International Journal of Bifurcation and Chaos, Vol. 14, No. 7 (2004) 2223–2228

PIECEWISE TWO-DIMENSIONAL MAPS AND APPLICATIONS TO CELLULAR NEURAL NETWORKS

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Received March 13, 2003; Revised September 9, 2003

Of concern is a two-dimensional map T of the form $T(x, y) = (y, F(y) - bx)$. Here F is a three-piece linear map. In this paper, we first prove a theorem which states that a semiconjugate condition for T implies the existence of Smale horseshoe. Second, the theorem is applied to show the spatial chaos of one-dimensional Cellular Neural Networks. We improve a result of Hsu [2000].

Keywords: Cellular Neural Networks; Smale horseshoe; piecewise two-dimensional map.

1. Introduction

We consider a piecewise two-dimensional map of the form

$$T(x, y) = (y, F(y) - bx), \tag{1}$$

where

$$F(y) = \begin{cases} a_1 y + a_0 - a_1 + c_1 & y \geq 1, \\ a_0 y + c_1 & |y| \leq 1, \\ a_{-1} y + a_{-1} - a_0 + c_1 & y \leq -1. \end{cases} \tag{2}$$

Here $a_0 < 0$, a_1, $a_{-1} > 1$, $b > 0$, and $c_1 \in \mathbb{R}$ is a biased term. The graph of F is given in Fig. 1.

The motivation for studying such a map is, in part, due to the form of the map is a generalized version of Lozi map [Lozi, 1978]. More importantly, the map arises in the study of complexity of a set of bounded stable stationary solutions of one-dimensional Cellular Neural Networks (CNNs) (see e.g. [Chua, 1998; Chua & Yang, 1998a, 1998b]). In this paper, we first prove a theorem which states that a semiconjugate condition for T implies the existence of Smale horseshoe. Second, we apply the theorem to show the spatial chaos of one-dimensional Cellular Neural Networks. Such CNNs are of the form (e.g. [Ban *et al.*, 2002, 2001; Hsu, 2000]).

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + a f(x_i) + \beta f(x_{i+1}), \quad i \in \mathbb{Z} \tag{3a}$$

where $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r & x \geq 1 \\ x & |x| \leq 1 \\ lx + l - 1 & x \leq -1, \end{cases} \tag{3b}$$

where r and l are positive constants. The quantity z is called threshold or bias term, related to independent voltage sources in electric circuits. The constants α, a and β are the interaction weights between neighboring cells. The study of problems for the case of $r = l = 0$ and $\alpha = \beta$ has been established in [Chua, 1998; Chua & Yang, 1998a; Juang & Lin, 2000]. Here we consider $r > 0$ and $l > 0$. Then the main results are the following. Given α and β, if (z, a) is in a certain parameter region $\Sigma_{\alpha,\beta}$ (see Theorem 3.1), then there exist r and l sufficiently small for which $\Lambda_{l,r}$ (see Theorem 3.1) is a hyperbolic invariant set. Consequently, the spatial entropy of the corresponding set of bounded, stable stationary solutions is ln 2.

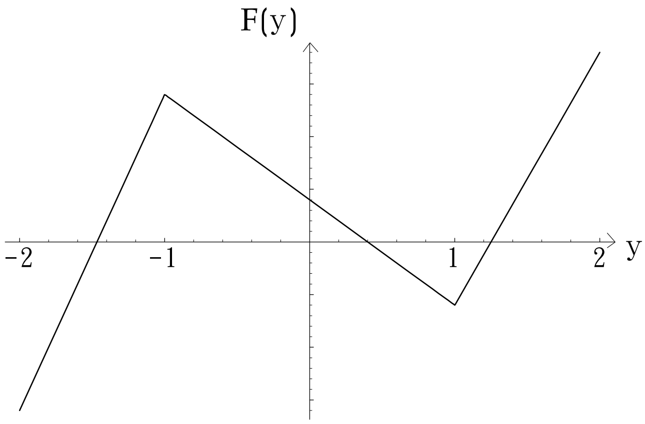

Fig. 1. $a_1 = 1.2$, $a_0 = -0.5$, $a_{-1} = 1.5$, $c_1 = 0.2$.

2. Main Results

We first introduce some notations. Let

$$S = \{(x, y) \in \mathbb{R}^2 : |x| \le p,\ |y| \le p\}. \tag{4}$$

Here $p > 1$. Let the four corners of S be labeled as

$$\begin{aligned} K &= (p, p), \quad L = (p, -p), \\ M &= (-p, -p), \quad N = (-p, p). \end{aligned} \tag{5a}$$

Set

$$\begin{aligned} \overline{K} &= (p, 1), \quad \overline{L} = (p, -1), \\ \overline{M} &= (-p, -1), \quad \overline{N} = (-p, 1). \end{aligned} \tag{5b}$$

The x and y coordinates of K are denoted, respectively, by K^x and K^y.

We next number the following conditions.

$$K_1^y \ge p > 1, \tag{6a}$$

$$\overline{N}_1^y \le -p, \tag{6b}$$

$$\overline{L}_1^y \ge p, \tag{6c}$$

and

$$M_1^y \le -p. \tag{6d}$$

Here the subscript denotes the iteration index under the map T. For instance, K_1^y denotes the y coordinate of $T(K) = K_1$. Suppose (6) holds. Then $T(S) \cap S$ has three vertical strips. See Fig. 2. Similarly, $T^{-1}(S) \bigcap S$ has three horizontal strips, and $T^{-1}(S) \bigcap S \bigcap T(S)$ has 9 components. By induction $\bigcap_{j=-n}^{n} T^j(S)$ has 9^n components. With this information we can define a semiconjugate

$$h : \Lambda \to \{0, 1, 2\}^2 \tag{7}$$

which is onto. Here $\Lambda = \bigcap_{j=-\infty}^{\infty} (T^j(S) \bigcap S)$. If the components of Λ are points, then Λ is a Cantor set.

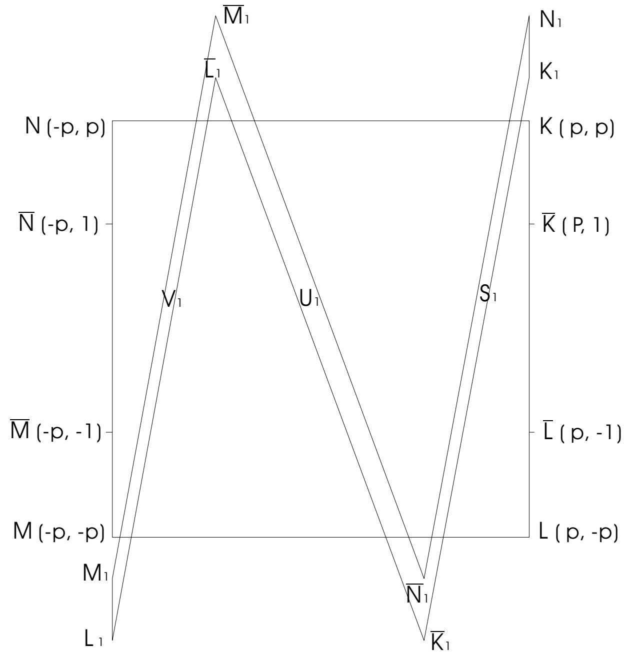

Fig. 2.

This, in turn, implies that the semiconjugacy h is one to one and so is a conjugacy. This motivates the following definition.

Definition 1.1. Conditions on b, a_{-1}, a_0, and a_1 so that there exists a $p > 1$ for which (6) holds are called a semiconjugate condition for T.

To prove the main theorem, we need to introduce more notations. Now, $T(S) \bigcap S$, has three vertical strips, say S_1, U_1 and V_1. The one on the right, see Fig. 2, is labeled as S_1. Clearly, $T(S_1) \bigcap S$ also has three vertical strips. The strip of $T(S_1) \bigcap S_1$ is to be denoted by S_2. We then define S_n inductively. Note that S_n, $n \in \mathbb{N}$, are all parallelograms. U_s and V_n are defined similarly.

The parallelogram $N_1 K_1 \overline{K}_1 \overline{N}_1$, see Fig. 2, is to be denoted by $\overline{S}_1$. Likewise, $\overline{S}_n$ denotes the parallelogram $N_n K_n \overline{K}_n \overline{N}_n$. The length of the shorter side of the parallelogram S_n (resp. $\overline{S}_n$) is to be denoted by

$$d_n (\text{resp. } c_n). \tag{8a}$$

The slope of the longer side of the parallelogram S_n is to be denoted by

$$m_n. \tag{8b}$$

Lemma 2.1. *The following recursive relations hold.*

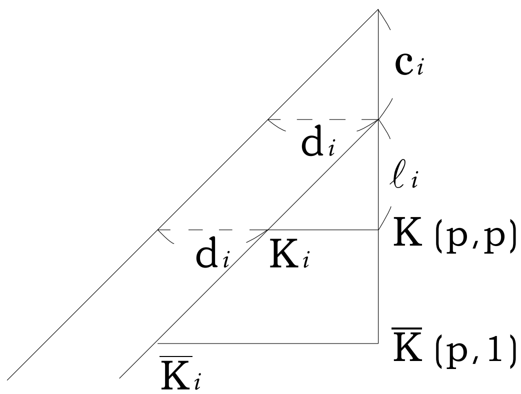

Fig. 3.

(i) $d_i = \frac{c_i}{m_i}$, $c_{i+1} = bd_i$,
(ii) $m_{i+1} = a_1 - \frac{b}{m_i}$, $m_1 = a_1$.

Proof. The first recursive relation is obvious. To see (ii), let l_i be given as in Fig. 3. We then see that $K_i = (p - (l_i/m_i), p)$ and $\overline{K}_i = (p - (l_i + p - 1)/m_i, 1)$. Now, the slope m_{i+1} = the slope of $\overline{T(K_i)T(\overline{K}_i)} = \overline{K_{i+1}\overline{K_{i+1}}} = F(p) - F(1) + b((1 - p)/m_i)/(p-1) = a_1 - (b/m_i)$. ■

Lemma 2.2. *If $b > 0$ and $a_1 \geq 2(1 + b)$, then* $\lim_{n\to\infty} c_n = 0$.

Proof. We first prove that $\lim_{n\to\infty} m_n = (a_1 + \sqrt{a_1^2 - 4b})/2$. To this end, we see that an induction would yield that $m_i \geq 1$ for all $i \in \mathbb{N}$ and that m_i is decreasing in i. Suppose x is the limit of $\{m_n\}$. Then x must satisfy equation $x = a_1 - (b/x)$. Upon using the fact that $m_1 = a_1$, we conclude that $x = a_1 + \sqrt{a_1^2 - 4b}/2$ as asserted. Now, using Lemma 2.1(i), we get $d_n = b^{n-1}d_1/\prod_{i=2}^n m_i$. Thus,

$$
\begin{aligned}
d_n &\leq \left(\frac{2b}{a_1 + \sqrt{a_1^2 - 4b}}\right)^{n-1} d_1 \\
&\leq \left(\frac{2b}{a_1}\right)^{n-1} d_1 \\
&\leq \left(\frac{b}{1+b}\right)^{n-1} d_1 \,.
\end{aligned}
$$

We have just completed the proof of the lemma. ■

Similarly, we have the following lemma.

Lemma 2.3. *If $b > 0$ and $a_{-1} > 2(1 + b)$, then the length of the shorter side of the parallelogram V_n shrinks to zero as $n \to \infty$.*

Using Lemmas 2.2 and 2.3, we have the following lemma.

Lemma 2.4. *If $b > 0$, $\min\{a_1, a_{-1}\} > 2(1 + b)$, then the length of the shorter side of the parallelogram U_n shrinks to zero as $n \to \infty$.*

Remark. The assumptions on Lemmas 2.2–2.4 would also yield that $\bigcap_{j=0}^{-\infty}(T^j(S)\bigcap S)$ are pairwise disjoint horizontal line segments.

We are now ready to state our main results.

Theorem 2.1. *Let F be a piecewise linear map defined as in (2) and the bias term c_1 satisfy the inequality*

$$
\begin{aligned}
&\max\{-1-b,\, a_0 + 1 + b\} \\
&\qquad < c_1 < \min\{1 + b,\, -a_0 - 1 - b\}\,, \qquad (9)
\end{aligned}
$$

then a semiconjugate condition for T implies the conjugate of h.

Proof. Note that $K_1^y \geq p$, (6b) and (6d) are equivalent to the following inequalities.

$$p(a_1 - 1 - b) \geq a_1 - a_0 - c_1\,, \qquad (10a)$$

$$-a_0 + c_1 \geq p(1 + b)\,, \qquad (10b)$$

$$-a_0 - c_1 \geq p(1 + b)\,, \qquad (10c)$$

and

$$p(a_{-1} - 1 - b) \geq a_{-1} - a_0 + c_1\,, \qquad (10d)$$

respectively. We remark (10b) and (10c) to ensure that $-a_0 - 1 - b > 0$, as a result, inequality (9) makes sense. Using (10a) and (10b), we see immediately that

$$\frac{-a_0 + c_1}{b+1} \geq p \geq \frac{a_1 - a_0 - c_1}{a_1 - b - 1}\,. \qquad (11)$$

Note that $a_1 - b - 1$ being positive is guaranteed by the fact that $p > 1$ and the assumptions on c_1. Using (10), we get that

$$a_1 \geq \frac{-2a_0(b+1)}{c_1 - a_0 - 1 - b} = \frac{2(b+1)}{1 + \dfrac{1 + b - c_1}{a_0}} \geq 2(b+1)\,. \qquad (12a)$$

The last inequality is justified by the assumptions on c_1. Similarly, we see that

$$a_{-1} \geq \frac{2a_0(b+1)}{c_1 + a_0 + 1 + b} = \frac{2(b+1)}{1 + \dfrac{1+b+c_1}{a_0}} \geq 2(b+1)\,. \tag{12b}$$

It then follows from Lemmas 2.2–2.4 that $\bigcap_{j=-\infty}^{\infty}(T^j(S)\bigcap S)$ is a Cantor set. We thus complete the proof of the main theorem. ■

Remarks

(1) If $F(y)$, as defined in 2, is such that $a_0 > 0$, and $a_1,\, a_{-1} < -1$, then a similar result can also be obtained.
(2) The theorem holds true in general for F being a finitely many piecewise linear map. Specifically, if the bias term c_1 is not "too biased", then a semiconjugate condition for T implies the existence of Smale horseshoe.

In the following, we give conditions on a_0, a_1, a_{-1}, b and c for which T has a semiconjugate condition.

Theorem 2.2. *Let $a_0 < 0$, $a_1,\, a_{-1} > 1$ and $b > 0$. Suppose $a_0 + 1 + b < 0$, $\min\{a_1, a_{-1}\} > 2(1+b)$. Let the bias term c_1 satisfy (9), and that*

$$a_1 \geq \frac{-2a_0(b+1)}{c_1 - a_0 - 1 - b} \tag{13a}$$

and

$$a_{-1} \geq \frac{2a_0(b+1)}{c_1 + a_0 + 1 + b}\,. \tag{13b}$$

then there exists a $p > 1$ such that T has a semiconjugate condition.

3. Applications to CNNs

A basic and important class of solutions of (1) is the bounded, stable stationary solutions. In the case that $r = l = 0$ and $\alpha = \beta$, the corresponding stable stationary solutions have been studied in [Chua & Yang, 1998a; Juang & Lin, 2000]. The case that r and l are positive is considered in [Ban *et al.*, 2002, 2001; Hsu, 2000]. The techniques in these two cases are quite different. Specifically, in the latter case, the question of complexity of a set of stable stationary solutions is converted to asking how chaotic is a map. If α or $\beta = 0$, then the resulting map is one-dimensional [Ban *et al.*, 2002, 2001]. If $\alpha,\, \beta \neq 0$, then the resulting map is a two-dimensional of the following form [Hsu, 2000]

$$T(x,\, y) = \left(y,\, \frac{1}{\beta}(\overline{F}(y) - ay - z) - \frac{\alpha}{\beta}x\right) =: (y,\, F(y) - bx)\,. \tag{14a}$$

Here,

$$\overline{F}(y) = \begin{cases} \dfrac{1}{r}y - \dfrac{1}{r} + 1 & y \geq 1 \\ y & |y| \leq 1 \\ \dfrac{1}{l}y - 1 + \dfrac{1}{l} & y \leq -1. \end{cases} \tag{14b}$$

Hsu [2000] used a theorem of Afraimovich (see e.g. [Afraimovich, 1993]) as well as a semiconjugate condition to show that in certain parameters' region, the map T has Smale horseshoe structure. However, Afraimovich's Theorem is not needed in this case. Only a semiconjugate condition is required.

To apply Theorem 2.2, we first note that $a_{-1} = \frac{1}{\beta}(\frac{1}{l} - a)$, $a_0 = \frac{1}{\beta}(1-a)$, $a_1 = \frac{1}{\beta}(\frac{1}{r} - a)$, $c_1 = \frac{-z}{\beta}$, $b = \frac{\alpha}{\beta}$. With the above identifications, we immediately have the following results concerning the complexity of a set of bounded, stable stationary mosaic solutions of (3). Here the stationary mosaic solutions $(x_i)_{i=-\infty}^{\infty}$ means that $(x_i)_{i=-\infty}^{\infty}$ is a stationary solution of (3) and that $|x_i| > 1$ for all $i \in \mathbb{Z}$. Moreover, the mosaic solutions obtained in the following theorem are bounded and stable (see e.g. [Chua & Yang, 1998a; Hsu, 2000]).

Define $s = \alpha + a + \beta$. Assume the bias term z satisfies the following inequality.

$$\max\{-s + a,\, s - 2a + 1\} < z < \min\{s - a,\, 2a - 1 - s\}\,. \tag{15}$$

Define, respectively, the regions $\Sigma_{\alpha,\beta}$ and $\Sigma_{\alpha,\beta,l,r}$ as follows.

$$\Sigma_{\alpha,\beta} = \{(z,\, a) \in \mathbb{R}^2 |\ (15) \text{ holds}\}\,, \tag{16}$$

and

$$\Sigma_{\alpha,\beta,l,r} = \{(z,\, a) \in \mathbb{R}^2 | r < r^+, \text{ and } l < l^+\}\,. \tag{17}$$

Here,

$$r^+_{z,\alpha,a,\beta} = \frac{2a - s - 1 - z}{a(1 + s - z) - 2s}\,, \tag{18a}$$

and

$$l^+_{z,\alpha,a,\beta} = \frac{2a - s - 1 + z}{a(1 + s + z) - 2s}\,. \tag{18b}$$

We are now in a position to state the following results.

Theorem 3.1. *Let α and β be positive numbers and let $a > 1 + \alpha + \beta$. Suppose $(z, a) \in \sum_{\alpha,\beta}$. Then there exist r and l sufficiently small, more precisely $0 < r < r^+ = r^+_{z,\alpha,a,\beta}$ and $0 < l < l^+ = l^+_{z,\alpha,a,\beta}$ for which T has a hyperbolic invariant set $\Lambda_{l,r}(z, \alpha, a, \beta) = \Lambda_{l,r}$ in the (x, y) plane such that $T|_{\Lambda_{l,r}}$ is topologically conjugate to a two-side Bernoulli shift of two symbols. Hence, the spatial entropy of the corresponding set of stationary solutions equals* ln 2.

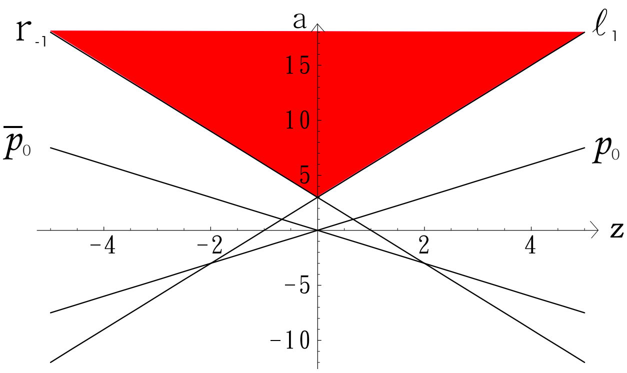

Fig. 4. $\epsilon = \frac{1}{3}$, $l_1 : -z + a(1 - 2\epsilon) = 1$, $p_0 : z = 2a\epsilon$, $r_{-1} : z + a(1 - 2\epsilon) = 1$, $\overline{p}_0 : z = -2a\epsilon$.

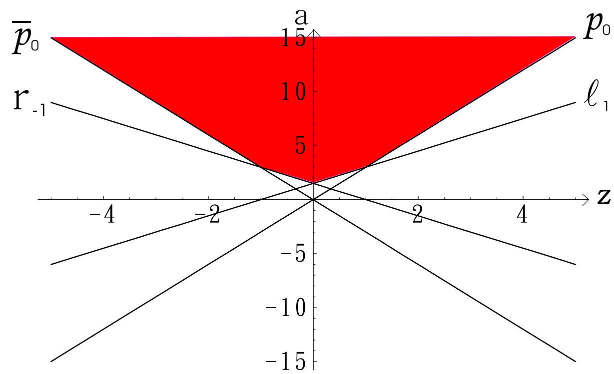

Fig. 5. $\epsilon = \frac{1}{6}$, $l_1 : -z + a(1 - 2\epsilon) = 1$, $p_0 : z = 2a\epsilon$, $r_{-1} : z + a(1 - 2\epsilon) = 1$, $\overline{p}_0 : z = -2a\epsilon$.

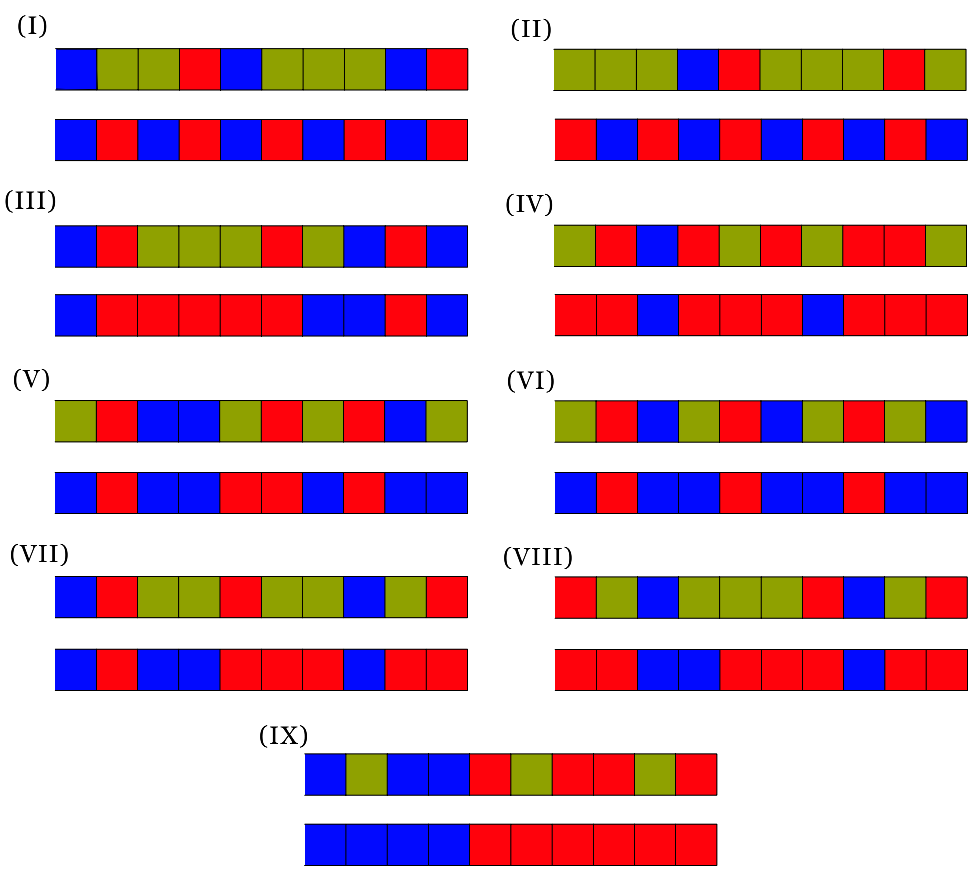

Fig. 6.

Remarks

(1) Note that if $(z, a) \in \Sigma_{\alpha,\beta}$, then $-2s + a(1+s-z) = a(-z-1-s+2a) + 2(a-1)(s-a) > 0$ and $-2s + a(1+s+z) = a(z-1-s+2a) + 2(a-1)(s-a) > 0$. Consequently, those r^+ and l^+ are positive.

(2) Adapting the notations in [Juang & Lin, 2000] we let $\alpha = \beta = a\epsilon$. Then the set $\Sigma_{\alpha,\beta} = \Sigma_\epsilon$ is given in the following figure.

Note that for $0 < \varepsilon < \frac{1}{4}$, $\Sigma_\epsilon \subsetneq [3,\ 3]_\epsilon$ (see Fig. 5.1 of [Juang & Lin, 2000] for the definition of $[3,\ 3]_\epsilon$), and for $\frac{1}{4} \le \epsilon < \frac{1}{2}$, $\Sigma_\epsilon = [3,\ 3]_\epsilon$ (see Figs. 4 and 5). Applying Theorem 3.1, we conclude that let $\alpha = \beta = a\epsilon$, $\frac{1}{4} \le \varepsilon < \frac{1}{2}$, and if $(z, a) \in \Sigma_\epsilon = [3,\ 3]_\epsilon$, then there exist r and l sufficiently small for which $\Lambda_{l,r}$ is a hyperbolic invariant set. This result generalized those in [Chua, 1998; Chua & Yang, 1998a; Juang & Lin, 2000]. For $0 < \epsilon < \frac{1}{4}$, if $(z, a) \in \Sigma_\epsilon$ and $r,\ l > 0$ is sufficiently small, then the corresponding set of stable, bounded stationary solutions also has spatial entropy ln 2.

(3) To get a feel of how small r and l are required to be, set $\epsilon = \frac{1}{4}$ and $z = 0$. We see easily that $r^+ = l^+$ has a maximum $\frac{1}{16}$ for $2 < a < \infty$.

(4) Figure 6 is a collection of a computer simulation with a set of parameters, satisfying $a > 1 + \alpha + \beta$, $0 < r < r^+ = r^+_{z,\alpha,a,\beta}$ and $0 < l < l^+ = l^+_{z,\alpha,a,\beta}$. Specifically, we choose $\alpha = \beta = 1$, $r = l = 0.005$, $z = 0$, $a = 4$. Each collection in Fig. 6 contains two arrays of colors. The first array is the initial outputs. The second array represents the final outputs. If the state x_j of a cell c_j is such that $|x_j| < 1$, then we color it green. If the state x_j of a cell c_j is less than -1 (greater than 1, respectively), then we color it blue (red, respectively).

Acknowledgment

We thank Dr. C. J. Yu for providing the simulation work in the paper.

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