $n = 2$, however, the value obtained by Eq. (5.7) is

$$\left(\frac{b}{a}\right)^2 e^{\frac{b^2}{2a}} + \frac{1}{a} e^{\frac{b^2}{2a}} = x^2 e^{-\frac{a}{2}x^2 + bx} \big|_{x = \frac{b}{a}} + \frac{1}{a} e^{\frac{b^2}{2a}}.$$

It is this extra $\frac{1}{a} e^{\frac{b^2}{2a}}$ term that was missing in the argument of Section 6 of Ref. [10]. Similar situations happen for $n \geqslant 3$ and even more terms were missed. The argument can be easily generalized to $\vec{b} \in R^3$ in the space of helicity decomposition. Eq. (5.6) corresponds to the case of $n = 3$. It can be checked that some terms with the same energy order as $\mathcal{T}_{TTT}$ survive in the calculation of Eq. (5.6). They will be missing if one misuses Eqs. (5.3)–(5.5). Similar wrong calculations will suppress many other should be nonzero high-energy amplitudes at mass level $M^2 = 6$ stated after Eq. (4.27). Another way to calculate Eq. (5.6) is to use Wick theorem. Again, naive uses of Eqs. (5.3)–(5.5) will miss some high-energy amplitudes which correspond to, for example, the contraction of $e^{ik_1 X}$ with $e_L \cdot \partial X e_L \cdot \partial X$. We stress here that Eqs. (5.1)–(5.5) are still valid as they stand.

6. Conclusion

We have shown that the physical origin of high-energy symmetries and the proportionality constants in Eqs. (2.19) and (4.27) are from zero-norm states in the OCFQ spectrum. Other related approaches of high-energy stringy symmetries can be found in [15]. The most challenging problem remained is the calculation of algebraic structure of these stringy symmetries derived from the complete zero-norm state solutions of Eqs. (2.1) and (2.2) with arbitrarily high spins. Presumably, it is a complicated 26D generalization of ω_∞ of the simpler toy 2D string model [8]. Our calculation in Eqs. (2.19) and (4.27) are, similar to the toy 2D string, purely algebraic without any integration which signal the powerfulness of zero-norm states and symmetries they imply. The results presented in this paper can be served as consistent checks of saddle point calculations [3] and as the realization of high-energy symmetries [4] of string theory. The simple idea of massive gauge invariance of our calculations correct the inconsistent high-energy calculation in Section 6 of Ref. [10].

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