Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Chaos, Solitons and Fractals 21 (2004) 101–111
CHAOS
SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Synchronization of unidirectional coupled chaotic systems via partial stability

Zheng-Ming Ge *, Yen-Sheng Chen

Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan, ROC

Accepted 22 September 2003

Abstract

Chaos synchronization can be achieved by several methods but there is no easy unified criterion in general. In this paper, a general scheme is proposed to achieve chaos synchronization via stability with respect to partial variables. Three theorems for synchronization of unidirectional coupled non-autonomous (also autonomous) systems by linear feedback are developed for systems with and without system structure perturbations. The system, fly-ball governor, is demonstrated as an example.

1. Introduction

Chaotic systems exhibit sensitive dependence on initial conditions. Because of this property, chaotic systems are difficult to be synchronized or controlled. From the earlier works [1–3] (especially [3]), the researchers have realized that synchronization of chaotic motions are possible, synchronization of chaos was of great interest in these years [4–16]. In particular, it was pointed out that chaos synchronization has the potential in secure communication. Many engineers and scientists were attracted to this discipline [17–25].

Two kinds of chaos synchronization are discussed the most often. (1) Duplication: the first method introduced by Pecora & Carroll [1] consists of a driving system and a response system. The former one evolves chaotic orbits and the latter is identical to the driving system except some partial states replaced by that of the driving one. (2) Coupling: the second kind consists of two identical chaotic systems except coupling term. Coupled systems can be unidirectional or mutual. Under certain conditions (appropriate coupling parameters and/or system parameters with enough evolution time) the response system will behave the same orbit with the driving system.

A more general case called generalized synchronization (GS) was studied in [48–53], this means that there is a functional relation between state variables of driving and response systems. This function need not be defined on the whole phase space but on the attractor only. Three methods were proposed to detect GS in [48–50] respectively while another method measuring the smooth degree of this function in [52].

Synchronization means that the state variables of response system approach eventually to the ones of driving system. There are many control methods to synchronize chaotic systems such as observer-based design methods [26–29], adaptive control [30–38] and other control methods [39–47]. Zero crossing of Lyapunov exponent was used as a criterion of chaos synchronization widely. There is a drawback that we can only calculate finite evolution time in computer simulation but infinite evolution time is needed by definition of Lyapunov exponent. On the other hand, it is difficult to use Lyapunov direct method since the state error equation is not a pure function of state error in general. In this paper, we propose a general scheme to achieve chaos synchronization via partial stability due to Rumjantsev [55]. The upper

* Corresponding author. Tel.: +866-35712121; fax: +886-35720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2003.10.004

obstacles will be overcome by our method and it serves as a criterion for chaos synchronization by control methods. Criterions of unidirectional coupled nonautonomous systems by linear feedback are developed for systems with and without system structure perturbation. The system, fly-ball governor, is demonstrated as an example.

2. Analysis

Consider the following unidirectional coupled nonautonomous systems

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1)\\ \dot{\mathbf{x}}_2 &= \mathbf{f}(t, \mathbf{x}_2) + \mathbf{g}(t, \mathbf{x}_2, \mathbf{x}_1)\end{aligned} \tag{1}$$

where $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbf{R}^n$ and $\mathbf{f} : \Omega_1 \subset \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{g} : \Omega_2 \subset \mathbf{R} \times \mathbf{R}^{2n} \to \mathbf{R}^n$ satisfy Lipschitz condition. Ω_1, Ω_2 are domains containing the origin. Assume that the solutions of Eq. (1) have a priori bound then they must exist for infinite time. That is, for given $(t_0, \mathbf{x}_{10}, \mathbf{x}_{20}) \in \Omega_1 \cap \Omega_2$ the solutions $\mathbf{x}_1(t, t_0, \mathbf{x}_{10}, \mathbf{x}_{20})$, $\mathbf{x}_2(t, t_0, \mathbf{x}_{10}, \mathbf{x}_{20})$ of Eq. (1) exist for $t \geqslant t_0$. At the first, we recall the definition of identical synchronization (complete synchronization).

Definition. The system (1) is identical synchronized if there is an invariant manifold $S \subset \mathbf{R} \times \mathbf{R}^{2n}$ s.t. $\lim_{t\to\infty} \|\mathbf{x}_1(t, t_0, \mathbf{x}_{10}, \mathbf{x}_{20}) - \mathbf{x}_2(t, t_0, \mathbf{x}_{10}, \mathbf{x}_{20})\| = 0$ with $(t_0, \mathbf{x}_{10}, \mathbf{x}_{20}) \in \Omega_1 \cap \Omega_2$.

In Eq. (1) $\mathbf{g}$ is the coupling function. Assume that $\mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_1) = \mathbf{0}$, i.e. the synchronized sub-manifold of Eq. (1) agrees with the original uncoupled one while synchronization occurs. In order to discuss the transversal stability of synchronization manifold, define $\mathbf{e} = \mathbf{x}_2 - \mathbf{x}_1$ to be the state error. Then the error equations can be written as

$$\dot{\mathbf{e}} = \mathbf{f}(t, \mathbf{e} + \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_1) + \mathbf{g}(t, \mathbf{e} + \mathbf{x}_1, \mathbf{x}_1) \tag{2}$$

Notice that the right hand side of Eq. (2) is not a pure function of t and error $\mathbf{e}$, as a result that the Lyapunov direct method might hardly be used. On the other hand, the variational equation or Lyapunov exponents can be used to clarify transversal stability. Josić [54] analyzed that synchronization manifolds will persist under perturbation if such manifolds possess a property of k-hyperbolicity. Herein, we add the upper half (lower half also works) of Eq. (1) with $\mathbf{x}_2$ replaced by $\mathbf{x}_2 = \mathbf{e} + \mathbf{x}_1$ to Eq. (2), then extended equations are obtained as following

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1)\\ \dot{\mathbf{e}} &= \mathbf{f}(t, \mathbf{e} + \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_1) + \mathbf{g}(t, \mathbf{e} + \mathbf{x}_1, \mathbf{x}_1)\end{aligned} \tag{3}$$

If the partial variables $\mathbf{e}$ in Eq. (3) are asymptotically stable about $\mathbf{e} = \mathbf{0}$, the synchronization manifold is stable in transversal directions. This can be done via stability with respect to partial variables. The theory of partial stability can be found in Appendix A.

In the following, three theorems will be derived for a special form of Eq. (1). The first theorem is suitable for the case without system structure perturbation and the other two are the cases for systems under structure perturbations. These theorems will be applied to an example, the fly-ball governor, in the next section. Consider unidirectional coupled nonautonomous systems as

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1)\\ \dot{\mathbf{x}}_2 &= \mathbf{f}(t, \mathbf{x}_2) + \Gamma(\mathbf{x}_1 - \mathbf{x}_2)\end{aligned} \tag{4}$$

where $\mathbf{f}$ satisfies Lipschitz condition with Lipschitz constant L and $\Gamma \in M_{n\times n}$ is a constant matrix whose entries represent the coupling strength of the linear feedback term $(\mathbf{x}_1 - \mathbf{x}_2)$. Define $\mathbf{e} = \mathbf{x}_2 - \mathbf{x}_1$, an extended equation can be obtained as

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1)\\ \dot{\mathbf{e}} &= \mathbf{f}(t, \mathbf{e} + \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_1) - \Gamma\mathbf{e}.\end{aligned} \tag{5}$$

Theorem 1. *The partial state* $\mathbf{e}$ *asymptotically approaches to* $\mathbf{0}$ *in Eq.* (5) *if* $L\sqrt{n}\mathbf{I}_n - \Gamma$ *is negative definite, i.e. the systems in the form of Eq.* (4) *are synchronized if* $L\sqrt{n}\mathbf{I}_n - \Gamma$ *is negative definite.*

Proof. Choose a function $V(\mathbf{x}_1, \mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$ positive definite with respect to $\mathbf{e}$ and with infinitesimal upper bound, then

$$\dot{V} = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} = \sum_{i=1}^{n} e_i[f_i(t, \mathbf{x}_1 + \mathbf{e}) - f_i(t, \mathbf{x}_1)] - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} \leqslant L\|\mathbf{e}\| \sum_{i=1}^{n} |e_i| - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} \leqslant L\sqrt{n}\|\mathbf{e}\|^2 - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} = \mathbf{e}^{\mathrm{T}}(L\sqrt{n}\mathbf{I}_n - \Gamma)\mathbf{e}$$

The state error $\mathbf{e}$ approaches $\mathbf{0}$ asymptotically if $L\sqrt{n}\mathbf{I}_n - \Gamma$ is negative definite by Theorem A.2 in Appendix A. In upper deviations, property of norm equivalent on finite dimensional vector space and Lipschitz condition were used. □

Remark 1. From the matrix theory, $L\sqrt{n}\mathbf{I}_n - \Gamma$ is negative definite if and only if all its eigenvalues are negative. For the case $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\gamma_i > 0$ for $i = 1, \ldots, n$, synchronization occurs if $\gamma_{\min} > L\sqrt{n}$, $\gamma_{\min} \leqslant \gamma_i$, $i = 1, \ldots, n$. This is because the time derivative of $V(\mathbf{x}, \mathbf{e})$ can be written as $\dot{V}(\mathbf{x}, \mathbf{e}) \leqslant (L\sqrt{n} - \gamma_{\min})n\|\mathbf{e}\|^2$. Moreover, the result is global by Theorem A.4 if $\mathbf{f}$ is globally Lipschitz.

Consider unidirectional coupled nonautonomous systems under system perturbation as

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1) \\ \dot{\mathbf{x}}_2 &= \mathbf{f}(t, \mathbf{x}_2) + \Delta\mathbf{f}(t, \mathbf{x}_1, \mathbf{e}) + \Gamma(\mathbf{x}_1 - \mathbf{x}_2) \end{aligned} \tag{6}$$

where $\Gamma \in M_{n\times n}$ is a constant matrix whose entries represent the coupling strength of the linear feedback term $(\mathbf{x}_1 - \mathbf{x}_2)$ and $\Delta\mathbf{f}(t, \mathbf{x}_1, \mathbf{e})$ is the system perturbation with $\Delta\mathbf{f}(t, \mathbf{x}_1, \mathbf{0}) = \mathbf{0}$. Define $\mathbf{e} = \mathbf{x}_2 - \mathbf{x}_1$, an extended equation can be obtained as

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}(t, \mathbf{x}_1) \\ \dot{\mathbf{e}} &= \mathbf{f}(t, \mathbf{e} + \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_1) + \Delta\mathbf{f}(t, \mathbf{x}_1, \mathbf{e}) - \Gamma\mathbf{e} \end{aligned} \tag{7}$$

Theorem 2. *Assume that* $\exists K_i > 0$ *such that* $|\Delta f_i| < K_i$, $i = 1, \ldots, n$, *i.e.* $\exists K > 0 \Rightarrow \|\Delta\mathbf{f}\| < K$. Γ *is a diagonal matrix such that* $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ *with* $\gamma_i > 0$ *for* $i = 1, \ldots, n$. *Then the Eq.* (7) *is asymptotically* $\mathbf{e}$*-stable if* $\gamma_{\min} > (L + K)\sqrt{n}$ *with* $\gamma_{\min} \leqslant \gamma_i$ *for* $i = 1, \ldots, n$, *i.e. the systems in the form of Eq.* (6) *are synchronized if* $\gamma_{\min} > (L + K)\sqrt{n}$.

Proof. Choose a function $V(\mathbf{x}_1, \mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$ positive definite with respect to $\mathbf{e}$ and with infinitesimal upper bound, then

$$\begin{aligned} \dot{V} &= \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} \\ &= \sum_{i=1}^{n} e_i[f_i(t, \mathbf{x}_1 + \mathbf{e}) - f_i(t, \mathbf{x}_1) + \Delta f_i] - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} \leqslant L\|\mathbf{e}\|\sum_{i=1}^{n}|e_i| + K\sum_{i=1}^{n}|e_i| - \gamma_{\min}\|\mathbf{e}\|^2 \leqslant (L\sqrt{n} - \gamma_{\min})\|\mathbf{e}\|^2 + K\sqrt{n}\|\mathbf{e}\| \end{aligned}$$

There are three cases to discuss. The first case: $\dot{V}(\mathbf{x}_1, \mathbf{e}) = 0$ for $\|\mathbf{e}\| = 0$; the second case: $\dot{V}(\mathbf{x}_1, \mathbf{e}) < [(L + K)\sqrt{n} - \gamma_{\min}]\|\mathbf{e}\|^2$ for $\|\mathbf{e}\| > 1$; the third case: $\dot{V}(\mathbf{x}_1, \mathbf{e}) < (L + K)\sqrt{n} - \gamma_{\min}$ for $\|\mathbf{e}\| \leqslant 1$. Hence, $\dot{V}(\mathbf{x}_1, \mathbf{e}) < 0$ if $(L + K)\sqrt{n} - \gamma_{\min} < 0$. □

Remark 2. This result is global by Theorem A.4 if $\mathbf{f}$ is globally Lipschitz.

Theorem 3. *Assume that* $\exists K > 0 \Rightarrow \|\Delta\mathbf{f}\| < K\|\mathbf{e}\|$. *Then the Eq.* (7) *is asymptotically* $\mathbf{e}$*-stable if* $(L + K)\sqrt{n}\mathbf{I}_n - \Gamma$ *is negative definite, i.e. the systems in the form of Eq.* (6) *are synchronized if* $(L + K)\sqrt{n}\mathbf{I}_n - \Gamma$ *is negative definite.*

Proof. Choose a function $V(\mathbf{x}_1, \mathbf{e}) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}\mathbf{e}$ positive definite with respect to $\mathbf{e}$, then

$$\dot{V} = \mathbf{e}^{\mathrm{T}}\dot{\mathbf{e}} = \sum_{i=1}^{n} e_i[f_i(t, \mathbf{x}_1 + \mathbf{e}) - f_i(t, \mathbf{x}_1) + \Delta f_i] - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} \leqslant (L + K)\|\mathbf{e}\|\sum_{i=1}^{n}|e_i| - \mathbf{e}^{\mathrm{T}}\Gamma\mathbf{e} \leqslant \mathbf{e}^{\mathrm{T}}[(L + K)\sqrt{n}\mathbf{I}_n - \Gamma]\mathbf{e}$$

Hence, the Eq. (7) is asymptotically $\mathbf{e}$-stable if $(L + K)\sqrt{n}\mathbf{I}_n - \Gamma$ is negative definite. □

Remark 3. $(L + K)\sqrt{n}\mathbf{I}_n - \Gamma$ is negative definite if and only if all its eigenvalues are negative. When $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\gamma_i > 0$ for $i = 1, \ldots, n$, synchronization occurs if $\gamma_{\min} > (L + K)\sqrt{n}$, where $\gamma_{\min}$ is the minimum one in γ_i. Furthermore, this result is global by Theorem A.4 if $\mathbf{f}$ is globally Lipschitz.

3. Examples

A system, fly-ball governor with and without system structure perturbation, is demonstrated as an example in this section. The system equation is as following

$$\dot{x} = y$$
$$\dot{y} = rz^2 \sin x \cos x - \sin x - Cy$$
$$\dot{z} = k \cos x - F$$

where $r = 0.25$, $C = 0.7$, $F = 1.942$ and $k = 5.13$ ensure that there exists chaotic behavior. The chaotic attractor is shown in Fig. 1.

3.1. Unidirectional coupled fly-ball governors without perturbation

Consider the following unidirectional coupled systems without system perturbation as in the form of Eq. (5)

$$\dot{x}_1 = y_1$$
$$\dot{y}_1 = rz_1^2 \sin x_1 \cos x_1 - \sin x_1 - Cy_1$$
$$\dot{z}_1 = k \cos x_1 - F$$
$$\dot{x}_2 = y_2 + \gamma(x_1 - x_2)$$
$$\dot{y}_2 = rz_2^2 \sin x_2 \cos x_2 - \sin x_2 - Cy_2 + \gamma(y_1 - y_2)$$
$$\dot{z}_2 = k \cos x_2 - F + \gamma(z_1 - z_2)$$

where $\gamma = 1$. The initial value $\mathbf{x_0} = (1, 1, 1, 3, 3, 3)^{\mathrm{T}}$ is adopted in all simulated results. In Fig. 2, three state errors versus time are shown and the state errors approach zero as time evolves. Fig. 3 shows that synchronization sub-manifolds represent diagonal-like since $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$ as $t \to \infty$. The three Lyapunov exponents versus coupling strength γ are shown in Fig. 4. There is a zero crossing when $\gamma \approx 0.155$. This γ is a threshold value which synchronization occurs.

3.2. Unidirectional coupled fly-ball governors with perturbation $\|\Delta \boldsymbol{f}\| < K$

Consider the following unidirectional coupled systems with system perturbation as in the form of Eq. (6)

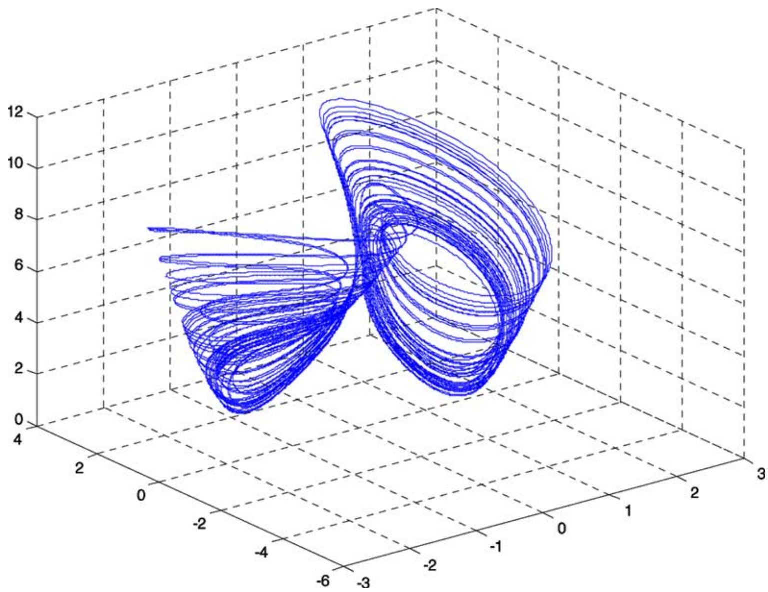

Fig. 1. Chaotic attractor of fly-ball governor.

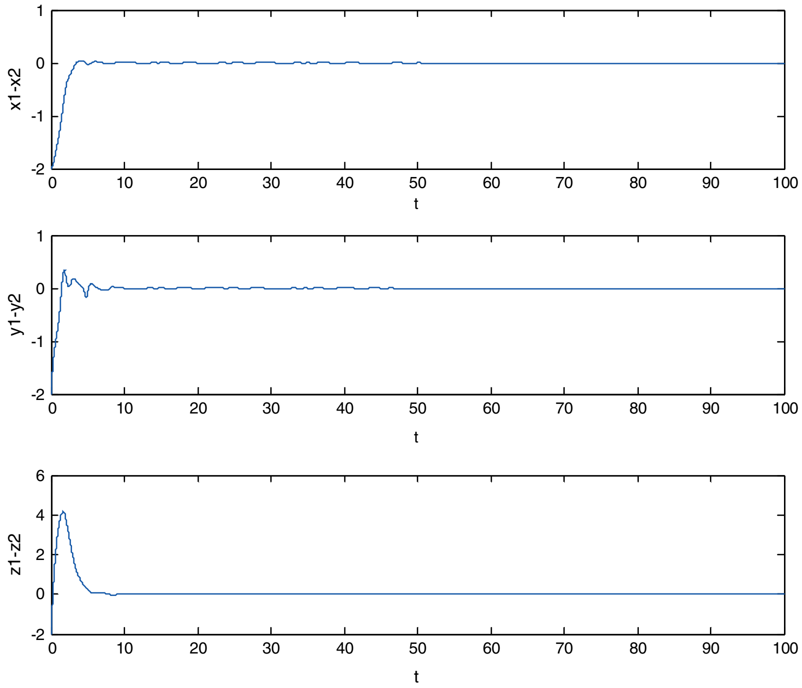

Fig. 2. State errors versus time of unidirectional coupled fly-ball governor without system perturbation.

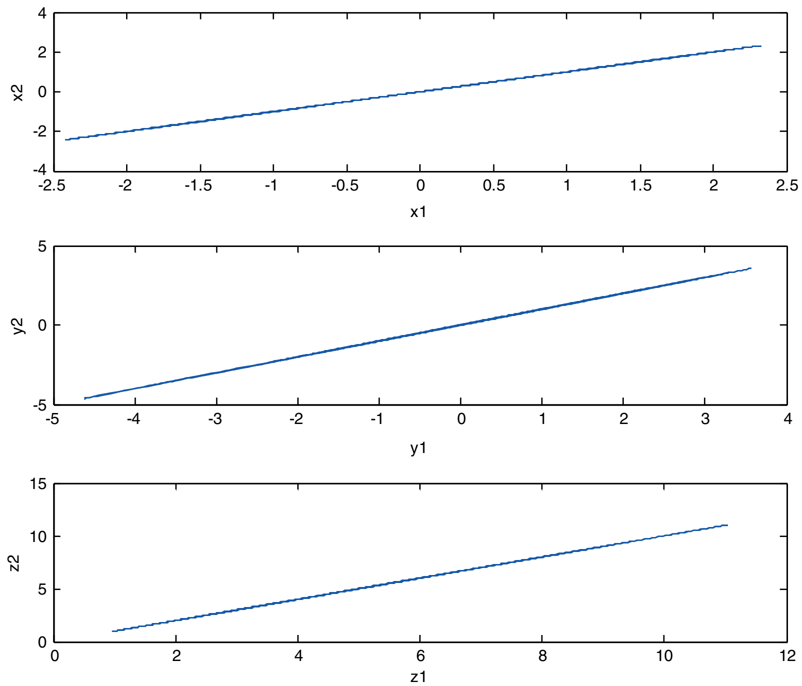

Fig. 3. Synchronization sub-manifold of unidirectional coupled fly-ball governor without system perturbation.

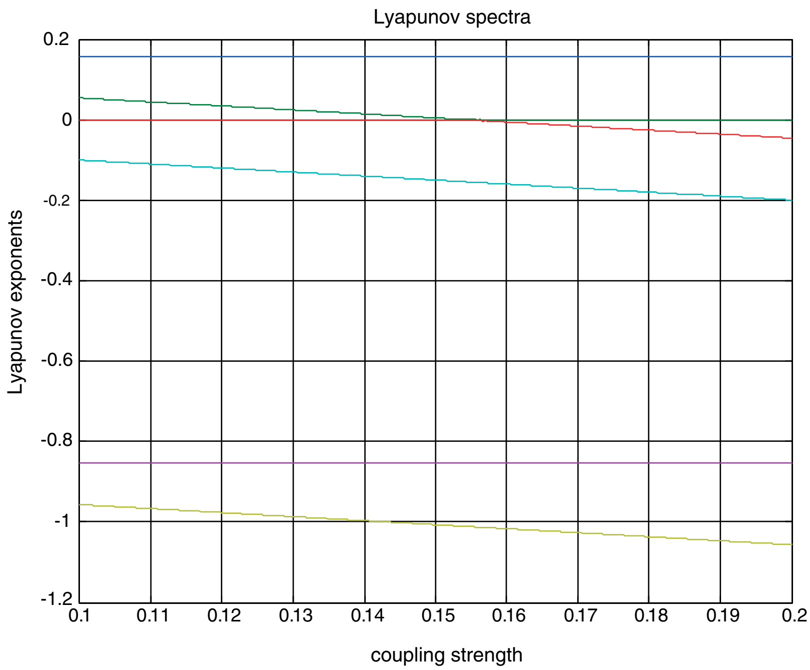

Fig. 4. Lyapunov spectra of unidirectional coupled fly-ball governor without system perturbation.

$$\begin{aligned}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= rz_1^2 \sin x_1 \cos x_1 - \sin x_1 - Cy_1 \\
\dot{z}_1 &= k \cos x_1 - F \\
\dot{x}_2 &= y_2 + \sin(tz_2(y_1 - y_2)) + \gamma(x_1 - x_2) \\
\dot{y}_2 &= rz_2^2 \sin x_2 \cos x_2 - \sin x_2 - Cy_2 + \gamma(y_1 - y_2) \\
\dot{z}_2 &= k \cos x_2 - F + \gamma(z_1 - z_2)
\end{aligned}$$

The first error dynamics is $\dot{e}_1 = e_2 - \gamma e_1 + \sin(tz_2(y_1 - y_2))$, then the system perturbation is $|\Delta f_1| = |\sin(tz_2(y_1 - y_2))| \leqslant 1$. For $\gamma = 7.3$, the state errors approach zero as time goes to infinite as shown in Fig. 5. Synchronization sub-manifolds are shown in Fig. 6. They represent diagonal-like since the state errors are asymptotically stable.

3.3. Unidirectional coupled fly-ball governors with perturbation $\|\Delta \mathbf{f}\| < K\|\mathbf{e}\|$

Consider the following unidirectional coupled systems with system perturbation as in the form of Eq. (7)

$$\begin{aligned}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= rz_1^2 \sin x_1 \cos x_1 - \sin x_1 - Cy_1 \\
\dot{z}_1 &= k \cos x_1 - F \\
\dot{x}_2 &= y_2 + 100(z_2 - z_1) + \gamma(x_1 - x_2) \\
\dot{y}_2 &= rz_2^2 \sin x_2 \cos x_2 - \sin x_2 - Cy_2 + \gamma(y_1 - y_2) \\
\dot{z}_2 &= k \cos x_2 - F + \gamma(z_1 - z_2)
\end{aligned}$$

where $\gamma = 5.1$. The system perturbation is $|\Delta f_1| = |100(z_2 - z_1)| \leqslant 100\|\mathbf{e}\|$. In Fig. 7, the state errors approach zero as time goes to infinite. Synchronization sub-manifolds are shown in Fig. 8. They represent diagonal-like since the state errors are asymptotically stable.

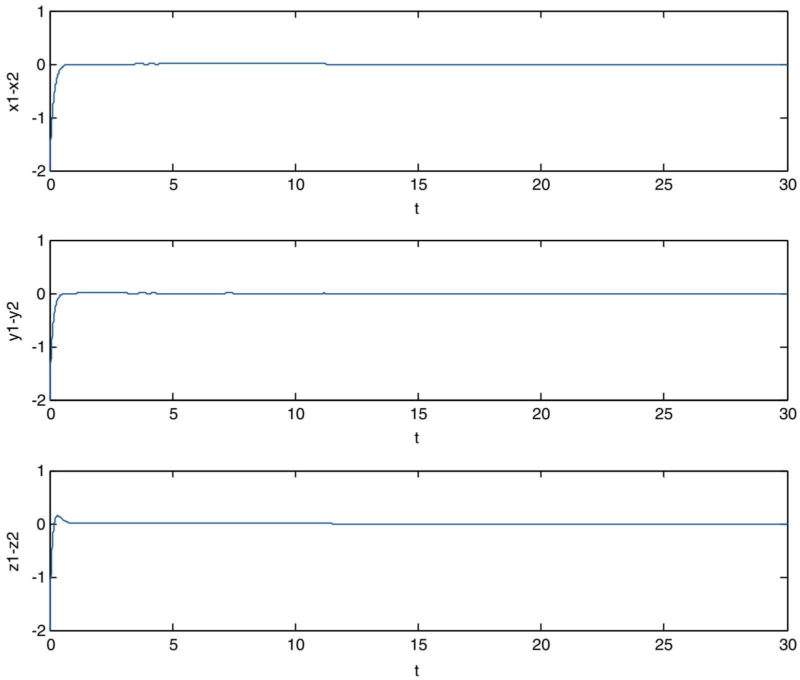

Fig. 5. State errors versus time of unidirectional coupled fly-ball governor with system perturbation $|\Delta f_1| \leqslant 1$ for $\gamma = 7.3$.

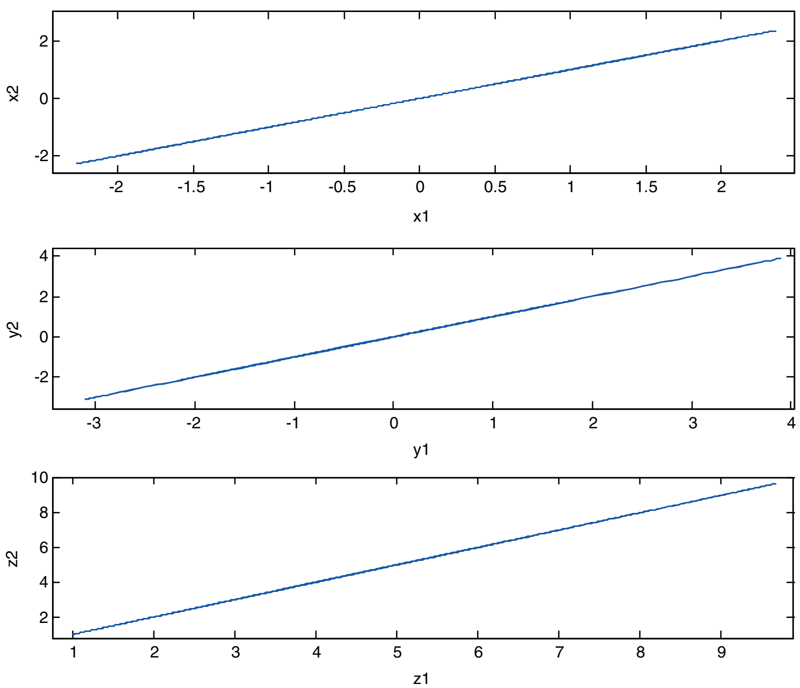

Fig. 6. Synchronization sub-manifold of unidirectional coupled fly-ball governor with system perturbation $|\Delta f_1| \leqslant 1$ for $\gamma = 7.3$.

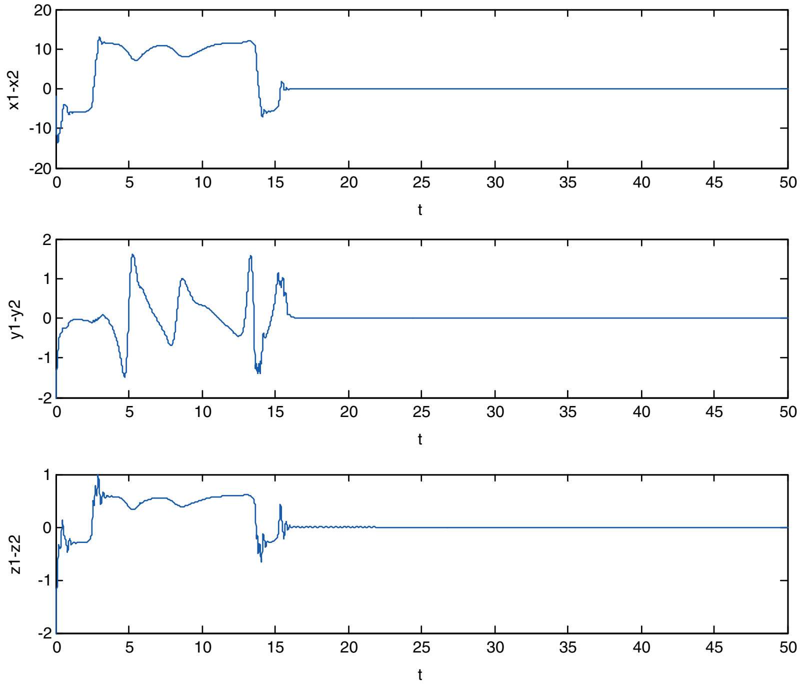

Fig. 7. State errors versus time of unidirectional coupled fly-ball governor with system perturbation $|\Delta f_1| \leqslant 100\|\mathbf{e}\|$ for $\gamma = 5.1$.

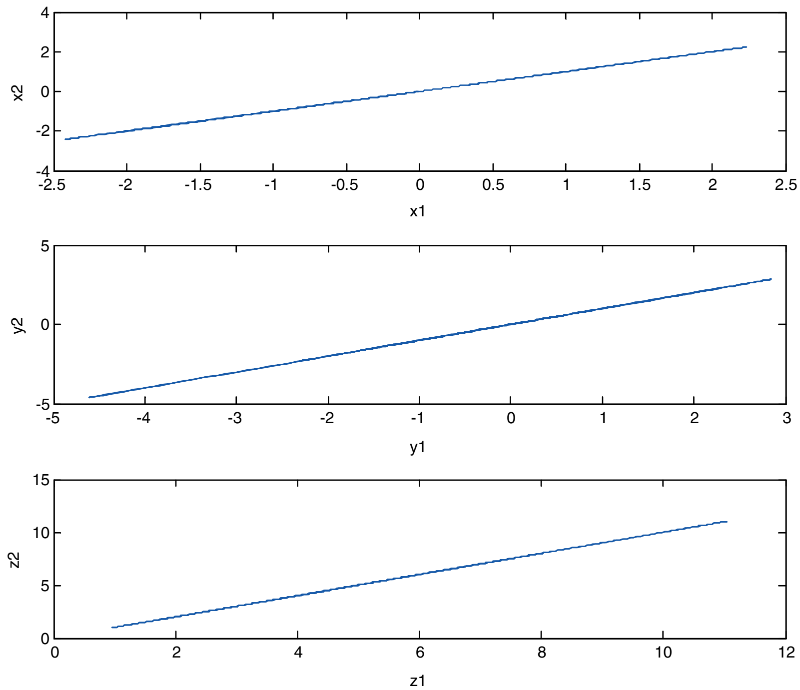

Fig. 8. Synchronization sub-manifold of unidirectional coupled fly-ball governor with system perturbation $|\Delta f_1| \leqslant 100\|\mathbf{e}\|$ for $\gamma = 5.1$.

4. Conclusions

There are many methods to ensure chaos synchronization such as zero crossing of Lyapunov spectra, Lyapunov direct method and control methods. The realization of Lyapunov exponent needs numerical calculation for infinite evolution time, therefore this method is not complete in practice. On the other hand, it is difficult to use Lyapunov direct method since the state error equation is not a pure function of time and state error in general. Control methods might be appropriate to some kinds of systems. In this paper, a general scheme to achieve chaos synchronization via partial stability was proposed. The upper drawbacks can be overcome by this method. Three theorems were proven to ensure chaos synchronization for a general kind of unidirectional coupled nonautonomous (also autonomous) systems by linear feedback coupling term. The first theorem is for the case without system perturbation and the other two theorems are for the case under perturbations. The fly-ball governor was illustrated as an example to show these results.

Acknowledgement

This research was supported by the National Science Council, Republic of China, under grant number NSC 91-2212-E-009-025.

Appendix A

The content of this appendix follows [55–57]. Consider a differential system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{A.1}$$

where $\mathbf{f} : [t_0, \infty) \times \Omega \to \mathbf{R}^n$, $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}\ \forall t \in [t_0, \infty)$ and $\Omega \subset \mathbf{R}^n$ is a region containing the origin. Assume that $\mathbf{f}$ is smooth enough to ensure that the solution of (A.1) exists uniquely. To shorten the notation, write $\mathbf{x} = (y_1, \dots, y_m, z_1, \dots, z_{n-m})^{\mathrm{T}}$, $\|\mathbf{y}\| = \left(\sum_{i=1}^{m} y_i^2\right)^{1/2}$, $\|\mathbf{z}\| = \left(\sum_{i=1}^{n-m} z_i^2\right)^{1/2}$ and $\|\mathbf{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = (\|\mathbf{y}\|^2 + \|\mathbf{z}\|^2)^{1/2}$ with $0 < m \leqslant n$. We assume that the solution of (A.1) is $\mathbf{z}$-extendable, i.e. any solution of (A.1) exists for all $t \geqslant t_0$ and $\|\mathbf{y}(t)\| \leqslant H$, H is a constant. Write $Q = \{(t, \mathbf{x}) | t \geqslant t_0, \|\mathbf{y}\| H, 0 \leqslant \|\mathbf{z}\| < +\infty\}$ and $\tilde{Q} = \{(t, \mathbf{x}) | t \geqslant t_0, \|\mathbf{x}\| < \infty\}$.

Definition A.1. The solution of (A.1) is stable with respect to $\mathbf{y}$ ($\mathbf{y}$-stable) if $\forall \varepsilon > 0$, $\forall t_0 \in [0, \infty)$, $\exists \delta(t_0, \varepsilon) > 0$, $\forall \mathbf{x}_0 \in B_\delta := \{\mathbf{x} | \|\mathbf{x}\| < \delta(t_0, \varepsilon)\}$ such that $\|\mathbf{y}(t, t_0, \mathbf{x}_0)\| < \varepsilon\ \forall t \geqslant t_0$. The solution of (A.1) is uniformly $\mathbf{y}$-stable if $\delta(t_0, \varepsilon)$ is independent of t_0 in the definition of $\mathbf{y}$-stable.

The solution of (A.1) is asymptotically stable with respect to $\mathbf{y}$ (asymptotically $\mathbf{y}$-stable) if it is (1) $\mathbf{y}$-stable and (2) $\mathbf{y}$-attractive, i.e. $\forall t_0 \in [0, \infty)$, $\exists \delta'(t_0) > 0$, $\forall \varepsilon' > 0$, $\forall \mathbf{x}_0 \in B_{\delta'} := \{\mathbf{x} | \|\mathbf{x}\| < \delta'(t_0)\}$, $\exists T(t_0, \mathbf{x}_0, \varepsilon')$ such that $\|\mathbf{y}(t, t_0, \mathbf{x}_0)\| < \varepsilon'$ $\forall t \geqslant t_0 + T$. The solution of (A.1) is uniformly asymptotically $\mathbf{y}$-stable if it is (1) uniformly $\mathbf{y}$-stable and (2) uniformly $\mathbf{y}$-attractive, i.e. $\delta'(t_0)$ is independent of t_0 and $T(t_0, \mathbf{x}_0, \varepsilon')$ is independent of $t_0, \mathbf{x}_0$ in the definition of $\mathbf{y}$-attractive.

The solution of (A.1) is globally $\mathbf{y}$-attractive if $B_\delta = \mathbf{R}^n$ in the definition of $\mathbf{y}$-attractive. Furthermore, if $B_\delta = \mathbf{R}^n$ and $\exists \delta'(t_0) > 0$ can be replaced by $\forall \delta'$ the solution of (A.1) is globally uniformly $\mathbf{y}$-attractive. The solution of (A.1) is globally asymptotically $\mathbf{y}$-stable if it is (1) $\mathbf{y}$-stable and (2) globally $\mathbf{y}$-attractive. The solution of (A.1) is globally uniformly asymptotically $\mathbf{y}$-stable if it is (1) uniformly $\mathbf{y}$-stable and (2) globally uniformly $\mathbf{y}$-attractive.

The next definition extends the notation of definite functions with respect to partial variables. Let $V(t, \mathbf{x}) \in C([t_0, \infty) \times \mathbf{R}^n, \mathbf{R})$ with $V(t, \mathbf{0}) = \mathbf{0}$ and V defined on Q.

Definition A.2. A t implicit positive (negative) semi-definite function $V(\mathbf{x})$ is called positive (negative) definite with respect to $\mathbf{y}$ if $V(\mathbf{x})$ can vanish only when $\mathbf{y} = \mathbf{0}$.

A positive (negative) semi-definite function $V(t, \mathbf{x})$ is called positive (negative) definite with respect to $\mathbf{y}$ if there is a positive (negative) definite function $W(\mathbf{y})$ such that $V(t, \mathbf{x}) \geqslant W(\mathbf{y})$ $(V(t, \mathbf{x}) \leqslant W(\mathbf{y}))$.

Definition A.3. A function $V(t, \mathbf{x})$ is called bounded if $\exists M > 0$ such that $|V(t, \mathbf{x})| \leqslant M$. A bounded function $V(t, \mathbf{x})$ possesses an infinitesimal upper bound if $\forall \tilde{\varepsilon} > 0$, $\exists \tilde{\delta}(\tilde{\varepsilon}) > 0$, for $t \geqslant t_0$ and $\|\mathbf{x}\| < \tilde{\delta}(\tilde{\varepsilon})$ such that $|V(t, \mathbf{x})| \leqslant \tilde{\varepsilon}$. A bounded function $V(t, \mathbf{x})$ possesses an infinitesimal upper bound with respect to $x_1, \dots, x_k (m \leqslant k \leqslant n)$ if $\forall \tilde{\varepsilon} > 0$, $\exists > \tilde{\delta}(\tilde{\varepsilon}) > 0$, for $t \geqslant t_0$, $\sum_{i=1}^{k} x_i^2 < \tilde{\delta}^2$, $-\infty < x_{k+1}, \dots, x_n < \infty$ such that $|V(t, \mathbf{x})| \leqslant \tilde{\varepsilon}$.

The following four theorems still hold when the undisturbed motion has nonzero $\mathbf{z}$.

Theorem A.1. *Suppose there exists a positive definite function* $V(t,\mathbf{x})$ *with respect to* $x_1,\ldots,x_k (k \leqslant n)$ *such that* $\dot{V}(t,\mathbf{x})$ *is negative semi-definite or vanishes, then the undisturbed motion is stable with respect to* $x_1,\ldots,x_k (k \leqslant n)$.

Theorem A.2. *Suppose there exists a positive definite function* $V(t,\mathbf{x})$ *with respect to* $x_1,\ldots,x_k (k \leqslant n)$ *such that* $V(t,\mathbf{x})$ *possesses an infinitesimal upper bound and* $\dot{V}(t,\mathbf{x})$ *is negative definite with respect to* $x_1,\ldots,x_k$, *then the undisturbed motion is asymptotically stable with respect to* $x_1,\ldots,x_k$.

Theorem A.3. *Suppose there exist a function* $V : [0,\infty) \times \Omega \to \mathbf{R}$ *such that for some functions* $a,b,c \in \mathscr{K}$ *and every* $(t,\mathbf{x}) \in Q$:

$$\text{(i)} \quad a(\|\mathbf{y}\|) \leqslant V(t,\mathbf{x}), V(t,\mathbf{0}) = \mathbf{0},$$

$$\text{(ii)} \quad V(t,\mathbf{x}) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right), \quad m \leqslant k \leqslant n,$$

$$\text{(iii)} \quad \dot{V}(t,\mathbf{x}) \leqslant -c\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right),$$

then the origin is uniformly asymptotically $\mathbf{y}$*-stable*.

Theorem A.4. *Suppose there exist a function* $V : [0,\infty) \times \Omega \to \mathbf{R}$ *such that for some functions* a, b, $c \in K$, $a : \mathbf{R}^+ \to \mathbf{R}^+$ *with* $r \to +\infty \Rightarrow a(r) \to +\infty$ *and every* $(t,\mathbf{x}) \in \tilde{Q}$:

$$\text{(i)} \quad a(\|\mathbf{y}\|) \leqslant V(t,\mathbf{x}), V(t,\mathbf{0}) = \mathbf{0},$$

$$\text{(ii)} \quad V(t,\mathbf{x}) \leqslant b\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right) \quad m \leqslant k \leqslant n,$$

$$\text{(iii)} \quad \dot{V}(t,\mathbf{x}) \leqslant -c\left(\left(\sum_{i=1}^{k} x_i^2\right)^{1/2}\right),$$

$$\text{(iv)} \quad \sum_{i=1}^{n} x_i^2 \to +\infty \Rightarrow V(t,\mathbf{x}) \to +\infty,$$

then the origin is globally asymptotically $\mathbf{y}$*-stable*.

References

[1] Fujisaka H, Yamada. Prog Theor Phys 1983;69:32.
[2] Afraimovich VS, Verichev NN, Robinovich MI. Radiophys Quantum Electron 1986;29:795.
[3] Pecora LM, Carroll TL. Phys Rev Lett 1990;64:821.
[4] Pecora LM, Carroll TL. Phys Rev A 1991;44:2374.
[5] He R, Vaidya PG. Phys Rev A 1992;46:7387.
[6] Pecora LM, Carroll TL. Physica D 1993;67:126.
[7] Ding M, Ott. Phys Rev E 1994;49:945.
[8] Murali K, Lakshmanan M. Phys Rev E 1994;49:4882.
[9] Wu CW, Chua LO. Int J Bifurcat Chaos 1994;4:979.
[10] Carroll TL, Pecora LM. IEEE Trans Circ Syst 1993;40:646.
[11] Pyragas K. Phys Rev E 1996;54:4508.
[12] Kapitaniak T, Sekieta M, Ogorzolek M. Int J Bifurcat Chaos 1996;6:211.
[13] Carroll TL, Pecora LM. Int J Bifurcat Chaos 1999;9:2315.
[14] Santoboni GS, Bishop SR, Varone A. Int J Bifurcat Chaos 1999;9:2345.
[15] Kocarev L, Parlitz U, Brown R. Phys Rev E 2000;61:3716.
[16] Carroll TL, Pecora LM. IEEE Trans CAS 1991;38:453.
[17] Cuomo KM, Oppenheim AV. Phys Rev Lett 1993;71:65.
[18] Wu CW, Chua L. Int J Bifurcat Chaos 1993;3:1619.

[19] Short KM. Int J Bifurcat Chaos 1994;4:959.
[20] Perez G, Cerdeira HA. Phys Rev Lett 1995;74:1970.
[21] Cuomo KM. Int J Bifurcat Chaos 1993;3:1327.
[22] Cuomo KM, Oppenheim AV, Strogatz SH. IEEE Trans Circ Syst 1993;40:626.
[23] Peng JH, Ding EJ, Ding M, Yang W. Phys Rev Lett 1996;76:904.
[24] Lü J, Zhou T, Zhang S. Chaos, Solitons & Fractals 2002;14:529.
[25] Vaidya PG. Chaos, Solitons & Fractals 2003;17:443.
[26] Morgül O, Solak E. Phys Rev E 1996;54:4803.
[27] Grassi G, Mascolo S. IEEE Trans Circ Syst-I 1997;44:1011.
[28] Grassi G, Mascolo S. Int J Bifurcat Chaos 1999;9:1175.
[29] Grassi G, Mascolo S. IEEE Trans Circ Syst-II 1999;46:478.
[30] Kozlov AK, Shalfeev VD, Chua LO. Int J Bifurcat Chaos 1999;6:569.
[31] di Bernardo M. Phys Lett A 1996;214:139.
[32] di Bernardo M. Int J Bifurcat Chaos 1996;6:557.
[33] Fradkov AL, Markov AYu. IEEE Trans Circ Syst-I 1997;44:905.
[34] Wang C, Ge SS. Int J Bifurcat Chaos 2001;11:1743.
[35] Hong Y-G, Qin H-S, Chen G-R. Int J Bifurcat Chaos 2001;11:1149.
[36] Wang C, Ge SS. Chaos, Solitons & Fractals 2001;12:1199.
[37] Chen S, Lü J. Chaos, Solitons & Fractals 2002;14:643.
[38] Tan X, Zhang J, Yang Y. Chaos, Solitons & Fractals 2003;16:37.
[39] Fang J-Q, Hong Y-G, Chen G-R. Phys Rev E 1999;59:R2523.
[40] Grassi G, Mascolo S. Int J Bifurcat Chaos 1999;9:705.
[41] Tonelli R, Lai Y-C, Grebogi C. Int J Bifurcat Chaos 2000;10:2611.
[42] Wang X-F, Wang Z-Q, Chen G-R. Int J Bifurcat Chaos 1999;9:1169.
[43] Yu X-H, Song Y-X. Int J Bifurcat Chaos 2001;11:1737.
[44] Hegazi AS, Agiza HN, Ei Dessoky MM. Chaos, Solitons & Fractals 2001;12:1091.
[45] Liu F et al. Chaos, Solitons & Fractals 2002;13:723.
[46] Codreanu S. Chaos, Solitons & Fractals 2003;15:507.
[47] Chen M, Han Z. Chaos, Solitons & Fractals 2003;17:709.
[48] Rulkov NF et al. Phys Rev E 1995;51:980.
[49] Abarbanel HDI, Rulkov NF, Sushchik MM. Phys Rev E 1996;53:4528.
[50] Kocarev L, Parlitz U. Phys Rev Lett 1996;76:1816.
[51] Pecora LM, Carroll TL, Heagy JF. Phys Rev E 1995;52:3420.
[52] Hunt BR, Ott E, Yorke JA. Phys Rev E 1997;55:4029.
[53] Lu J, Xi Y. Chaos, Solitons & Fractals 2003;17:825.
[54] Josić K. Phys Rev Lett 1996;80:3053.
[55] Rumjantsev VV. Vestnik Moskov Univ Ser I Mat Meh 1957;4:9.
[56] Rouche N, Habets P, Laloy M. Stability Theory by Liapunov's direct method, 1977.
[57] Rumjantsev VV, Oziraner AS. Stability and stabilization of motion with respect to part of the variables. Nauka; 1987 [in Russian].