On the Positive Solutions for Semilinear Elliptic Equations on Annular Domain with Non-homogeneous Dirichlet Boundary Condition*

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1. INTRODUCTION

In this paper we consider the existence and the multiplicity of positive radial solutions of the equation

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

with some non-homogeneous Dirichlet boundary conditions, where Ω is annular or in multiply connected bounded smooth domains in $\mathbb{R}^n$, $n > 2$. We assume that f satisfies the following assumptions throughout the paper, without further mention.

(A-1) $f \in C^2(\mathbb{R}^1)$ and $f(t) > 0$ for $t > 0$,

(A-2) $\lim_{t \to 0^+} (f(t)/t) = 0$,

(A-3) $\lim_{t \to \infty} (f(t)/t) = \infty$.

This paper was motivated by the recent work of Bandle and Peletier [2]. In [2], they consider the Dirichlet problem

$$\Delta u + u^{(n+2)/(n-2)} = 0, \qquad u > 0 \text{ in } \Omega,$$

$$u = 0 \qquad \text{on } \Gamma_0, \tag{1.2}$$

$$u = b, \qquad b > 0 \text{ in } \Gamma_1,$$

where Ω is a doubly connected bounded smooth domain in $\mathbb{R}^n$, $n > 2$, with inner boundary Γ_1 and outer boundary Γ_0. They proved that if Ω is annular or in general domains for which there exist constants

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$0 < R_2 < R_1 < R_0 < \infty$ such that $\bar{\Omega} \subseteq \{x; R_2 < |x| < R_0\}$ and $\Gamma_1 \subseteq \{x : |x| < R_1\}$, then there exists a positive constant b^* such that problem (1.2) has a solution for all $b < b^*$ and none for $b > b^*$.

In this paper, on the annulus $\Omega = \{x \in \mathbb{R}^n : 0 < R_1 < |x| < R_0\}$, we consider (1.1) with one of the Dirichlet boundary conditions

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = b \quad \text{on} \quad |x| = R_0, \quad (1.3a)$$

$$u = b \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_0; \quad (1.3b)$$

here $b > 0$. We shall prove that there exists a positive constant b^* such that problem (1.1), (1.3a) has at least two positive radial solutions for all $b < b^*$ and none for $b > b^*$; a similar result also holds for (1.1), (1.3b). Moreover, in addition to (A-1)–(A-3), if f satisfies

(A-4) $f''(t) > 0$ for all $t > 0$,

(A-5) $[n/(n-2) + 2R_1^{n-2}/(R_0^{n-2} - R_1^{n-2})] f(t) \geqslant tf'(t) > f(t) > 0$ for all $t > 0$,

then we prove that there are exactly two positive radial solutions of (1.1), (1.3a) on annulus. Furthermore, we prove that a similar result holds for the equation

$$\Delta u + u^p = 0, \qquad p > 1, \quad \text{in } \Omega, \quad (1.4)$$

with boundary condition (1.3a). We shall also generalize the result of Bandle and Peletier [2] to problem (1.1) with the multiple-connected bounded smooth domains.

Equation (1.1) on the annular domain $\Omega = \{x : 0 < R_1 < |x| < R_0\}$ with one of the following sets of boundary conditions,

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_0, \quad (1.5a)$$

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_0, \quad (1.5b)$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_0; \quad (1.5c)$$

here $r = |x|$ has been considered by many authors (see, e.g., [1, 3, 4]). In [1], Bandle et al. proved that problem (1.1), (1.5) has positive radial solutions for any annulus in $\mathbb{R}^n$, $n > 2$, by assuming (A-1)–(A-3), and f is nondecreasing in $(0, \infty)$. In [3], Garaizar proved the various existence and non-existence results of problem (1.1), (1.5a) under the various assumptions of f at infinity and at 0. In [4], Lin proved that under

assumptions (A-1)–(A-3), problem (1.1), (1.5) has at least one positive radial solution for any annulus in $\mathbb{R}^n$, $n > 2$.

The paper is organized as follows: In Section 2, we study the existence of positive radial solutions for (1.1), (1.3a) and (1.1), (1.3b) on annulus. In Section 3, we prove that there are exactly two positive radial solutions of (1.1), (1.3a) if (A-1)–(A-5) are satisfied. In Section 4, we study the existence of solutions for (1.1), (1.3a) on the general domains.

2. Existence of Positive Radial Solutions on Annulus

Since we are interested in the positive radial solutions of (1.1), (1.3a) on the annulus $\Omega = \{x \in \mathbb{R}^n : 0 < R_1 < |x| < R_0\}$, $u = u(r)$ satisfying

$$u''(r) + \frac{n-1}{r} u'(r) + f(u(r)) = 0 \qquad \text{in } (R_1, R_0), \tag{2.1}$$

$$u(R_1) = 0, \qquad u(R_0) = b. \tag{2.2}$$

Thus, in terms of variables

$$s = r^{2-n} \qquad \text{and} \qquad u(s) = u(r),$$

problem (2.1), (2.2) can be rewritten as

$$u''(s) + \rho(s) f(u(s)) = 0 \qquad \text{in } (s_0, s_1), \tag{2.3}$$

$$u(s_0) = b, \qquad u(s_1) = 0, \tag{2.4}$$

where $\rho(s) = (n-2)^{-2} s^{-k}$, $k = 2 + 2/(n-2)$, $s_0 = R_0^{2-n}$, and $s_1 = R_1^{2-n}$.

We shall use the method of backward shooting, i.e., consider the family of solutions of the initial value problems

$$u''(s) + \rho(s) f(u(s)) = 0 \qquad \text{for } s < s_1, \tag{2.5}$$

$$u(s_1) = 0, \qquad u(s_1) = -\beta, \tag{2.6}$$

where $\beta > 0$ is the shooting parameter. For every $\beta > 0$, problem (2.5), (2.6) has a unique solution $u(\cdot) \equiv u(\cdot, \beta)$ with the maximal domain of existence $(\tilde{s}(\beta), s_1)$. It is easy to check that (2.5), (2.6) is equivalent to the integral equation

$$u(s) = \beta(s_1 - s) - \int_s^{s_1} (t - s) \rho(t) f(u(t)) \, dt, \qquad s \in (\tilde{s}(\beta), s_1), \tag{2.7}$$

and the solution u also satisfies

$$u(s) = u(\bar{s}) + u'(\bar{s})(s - \bar{s}) + \int_s^s (t - s)\rho(t) f(u(t)) \, dt, \qquad s, \bar{s} \in (\tilde{s}(\beta), s_1). \quad (2.8)$$

From (2.7), if u is positive in some interval (α, s_1) with $\alpha \geqslant 0$, then

$$u(s) \leqslant \beta(s_1 - s) \qquad \text{in } (\alpha, s_1). \quad (2.9)$$

If u has a zero in $(\tilde{s}(\beta), s_1)$, denote

$$s_0(\beta) = \inf\{s_0 : u(s, \beta) > 0 \text{ in } (s_0, s_1)\},$$

and in this case, there exists a unique $\tau(\beta) \in (s_0(\beta), s_1)$, such that $u'(\tau(\beta), s_1) = 0$. Since $u''(s, \beta) < 0$ in $(s_0(\beta), s_1)$ and $u'(s_0(\beta), \beta) > 0$, by the implicit function theorem, $s_0(\beta)$ and $\tau(\beta)$ are C^1 for those β such that $s_0(\beta) > 0$.

For each fixed $s_0 > 0$, denote

$$\Lambda(s_1) = \{\beta > 0 : (2.5), (2.6) \text{ has a positive solution in } (s_0, s_1)\} \quad (2.10)$$

$$B(s_1) = \sup\{u(s_0, \beta) : \beta \in \Lambda(s_1)\}. \quad (2.11)$$

The following two lemmas, which were proved in Lemma 2.7 of [1] and Lemmas 2.1 and 2.2 of [4], indicated that $\Lambda(s_1) \neq \phi$.

LEMMA 2.1. *If (A-1) and (A-2) are satisfied, then $s_0(\beta)$ is nonnegative for any $\beta > 0$ and*

$$\lim_{\beta \to 0^+} s_0(\beta) = 0. \quad (2.12)$$

LEMMA 2.2. *If (A-1) and (A-2) are satisfied, then $\tau(\beta)$ and $s_0(\beta)$ are well-defined, when β is sufficiently large and*

$$\lim_{\beta \to \infty} \tau(\beta) = s_1, \quad (2.13)$$

$$\lim_{\beta \to \infty} u(\tau(\beta), \beta) = \infty, \quad (2.14)$$

$$\lim_{\beta \to \infty} s_0(\beta) = s_1. \quad (2.15)$$

LEMMA 2.3. *If (A-1)–(A-3) are satisfied, then $B(s_1)$ is finite for any $s_1 > s_0$, and problem (2.3), (2.4) has a positive solution for any $b \leqslant B(s_1)$ and none for $b > B(s_1)$.*

Proof. First, we prove that $B(s_1) < \infty$ for any $s_1 > s_0$. Otherwise, there would be a sequence $\beta_k > 0$ with $u(s_0, \beta_k) \to \infty$ as $k \to \infty$. Let $u_k(\cdot) \equiv u(\cdot, \beta_k)$. Then

$$\beta_k > \left| \frac{u_k(s_1) - u_k(s_0)}{s_1 - s_0} \right| = \frac{u_k(s_0)}{s_1 - s_0}$$

implies $\lim_{k \to \infty} \beta_k = \infty$. By Lemma 2.2, we have a contradiction.

Second, we prove that (2.3), (2.4) has a positive solution for any $b \leqslant B(s_1)$ and none for $b > B(s_1)$. It is clear that (2.3), (2.4) has a solution for $b = B(s_1)$, and none for $b > B(s_1)$. Let v be a solution of (2.3), (2.4) with $b = B(s_1)$. Then for each $b \in (0, B(s_1))$ v is a supersolution of (2.3), (2.4). Since $w = 0$ is a subsolution of (2.3), (2.4), therefore (2.3), (2.4) has a solution by the method of monotone iterration scheme (see, e.g., [7]). The proof is complete.

Instead of using the method of backward shooting, we shall use the method of forward shooting to study the multiplicity problem of solutions of (2.3), (2.4), i.e., consider the family of solutions of the initial value problems

$$u''(s) + \rho(s) f(u(s)) = 0, \qquad s > s_0, \tag{2.16}$$

$$u(s_0) = b, \qquad u'(s_0) = \alpha; \tag{2.17}$$

here $\alpha \in \mathbb{R}^1$.

For every $\alpha \in \mathbb{R}^1$, (2.16), (2.17) has a unique solution $u(\cdot) \equiv u(\cdot, \alpha)$ with the maximal domain $(s_0, \tilde{s}(\alpha))$, and it is easy to check that (2.16), (2.17) is equivalent to the integral equation

$$u(s) = b + \alpha(s - s_0) + \int_{s_0}^{s} (t - s) \rho(t) f(u(t)) \, dt, \qquad s \in (s_0, \tilde{s}_1(\alpha)). \tag{2.18}$$

Denote

$$s_1(\alpha, b) = \sup\{s_1 : u(\cdot, \alpha) > 0 \text{ in } (s_0)\}, \tag{2.19}$$

$$s_1(b) = \sup\{s_1(\alpha, b) : \alpha \in \mathbb{R}^1\}. \tag{2.20}$$

LEMMA 2.4. *If* (A-1) *and* (A-3) *are satisfied, then*

$$\lim_{\alpha \to \pm \infty} s_1(\alpha, b) = s_0. \tag{2.21}$$

Proof. First, we prove (2.21) when $\alpha \to -\infty$. From (2.18), we have

$$u(s) \leqslant b + \alpha(s - s_0), \qquad s \in (s_0, s_1(\alpha, b)).$$

Let $\hat{s}(\alpha) = s_0 - b/\alpha$, since $s_0 \leqslant s_1(\alpha, b) \leqslant \hat{s}(\alpha)$. This implies (2.21) when $\alpha \to -\infty$.

Next, by arguments similar to those in Lin [4], we shall prove the following results,

$$\lim_{\alpha \to \infty} \tau(\alpha, b) = s_0, \tag{2.22}$$

$$\lim_{\alpha \to \infty} u(\tau(\alpha, b), \alpha) = \infty, \tag{2.23}$$

$$\lim_{\alpha \to \infty} \hat{s}_1(\alpha, b) = s_0, \tag{2.24}$$

$$\lim_{\alpha \to \infty} u'(\hat{s}_1(\alpha, b), \alpha) = -\infty, \tag{2.25}$$

where $\hat{s}_1(\alpha, b)$ satisfies $u(\hat{s}_1(\alpha, b), \alpha) = b$ and $u'(\hat{s}_1(\alpha, b), \alpha) < 0$, $\tau(\alpha, b)$ satisfies $u'(\tau(\alpha, b), \alpha) = 0$, $u'(\cdot, \alpha) > 0$ in $(s_0, \tau(\alpha, b))$. If (2.22), (2.23) hold, then $\tau(\alpha, b)$, $\hat{s}_1(\alpha, b)$ are well-defined for α sufficiently large. Hence, (2.21) with $\alpha \to \infty$ follows from (2.24), (2.25).

If (2.22) were false, there would be a point $\tau_0 > s_0$ and a sequence $\alpha_k \to \infty$ with

$$u_k(s) > 0 \quad \text{and} \quad u'_k(s) > 0 \quad \text{in } (s_0, \tau_0), \tag{2.26}$$

where $u_k(\cdot) = u_k(\cdot, \alpha_k)$. Let $\bar{s} = (s_0 + \tau_0)/2$. We claim that

$$\lim_{k \to \infty} \sup u_k(\bar{s}) = \infty. \tag{2.27}$$

Suppose that this is not the case. Then there exists a constant $M > 0$, such that

$$u_k(\bar{s}) \leqslant M \quad \text{for all } k. \tag{2.28}$$

By (2.18) and (2.28), we have

$$u_k(\bar{s}) \geqslant b + \alpha_k(\tau_0 + s_0)/2 - C$$

for some constant $C \geqslant 0$. But this contradicts (2.28). Therefore, (2.27) holds.

By choosing a subsequence of α_k, if necessary, we may assume that

$$\lim_{k \to \infty} u_k(\bar{s}) = \infty. \tag{2.29}$$

Denote

$$M_k = \inf\{f(u_k(s))/u_k(s) : s \in [\bar{s}, \tau_0]\}$$

$$\geqslant \inf\{f(u)/u : u \geqslant u_k(\bar{s})\}.$$

By (2.29) and (A-3), we have

$$\lim_{k \to \infty} M_k = \infty. \tag{2.30}$$

Let u_k and v_k be the solutions of

$$u''(s) + \rho(s) h_k(s) u(s) = 0 \qquad \text{in } (\bar{s}, \tau_0),$$
$$v''(s) + \rho(s) M_k v(s) = 0 \qquad \text{in } (\bar{s}, \tau_0),$$

respectively, where $h_k(s) = f(u_k(s))/u_k(s)$. Then

$$\rho(s) h(s) \geqslant \rho(\tau_0) M_k \tag{2.31}$$

and (2.30) implies that v_k has at least two zeros in $(\bar{s}, \tau_0)$, when k is sufficiently large. By (2.31) and the Sturm Comparison Theorem, u_k has at least one zero in $(\bar{s}, \tau_0)$. But this contradicts (2.26). Hence, (2.22) holds.

Next, we prove (2.23). Suppose that (2.23) does not hold. Then there exist a constant $M > 0$ and a sequence $\alpha_k \to \infty$ such that

$$u_k(\tau_k) \leqslant M \qquad \text{for all } k, \tag{2.32}$$

where $u_k(s) = u_k(s, \alpha_k)$ and $\tau_k = \tau(\alpha_k)$. Denote

$$F(u) = \int_0^u f(t)\, dt$$

and define

$$V(s) = V(s, \alpha) \equiv (u'(s))^2/2 + \rho(s) F(u(s)). \tag{2.33}$$

Since $V'(s) = \rho'(s) F(u(s))$,

$$V(\tau_k) = V(s_0) + \int_{s_0}^{\tau_k} \rho'(t) F(u(t))\, dt.$$

Therefore, we have

$$\rho(\tau_k) F(u_k(\tau_k)) = \alpha_k^2/2 + \rho(s_0) F(b) + \int_{s_0}^{\tau_k} \rho'(t) F(u_k(t))\, dt. \tag{2.34}$$

Now, (2.32) implies the left-hand side of (2.34) is bounded, but this is impossible when $k \to \infty$. Hence, (2.3) holds.

Next, we prove (2.24). Suppose that (2.24) does not hold. Then there exist a constant $\bar{s}_1 > s_0$ and a sequence $\alpha_k \to \infty$ such that

$$u_k(s) > b \qquad \text{and} \qquad u_k'(s) < 0 \qquad \text{in } (\tau_k, \bar{s}_1), \tag{2.35}$$

where u_k and τ_k are given after (2.32). Denote

$$\bar{s} = (s_0 + \tilde{s}_1)/2. \tag{2.36}$$

By (2.22), we may assume $\bar{s} > \tau_k$ for any k. We claim that

$$\lim_{k \to \infty} \sup u_k(\bar{s}) = \infty. \tag{2.37}$$

Otherwise, there exists a constant $M > 0$ such that

$$u_k(\bar{s}) \leqslant M, \qquad \text{for all } k. \tag{2.38}$$

Since $u'' < 0$ in $[s_0, \tilde{s}_1]$, therefore (2.23) and (2.35) imply that

$$\lim_{k \to \infty} u'_k(\bar{s}) = -\infty.$$

Moreover, $u'' < 0$ also implies that

$$\lim_{k \to \infty} u'_k(\tilde{s}_1) = -\infty, \tag{2.39}$$

Hence

$$u_k(\bar{s}) \, u_k(\tilde{s}_1) + u'_k(\tilde{s}_1)(\bar{s} - \tilde{s}_1) + \int_{\tilde{s}_1}^{\bar{s}} (t - \bar{s}) \rho(t) f(u_k(t)) \, dt$$

$$\geqslant b - u'_k(\tilde{s}_1)(\tilde{s}_1 - s_0)/2 - C$$

for some constant C. Therefore (2.39) implies

$$\lim_{k \to \infty} u_k(\bar{s}) = \infty,$$

a contradiction to (2.38). Hence, (2.37) holds. By (2.37) and the Sturm Comparison Theorem again, u_k has a zero in $(\tau_k, \bar{s})$ when k is sufficiently large. But, this contradicts (2.35). Hence, (2.24) holds.

Finally, we prove (2.25). Let m_α be the slope of the straight line passing through the points $(\tau(\alpha, b), u(\tau(\alpha, b)))$ and $(\hat{s}(\alpha, b), b)$. Then (2.23) and (2.24) implies that

$$\lim_{\alpha \to \infty} m_\alpha = -\infty. \tag{2.40}$$

Since $u'' < 0$, we have $u'(\hat{s}_1(\alpha, b), \alpha) < m_\alpha$. Hence, (2.40) implies (2.25). The proof is complete.

An immediate consequence of Lemma 2.4 is the following result.

COROLLARY 2.5. *If* (A-1) *and* (A-3) *are satisfied, then* (2.3), (2.4) *has at least two positive solutions for any* $s_1 \in (s_0, s_1(b))$, *at least one for* $s_1 = s_1(b)$ *if* $s_1(b) < \infty$, *and none for* $s_1 > s_1(b)$.

In the remaining part of this section, we shall prove that $B(s_1)$ and $s_1(b)$ which are defined by (2.11), (2.20), respectively, are inverse to each other. Then, we can easily obtain that there exist $b^* > 0$ such that problem (2.3), (2.4) has at least two positive solutions for any $b < b^*$.

LEMMA 2.6. *If* (A-1)–(A-3) *are satisfied and* $s_1(b_0) < \infty$, *then* $s_1(b) < \infty$ *for any* $b > b_0$ *and* $s_1(b)$ *is strictly decreasing in* (b_0, ∞).

Proof. First, we claim that $s_1(b) < \infty$ for $b > b_0$. Otherwise, there exist $\bar{b} > b_0$ such that $s_1(\bar{b}) = \infty$. By Corollary 2.5, (2.3), (2.4) with $u(s_0)$ with $u(s_0) = \bar{b}$, has a solution for any $s_1 \in (s_1(b_0), \infty)$. Hence $B(s_1) \geqslant \bar{b}$. Then by Theorem 2.3, (2.3), (2.4) with $u(s_0) = b_0$ has a solution, but this is impossible. Therefore $s_1(b) < \infty$ for $b > b_0$.

Similarly, it is easy to prove that if $b_1 < b_2$, then $s_1(b_1) \geqslant s_1(b_2)$.

Finaly, we prove that if $b_1 < b_2$, then $s_1(b_1) > s_1(b_2)$. Otherwise, there is a constant s_1 such that $s_1(b) = s_1$ for any $b \in [b_1, b_2]$. Let $u(s)$ be the solution of (2.3), (2.4) with $u(s_0) = b_2$, $u(s_1) = 0$. Then there exists a constant β such that $\beta = -u'(s_1)$. By the continuous dependence of o.d.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that $s_1 < \bar{s}_1 < s_1 + \delta$, and $v(s)$ is the solution of (2.5), (2.6) with $s_1 = \bar{s}_1$, and $|u(s) - v(s)| < \varepsilon$, for any $s \in [s_0, s_1]$. By the assumption, $b_2 < v(s_0)$; hence $B(\bar{s}_1) \geqslant v(s_0) > b_2$. By Theorem 2.3 there exists a solution of (2.3), (2.4) with $b = b_2$, $s_1 = \bar{s}_1$, but this is impossible. The proof is complete.

LEMMA 2.7. $B(s_1)$ *is strictly decreasing and continuous in* (s_0, ∞).

Proof. By an argument similar to that in Lemma 2.6, we can prove that $B(s_1)$ is strictly decreasing in (s_0, ∞). The details are omitted. To prove $B(s_1)$ is continuous, we prove it by contradiction. If there exists $s_1 \in (s_0, \infty)$ such that $\lim_{s \to s_1} B(s) = B \neq B(s_1)$, let $B - B(s_1) = \varepsilon > 0$. Therefore, there exists $\delta > 0$, such that for $\bar{s}_1 \in (s_1 - \delta, s_1)$, $B(\bar{s}_1) \geqslant B$. Now, for any $\bar{s}_1 \in (s_1 - \delta, s_1)$, let $u(\cdot, s_1, \beta)$ be the solution of (2.5), (2.6) and let $\beta(\bar{s}_1)$ be a slope such that $u(s_0, \bar{s}_1, \beta(\bar{s}_1)) = B(\bar{s}_1)$. By an argument similar to that used in proving Lemma 2.3, it can be proved that $\beta(\bar{s}_1)$ is bounded above for $\bar{s}_1 \in (s_1 - \delta, s_1)$. Therefore, there exist a sequence $\{\tilde{s}_k\} \subset (s_1 - \delta, s_1)$ and $\tilde{\beta} \in \mathbb{R}^1$ such that $\tilde{s}_k \to s_1$ and $\beta(\tilde{s}_k) \to \tilde{\beta}$, when $k \to \infty$. Hence, by the continuous dependence of o.d.e., $u(s_0, s_1, \tilde{\beta}) \geqslant B - \varepsilon/2 > B(s_1)$, a contradiction. The proof is complete.

THEOREM 2.8. *The functions* $B: (s_0, \infty) \to (b_0, \infty)$ *and* $s_1: (b_0, \infty) \to (s_0, \infty)$, *which are defined by* (2.11), (2.20), *respectively, are inverse to each other, where* $b_0 = \lim_{s_1 \to \infty} B(s_1)$.

Proof. We claim that $B(s_1(b)) = b$, for any $b > b_0$. Otherwise, $B(s_1(b)) > b$. Let $b_1 = B(s_1(b))$. Then $s_1(b_1) \geqslant s_1(b)$, but this is impossible by Lemma 2.6. Similarly, we can prove $s_1(B(s)) = s$, for any $s > s_0$. The proof is complete.

THEOREM 2.9. *Let* (A-1)–(A-3) *be satisfied. Given* $s_0 < s_1$ *there exists a positive constant* b^* *such that* (2.3), (2.4) *has at least two positive solutions for any* $b \in (0, b^*)$ *and none for* $b > b^*$.

Proof. Let $b^* = B(s_1)$. Then Theorem 2.8, Lemma 2.6, and Corollary 2.5 imply the result. The proof is complete.

By Theorem 2.9, we can obtain the following generalization of Theorem 5.1 of Bandle and Peletier [2].

COROLLARY 2.10. *If* (A-1)–(A-4) *are satisfied and given* $R_1 < R_0$, *then there exists a positive constant* b^* *such that* (1.1), (1.3a) *has at least two positive solutions for any* $b \in (0, b^*)$ *and none for* $b > b^*$.

Proof. By Theorem 2.9, there exists a positive constant $b^* > 0$ such that (1.1), (1.3a) has positive radial solutions for $b < b^*$ and none $b > b^*$.

To prove there is no positive solution for any $b \in (b^*, \infty)$, we shall use an argument similar to one in Theorem 5.1 of [2], i.e., we shall prove that if there exists a non-radially symmetric solution for (1.1), (1.3a), then there also exists a radially symmetric solution for (1.1), (1.3a). Let $u(x)$ be a solution of (1.1), (1.3a) and let $\bar{u}(r)$ be its spherical mean

$$\bar{u}(r) = \frac{1}{\text{meas } S(r)} \int_{S(r)} u(x)\, ds,$$

where $S(r) = \{x \in \mathbb{R}^n : |x| = r\}$. By (A-4), Jensen's inequality is applicable. Therefore, we have

$$0 = \Delta \bar{u} + \frac{1}{\text{meas } S(r)} \int_{S(r)} f(u(x))\, ds$$

$$\geqslant \Delta \bar{u} + f(\bar{u}) \qquad \text{in } \Omega.$$

Since $\bar{u}(R_1) = 0$ and $\bar{u}(R_0) = b$, $\bar{u}(r)$ is a supersolution of (1.1), (1.3a). On the other hand, $v \equiv 0$ is the subsolution of (1.1), (1.3a). By monotone iteration scheme, there exists a positive radial solution for (1.1), (1.3a), but this is impossible. The proof is complete.

With a slight modification of the previous argument, we can obtain a similar result for (1.1), (1.3b) as follows. The details are omitted.

THEOREM 2.11. *If* (A-1)–(A-3) *are satisfied, given* $s_1 > s_0$, *there exists a positive constant* b^* *such that* (2.3) *with boundary conditions*

$$u(s_0) = 0, \qquad u(s_1) = b,$$

has at least two positive solutions for any $b \in (0, b^*)$ *and none for* $b \in (b^*, \infty)$. *Moreover, if* f *satisfies* (A-1)–(A-4), *then for any* $R_0 > R_1$, *there exists* $b^* > 0$ *such that* (1.1), (1.3b) *has at least two positive solutions for any* $b \in (0, b^*)$ *and none for* $b \in (b^*, \infty)$.

Remark 2.12. If (A-1) and (A-2) are replaced by the condition

(A-1)′ $f \in C^2(\mathbb{R}^1)$ and $f(t) > 0$ for $t \geqslant 0$,

then similar results hold for (1.1), (1.3a) and (1.1), (1.3b). The details will be given elsewhere.

3. EXACTLY TWO SOLUTIONS ON ANNULUS

In this section, we shall give some sufficient conditions which imply that (1.1), (1.3a) has exactly two positive radial solutions for any $b \in (0, b^*)$.

THEOREM 3.1. *If* (A-1)–(A-5) *are satisfied, then* (1.1), (1.3a) *has exactly two positive radial solutions for any* $b \in (0, b^*)$, *where* b^* *is given in Theorem* 2.9.

Proof. As in Theorem 2.15 of [6], we let $w(t) = tu(r)$ and $t = r^{n-2}$. Then (2.1), (2.2) take the forms

$$w''(t) + \bar{\rho}(t) f(w/t) = 0, \qquad \alpha < t < \beta, \tag{3.1}$$

$$w(\alpha) = 0, \qquad w(\beta) = \beta b, \tag{3.2}$$

where $\alpha = R_1^{n-2}$, $\beta = R_0^{n-2}$, $\bar{\rho}(t) = (n-2)^{-2} t^{-k}$, $k = 1 - 2/(n-2)$.
Consider the following initial value problems

$$w''(t) + \bar{\rho}(t) f(w/t) = 0, \qquad t > \alpha, \tag{3.3}$$

$$w(\alpha) = 0, \qquad w'(\alpha) = d > 0. \tag{3.4}$$

Let $\varphi(t, d) = (\partial w / \partial d)(t, d)$ and $\psi(t, d) = (\partial \varphi / \partial d)(t, d)$. Then

$$\varphi'' + \bar{\rho}(t) f'(w/t) \frac{\varphi}{t} = 0, \tag{3.5}$$

$$\varphi(\alpha) = 0, \qquad \varphi'(\alpha) = 1, \tag{3.6}$$

$$\psi'' + \bar{\rho}(t) f'(w/t) \frac{\psi}{t} = -\bar{\rho}(t) f''(w/t) \frac{\varphi^2}{t^2}, \tag{3.7}$$

$$\psi(\alpha) = 0, \qquad \psi'(\alpha) = 0. \tag{3.8}$$

For each $d > 0$, define $z(d)$ to be the first $t > \alpha$ such that $w(t, d) = 0$ if such t exists. By Lemma 2.1, $z(d)$ exists for any $d > 0$. Since w is concave in $(\alpha, z(d))$, therefore there exists $\gamma(d) \in (\alpha, z(d))$ such that $w'(t, d) > 0$ for $t \in (\alpha, \gamma(d))$ and $w'(t, d) < 0$ for $t \in (\gamma(d), z(d))$. Let $y(d)$ be the first zero of $\varphi(t, d)$ in $(\alpha, z(d))$. By Theorem 2.15 of Ni and Nussbaum [6], $y(d)$ is unique in $(\alpha, z(d))$ and $\gamma(d) < y(d)$.

We claim that $y'(d) < 0$ for any $d > 0$. Multiply (3.5) by ψ, (3.7) by φ, subtract and integrate from α to y. We obtain

$$\int_\alpha^y (\psi \varphi'' - \psi'' \varphi) \, dt = \int_\alpha^y \bar{\rho}(t) f''(w/t) \frac{\varphi^3}{t^2} \, dt.$$

By (3.6), (3.8), and (A-5), $\psi(y) \varphi'(y) > 0$. Hence, $\psi(y) < 0$ and $y'(d) = -\psi(y, d)/\varphi'(y, d) < 0$ for any $d > 0$.

For any $b \in (0, b^*)$, there exists a unique $d^*(b)$ such that $\varphi(\beta b, d^*(b)) = 0$. Hence $\varphi(\beta b, d) > 0$ for $d < d^*$ and $\varphi(\beta b, d) < 0$ for $d > d^*$. The proof is complete.

THEOREM 3.2. *Problem* (1.1), (1.4) *has exactly two positive radial solutions for any* $b \in (0, b^*)$, *where* b^* *is given in Theorem* 2.9.

Proof. Since the proof is similar to the previous one, we only sketch it. As in Theorem 3.1 of [6], let $t = \log r$, $w(t) = r^{(p-1)/2} u(r)$. Then (2.1), (2.2) with $f(u) = u^p$ can be rewritten as

$$w''(t) + \mu w'(t) + \nu w(t) + w^p(t) = 0, \qquad \alpha < t < \beta,$$
$$w(\alpha) = 0, \qquad w(\beta) = b',$$

where $\alpha = \log R_1$, $\beta = \log R_0$, $\mu = n - 2 - 4/(p-2)$, $\nu = 2(n - 2 - 2/(p-2))/(p-1)$, $b' = bR_0^{(p-1)/2}$.

Consider the following initial value problems

$$w''(t) + \mu w'(t) + \nu w(t) + w^p(t) = 0, \qquad t > \alpha,$$
$$w(\alpha) = 0, \qquad w'(\alpha) = d.$$

Let $\varphi(t, d) = (\partial w/\partial d)(t, d)$, $\psi(t, d) = (\partial \varphi/\partial d)(t, d)$, and let $z(d)$, $\gamma(d)$, and $y(d)$ be defined as in the proof of Theorem 3.1. By Theorem 3.1 of Ni and Nussbaum [6], $y(d)$ is unique in $(\alpha, z(d))$ and $\gamma(d) < y(d)$. By an argument similar to that of the previous theorem, we have $y'(d) = -\varphi'(y(d), d)/\psi(y(d), d) < 0$ for any $d > 0$. The proof is complete.

Remark 3.3. For the existence of at least two positive radial solutions, problems (1.1), (1.3a) and (1.1), (1.3b) have the same result. But for the problem of "exactly two positive radial solutions," they are quite different. In fact, if $f(u) = u^p$, $1 < p < (n + 2)/(n - 2)$, we can prove that there exists $b > 0$ such that (1.1), (1.3b) has at least three positive radial solutions.

4. GENERAL DOMAINS

In this section, we shall generalize the result of Bandle and Peletier [2] on problem (1.2). Let Ω be a multiply connected domain in $\mathbb{R}^n$ such that $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$, where $\partial\Omega_0$ is connected and satisfies the following assumptions

(D-1)   There exists a constant ρ_0 such that $\{x \in \mathbb{R}^n : |x| < \rho_0\} \subseteq \Omega_0$.

(D-2)   There exist constants $0 < \rho_1 < \rho_2$ and $x_0 \in \mathbb{R}^n$ such that $\{x \in \mathbb{R}^n : \rho_1 < |x - x_0| < \rho_2\} \subseteq \Omega$.

THEOREM 4.1. *Let assumptions* (A-1)–(A-4) *be satisfied and let Ω be a domain for which assumptions* (D-1) *and* (D-2) *hold. Then there exists a positive constant b^* such that problem* (1.1) *with the boundary conditions*

$$u = b \quad \text{on } \partial\Omega_0 \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega_1, \tag{4.1}$$

has at least one positive solution for all $b < b^*$ and none for $b > b^*$.

Proof. Since Ω is bounded, there exist R such that $\Omega \subset \{x \in \mathbb{R}^n : |x| < R\}$. By Lin [4], the problem

$$\Delta u + f(u) = 0, \qquad \rho_0 < |x| < R,$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = \rho_0 \qquad \text{and} \qquad u = 0 \quad \text{on} \quad |x| = R,$$

has a positive radial solution. Let it be $\bar{u}$, and $\bar{\rho}_0 = \sup\{x : x \in \partial\Omega_0\}$. Choose $b \leqslant \bar{u}(\bar{\rho}_0)$. Then $u(x) \leqslant \bar{u}(\bar{\rho}_0) \leqslant \bar{u}(x)$ for $x \in \partial\Omega_0$ and $\bar{u}$ is a supersolution of problem (1.1), (4.1). By the monotone iteration scheme (see, e.g., [7]), (1.1), (4.1) has a solution for $b \leqslant \bar{u}(\bar{\rho}_0)$. Hence $b^* \geqslant \bar{u}(\bar{\rho}_0)$.

Next, we claim that $b^* < \infty$. Consider the function $h \in C^2(\bar{\Omega})$ which satisfies

$$\Delta h = 0 \qquad \text{in } \Omega,$$

$$h = 1 \quad \text{on } \partial\Omega_0 \qquad \text{and} \qquad h = 0 \quad \text{on } \partial\Omega_1.$$

Let u be the solution of (1.1), (4.1). Then

$$u(x) \geqslant b h(x) \qquad \text{on } \bar{\Omega}.$$

For simplicity, let $x_0 = 0$ in assumption (D-2). By the strong maximum principle $h(x) > 0$ for all $x \in \Omega$; hence $\delta = \min\{h(x) : |x| = \rho_1\} > 0$ and $u(x) \geqslant b\delta$ on $|x| = \rho_1$. Let $b' = b\delta$. Consider the following problem

$$\Delta v + f(v) = 0, \qquad \rho_1 < |x| < \rho_2, \tag{4.2}$$

$$v = b' \quad \text{on} \quad |x| = \rho_1 \qquad \text{and} \qquad v = 0 \quad \text{on} \quad |x| = \rho_2. \tag{4.3}$$

If $b^* = \infty$, then for any $b' = b\delta$, there is a positive solution of (1.1), (4.1), which is also a supersolution of (4.2), (4.3). Hence, by te monotone iteration scheme, (4.2), (4.3) has a solution. This is a contradiction to Corollary 2.10; hence we must conclude that b is bounded above. The proof is complete.

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