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J. Differential Equations 201 (2004) 1–24
Journal of Differential Equations
http://www.elsevier.com/locate/jde

On spherically symmetric solutions of the relativistic Euler equation

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Received May 15, 2001; revised May 9, 2003

Abstract

This work investigates the spherical symmetric solutions of more realistic equation of states. We generalize the method of Hsu et al. (Methods Appl. Anal. 8 (2001) 159) to show the existence of spherical symmetric weak solution of the relativistic Euler equation with initial data containing the vacuum state.
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MSC: 35L45; 35Q05; 35Q75

Keywords: Riemann invariant; Darboux entropy; Lax–Friedrichs scheme

1. Introduction

The motion of a perfect fluid in the Minkowski space–time is described by the relativistic Euler equation

$$\frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\rho + P/c^2}{1-u^2/c^2}u_k\right) = 0,$$

$$\frac{\partial}{\partial t}\left(\frac{\rho + P/c^2}{1-u^2/c^2}u_j\right) + \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\left(\frac{\rho + P/c^2}{1-u^2/c^2}u_j u_k + P\delta_{jk}\right) = 0, \quad j = 1,2,3.$$

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[1] Work partially supported by The National Science Council of Taiwan and National Center for Theoretical Sciences Mathematics Division, Taiwan.
[2] Work partially supported by The National Science Council of Taiwan and National Center for Theoretical Sciences Mathematics Division, Taiwan.

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doi:10.1016/j.jde.2004.03.003

Here ρ is the density and (u_1, u_2, u_3) is the velocity. The speed of light c is a positive constant. The pressure P is supposed to be a given function of ρ.

Smoller–Temple [7] studied one-dimensional motions under the assumption $P = \sigma^2\rho$, σ being a positive constant smaller than c. In the previous paper, Hsu et al. [4], we studied one-dimensional motions under more realistic equation of states. In this article we discuss on spherically symmetric motions. Suppose

$$\rho = \rho(r,t), \quad u_j = \frac{x_j}{r}u(r,t), \quad r = |x|.$$

Then the equation is reduced to

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial r}\frac{(\rho + P/c^2)u}{1-u^2/c^2} + \frac{2}{r}\frac{(\rho + P/c^2)u}{1-u^2/c^2} = 0, \\ &\frac{\partial}{\partial t}\frac{(\rho + P/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial r}\frac{\rho u^2 + P}{1-u^2/c^2} + \frac{2}{r}\frac{(\rho + P/c^2)u^2}{1-u^2/c^2} = 0. \end{aligned} \tag{1.1}$$

In order to avoid the singularity at $r = 0$, we consider Eq. (1.1) on $r \geqslant 1$ with the initial condition

$$\rho|_{t=0} = \rho^0(r), \quad u|_{t=0} = u^0(r) \tag{1.2}$$

and the boundary condition

$$\rho u|_{r=1} = 0. \tag{1.3}$$

Under the assumption $P = \sigma^2\rho$, Mizohata [6] proved the existence of global weak solutions. However, we would like to consider a more realistic equation of states. We keep in mind the equation of state for a neutron stars, which is given by

$$P = Kc^5 f(y), \quad \rho = Kc^3 g(y),$$

$$f(y) = \int_0^y \frac{q^4}{\sqrt{1+q^2}}\,dq \quad \text{and} \quad g(y) = 3\int_0^y q^2\sqrt{1+q^2}\,dq.$$

For this equation of state, we have $P \sim \frac{c^2}{3}\rho$ as $\rho \to \infty$ but $P \sim \frac{1}{5}K^{2/3}\rho^{5/3}$ as $\rho \to 0$. So we assume the following properties of the function $P(\rho)$:

(A)

$$P(\rho) > 0, \quad 0 < dP/d\rho < c^2, \quad 0 < d^2P/d\rho^2 \quad \text{for } \rho > 0,$$

and

$$P = A\rho^\gamma(1 + [\rho^{\gamma-1}/c^2]_1) \quad \text{as} \quad \rho \to 0.$$

Here A and γ are positive constants and $\gamma = 1 + \frac{2}{2N+1}$, N being a positive integer, and $[X]_1$ denotes a convergent power series of the form $\sum_{k\geqslant 1} a_k X^k$.

We put

$$E = \frac{\rho + Pu^2/c^4}{1 - u^2/c^2}, \quad F = \frac{(\rho + P/c^2)u}{1 - u^2/c^2}, \quad G = \frac{\rho u^2 + P}{1 - u^2/c^2},$$
$$H_1 = -\frac{2}{r}F, \quad H_2 = -\frac{2}{r}Fu,$$
$$U = (E, F)^T, \quad f(U) = (F, G)^T, \quad H(r, U) = (H_1, H_2)^T.$$

Then the problem can be written as

$$U_t + f(U)_r = H(r, U), \tag{1.4}$$

$$U|_{t=0} = U^0(r), \tag{1.5}$$

$$F|_{r=1} = 0. \tag{1.6}$$

A weak solution is defined as a field $U \in L^\infty([1, +\infty) \times [0, T))$ such that $0 \leqslant \rho, |u| < c$ which satisfies

$$\int_0^T \int_1^\infty (\Phi_t U + \Phi_r f(U) + \Phi H(r, U))\, dr\, dt + \int_1^\infty \Phi(r, 0) U^0(r)\, dr = 0,$$

for any test function $\Phi = (\phi_1, \phi_2)^T \in C_0^\infty([1, +\infty) \times [0, T))$ such that $\phi_2|_{r=1} = 0$. Our conclusion is the following:

Theorem 1. *For any* C_0 *there is* $\varepsilon(C_0) > 0$ *such that if* $\rho^0(r) \geqslant 0$, $0 \leqslant u^0(r) < c$ *and*

$$\int_0^{\rho^0(r)} \frac{\sqrt{P'}}{\rho + P/c^2}\, d\rho \leqslant \frac{c}{2} \log \frac{c + u^0(r)}{c - u^0(r)} \leqslant C_0, \tag{1.7}$$

and if $1/c^2 \leqslant \varepsilon(C_0)$, *then there exists a global* ($T = \infty$) *weak solution of* (1.1)–(1.3).

Condition (1.7) is an analogy from the work Chen [1] on the non-relativistic problem.

The paper is organized as follows. Some results of the Riemann problem and estimations of entropy–entropy flux obtained in [4] are recalled in Section 2. In Section 3, we prove the key lemma for constructing the approximation solutions of (1.4). By using the Lax–Friedrichs scheme, the main theorem is proved in Section 4 and the entropy condition for such weak solution is also illustrated. In Section 5 we will discuss the problem including the co-ordinate origin.

2. Preliminary

Let us recall the results of Hsu et al. [4] on the one-dimensional equation, that is, the equation without the source term H:

$$U_t + f(U)_r = 0. \tag{2.1}$$

The Riemann problem to (2.1) on $t \geqslant t_0$ with center r_0 and data (U_L, U_R) is the Cauchy problem for the initial data

$$U = \begin{cases} U_L & \text{if } r < r_0, \\ U_R & \text{if } r_0 < r, \end{cases} \quad \text{at } t = t_0.$$

The Riemann invariants are

$$w = x + y, \quad z = x - y, \tag{2.2}$$

where

$$x = \frac{c}{2} \log \frac{c+u}{c-u}, \quad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} \, d\rho. \tag{2.3}$$

Condition (1.7) implies $0 \leqslant z^0 \leqslant w^0 \leqslant B = 2C_0$.

Proposition 2.1 (See Chen [2]). *The Riemann problem has a unique entropy solution which consists of rarefaction waves and shock waves provided that* $z_L \leqslant w_L, z_R \leqslant w_R$.

We put

$$\Sigma(\alpha, \beta) = \{(w, z) : \alpha \leqslant z \leqslant w \leqslant \beta\}.$$

Proposition 2.2. *The region* $\Sigma(\alpha, \beta)$ *is invariant with respect to the Riemann problem, that is, if the data* U_L, U_R *belong to* $\Sigma(\alpha, \beta)$ *the solution is confined to* $\Sigma(\alpha, \beta)$.

Proposition 2.3. *The invariant region* $\Sigma(\alpha, \beta)$ *is convex in the* (E, F)*-plane. Therefore if* $U(s) \in \Sigma(\alpha, \beta)$ *for* $s \in [a, b]$, *then the average*

$$\frac{1}{b-a} \int_a^b U(s) \, ds$$

belongs to $\Sigma(\alpha, \beta)$.

Proof. Let us consider the above hedge $F = F(E)$ which corresponds to $w = \beta, \alpha < z < \beta$. We have to show $d^2F/dE^2 < 0$. Along the hedge $w = \beta$, we have

$$u = c \tanh \frac{1}{c} \left(\beta - \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2} \, d\rho \right),$$

from which

$$\frac{du}{d\rho} = -(1 - u^2/c^2)\frac{\sqrt{P'}}{\rho + P/c^2}.$$

By a direct calculation we have

$$\frac{dF}{dE} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}.$$

Differentiating once more we have

$$\frac{d^2F}{dE^2} = -\frac{1 - u^2/c^2}{(1 - \sqrt{P'}u/c^2)^4}\left(\frac{P''}{2\sqrt{P'}} + \left(1 - \frac{P'}{c^2}\right)\frac{\sqrt{P'}}{\rho + P/c^2}\right) < 0.$$

The proof is complete.

A pair of functions $(\eta(U), q(U))$ is an entropy–entropy flux if

$$D_U q = D_U \eta \cdot D_U f. \tag{2.4}$$

Using the Riemann invariants, we can write (2.4) as

$$\frac{\partial q}{\partial w} = \frac{u + \sqrt{P'}}{1 + \sqrt{P'}u/c^2}\frac{\partial \eta}{\partial w}, \quad \frac{\partial q}{\partial z} = \frac{u - \sqrt{P'}}{1 - \sqrt{P'}u/c^2}\frac{\partial \eta}{\partial z}. \tag{2.5}$$

By eliminating q from the equation, we get the following second-order equation:

$$\frac{\partial^2 \eta}{\partial w \partial z} + Q\left(J\frac{\partial \eta}{\partial w} - \frac{1}{J}\frac{\partial \eta}{\partial z}\right) = 0, \tag{2.6}$$

where

$$Q = \frac{1}{4\sqrt{P'}}\left(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P''\right) \quad \text{and} \quad J = \frac{1 - \sqrt{P'}u/c^2}{1 + \sqrt{P'}u/c^2}.$$

Since this equation tends to the Euler–Poisson–Darboux equation

$$\frac{\partial^2 \eta}{\partial w \partial z} + \frac{N}{w - z}\left(\frac{\partial \eta}{\partial w} - \frac{\partial \eta}{\partial z}\right) = 0 \tag{2.7}$$

as $c \to \infty$, we shall call (2.6) as the relativistic Euler–Poisson–Darboux equation. According to [4], the entropy can be solved as follows.

Proposition 2.4. *There exists a kernel* $K(x,y,\xi)$ *of* C^{N+2}*-class in* $|x|<\infty, 0\leqslant y, |\xi - x|\leqslant y$ *such that*

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi)\phi(\xi)\,d\xi \tag{2.8}$$

is an entropy for any smooth ϕ. *Furthermore,*

$$K(x,y,\xi) = (y^2 - (x-\xi)^2)^N(1 + O(y/c^2)). \tag{2.9}$$

Such an entropy will be called a Darboux entropy. The standard entropy–entropy flux is

$$\begin{aligned}
\eta^* &= -\frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}} + c^2\frac{\rho + Pu^2/c^4}{1-u^2/c^2},\\
\Psi(\rho) &= \exp\int^{\rho}\frac{d\rho}{\rho + P/c^2},\\
q^* &= \left(-\frac{\Psi(\rho)}{(1-u^2/c^2)^{1/2}} + c^2\frac{\rho + P/c^2}{1-u^2/c^2}\right)u.
\end{aligned}$$

Therefore, the Hessian of the standard entropy $D^2\eta^*(U)$ is positive definite as follows.

Proposition 2.5. *The standard entropy is strictly convex in the sense*

$$(\Xi \mid D^2\eta^*(U)\cdot\Xi) \geqslant k_{\alpha,\beta}|\Xi|^2$$

for any $U \in \Sigma(\alpha,\beta)$ *and* $\Xi = (\xi_0, \xi_1)$. *Here* $k_{\alpha,\beta}$ *is a positive constant.*

Proof. By elementary computation, we obtain

$$\begin{aligned}
\frac{\partial\eta^*}{\partial E} &= -\frac{\Psi}{(\rho + P/c^2)(1-u^2/c^2)^{1/2}} + c^2,\\
\frac{\partial\eta^*}{\partial F} &= \frac{\Psi u/c^2}{(\rho + P/c^2)(1-u^2/c^2)^{1/2}},\\
\frac{\partial^2\eta^*}{\partial E^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho + P/c^2)^2}(P' + 2P'u^2/c^2 + u^2),\\
\frac{\partial^2\eta^*}{\partial E\partial F} &= \frac{-\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho + P/c^2)^2}(2P'/c^2 + 1 + P'u^2/c^4)u,\\
\frac{\partial^2\eta^*}{\partial F^2} &= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho + P/c^2)^2}(1 + 3P'u^2/c^4).
\end{aligned}$$

Therefore we get

$$
\begin{aligned}
(\xi|D_U^2\eta^* \cdot \xi) &= \eta^*_{EE}\xi_0^2 + 2\eta^*_{EF}\xi_0\xi_1 + \eta^*_{FF}\xi_1^2 \\
&= \frac{\Psi/c^2}{(1-P'u^2/c^4)(1-u^2/c^2)^{1/2}(\rho+P/c^2)^2} Z, \\
Z &= (P' + 2P'u^2/c^2 + u^2)\xi_0^2 - 2(2P'/c^2 + 1 + P'u^2/c^4)u\xi_0\xi_1 \\
&\quad + (1 + 3P'u^2/c^4)\xi_1^2 \\
&\geqslant \frac{2P'(1-u^2/c^2)^2(1-P'u^2/c^4)}{A + C + \sqrt{(A-C)^2 + 4B^2}}(\xi_0^2 + \xi_1^2), \\
A &= P' + 2P'u^2/c^2 + u^2, \quad B = (2P'/c^2 + 1 + P'u^2/c^4)u, \\
C &= 1 + 3P'u^2/c^4.
\end{aligned}
$$

This completes the proof. □

Furthermore, we are going to show that the Hessian $D_U^2\eta^*$ dominates any $D_U^2\eta$.

Proposition 2.6. *For any Darboux entropy η we have*

$$
|(\Xi \mid D^2\eta(U)\cdot\Xi)| \leqslant C_\phi(\Xi \mid D^2\eta^*(U)\cdot\Xi)
$$

for $U \in \Sigma(\alpha,\beta)$ provided that $1/c^2 \leqslant \varepsilon$, ε being a positive constant independent of ϕ. Moreover across any shock with speed σ, we have

$$
|\sigma[\eta] - [q]| \leqslant C(\sigma[\eta^*] - [q^*]),
$$

where $[\eta] = \eta(U_R) - \eta(U_L), [q] = q(U_R) - q(U_L)$.

Proof. Let $R = y^{2N+1}$ and $M = xy^{2N+1}$. Direct computation gives

$$
\begin{aligned}
(\xi \mid D_U^2\eta\cdot\xi) = &\frac{2^{2N+1}K^2}{y^{2N+1}}\int_0^1 (s-s^2)^N Z[\xi]D^2\phi\, ds - \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2) \\
&\times (u\xi_0 - \xi_1)^2\frac{\partial\eta}{\partial R} - \frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u + x(1-u^2/c^2))(u\xi_0-\xi_1)^2\frac{\partial\eta}{\partial M} \\
&+ O(y^{-2N+1}/c^2),
\end{aligned}
$$

where

$$Z[\xi] = Z_{00}\xi_0^2 + 2Z_{01}\xi_0\xi_1 + Z_{11}\xi_1^2,$$

$$\begin{aligned} Z_{00} = {} & (1+u^2/c^2)^2\left(\left(-x+\frac{y}{2N+1}(2s-1)\right)^2 + \frac{4}{(2N+1)^2}s(1-s)y^2\right) \\ & + 2(1+u^2/c^2)(-u+x(1+u^2/c^2))\left(-x+\frac{y}{2N+1}(2s-1)\right) \\ & + (-u+x(1+u^2/c^2))^2, \end{aligned}$$

$$\begin{aligned} Z_{01} = {} & -2(1+u^2/c^2)u/c^2\left(\left(-x+\frac{y}{2N+1}(2s-1)\right)^2 + \frac{4}{(2N+1)^2}s(1-s)y^2\right) \\ & + (1+3u^2/c^2 - 4x(1+u^2/c^2)u/c^2)\left(-x+\frac{y}{2N+1}(2s-1)\right) \\ & + (-u+x(1+u^2/c^2))(1-2xu/c^2), \end{aligned}$$

$$\begin{aligned} Z_{11} = {} & \frac{4u^2}{c^4}\left(\left(-x+\frac{y}{2N+1}(2s-1)\right)^2 + \frac{4}{(2N+1)^2}s(1-s)y^2\right) \\ & - \frac{4u}{c^2}(1-2xu/c^2)\left(-x+\frac{y}{2N+1}(2s-1)\right) + (1-2xu/c^2)^2. \end{aligned}$$

It can be shown that

$$Z[\xi] \geqslant \kappa s(1-s)y^2,$$

where κ is a positive constant depending on the compact subset of $\{\rho \geqslant 0\}$. In fact we see

$$Z_{00}Z_{11} - Z_{01}^2 = (1-u^2/c^2)\frac{4}{(2N+1)^2}s(1-s)y^2.$$

On the other hand, we can estimate

$$\begin{aligned} \left|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(1-u^2/c^2)\frac{\partial\eta}{\partial R}\right| & \leqslant \frac{\varepsilon}{y^{2N+1}}, \\ \left|\frac{2K^2}{y^{2N+1}}\frac{1}{c^2}(u+x(1-u^2/c^2))\frac{\partial\eta}{\partial M}\right| & \leqslant \frac{\varepsilon}{y^{2N+1}}, \end{aligned}$$

where $\varepsilon = K'/c^2$. Let us introduce the parameters $\zeta_0 = \xi_0$ and $\zeta_1 = \xi_1 - u\xi_0$. Then we have

$$Z[\xi] = Q_{00}\zeta_0^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{11}\zeta_1^2$$

and

$$Q_{00} = Q_{00}^{(1)}(x)(2s-1)y + Q_{00}^{(2)}(x,s)y^2,$$

$$Q_{01} = Q_{01}^{(1)}(x)(2s-1)y + Q_{01}^{(2)}(x,s)y^2,$$

$$Q_{11} = Z_{11} = 1 + O(1/c^2) > 0.$$

Therefore if $|D^2\phi| \leqslant C$, we see

$$\begin{aligned}|(\xi \mid D_U^2\eta \cdot \xi)| \leqslant & \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi]\, ds \\ & + \frac{12\varepsilon}{y^{2N+1}} \int_0^1 (s-s^2)^N \zeta_1^2\, ds + O(y^{-2N+1}/c^2) \\ \leqslant & \frac{2^{2N+1}K^2C}{y^{2N+1}} \int_0^1 (s-s^2)^N (Q_{11}(1+\varepsilon')\zeta_1^2 + 2Q_{01}\zeta_0\zeta_1 + Q_{00}\zeta_0^2)\, ds \\ & + O(y^{-2N+1}/c^2).\end{aligned}$$

But since $Q_{00}^{(0)} = Q_{01}^{(0)} = 0$, $\int_0^1 (s-s^2)^N (2s-1)\, ds = 0$, we see

$$\int_0^1 (s-s^2)^N (-2\varepsilon' Q_{01}\zeta_0\zeta_1 - \varepsilon' Q_{00}\zeta_0^2)\, ds = O(y^{-2N+1}/c^2).$$

Therefore we get

$$|(\xi \mid D_U^2\eta \cdot \xi)| \leqslant \frac{2^{2N+1}K^2C(1+\varepsilon')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi]\, ds + O(y^{-2N+1}/c^2).$$

Similarly, if $D^2\phi^* \geqslant \mu$, we have

$$(\xi \mid D_U^2\eta^* \cdot \xi) \geqslant \frac{2^{2N+1}K^2\mu(1-\varepsilon'')}{y^{2N+1}} \int_0^1 (s-s^2)^N Z[\xi]\, ds + O(y^{-2N+1}/c^2).$$

Thus we get

$$|(\xi \mid D_U^2\eta \cdot \xi)| \leqslant \frac{C(1+\varepsilon')}{\mu(1-\varepsilon'')} (\xi \mid D_U^2\eta^* \cdot \xi) + O(y^{-2N+1}/c^2).$$

But we know

$$(\xi \mid D_U^2\eta^* \cdot \xi) \geqslant \kappa|\xi|^2 y^{-2N+1}.$$

Hence if c is sufficiently large we get the required estimate. □

3. Key Lemma

The key point of this article is the following observation.

Proposition 3.1. *Suppose* $(w_0, z_0) \in \Sigma(0, B)$, *that is*, $0 \leqslant z_0 \leqslant w_0 \leqslant B$. *Let* $E_0 = E(w_0, z_0), F_0 = F(w_0, z_0)$ *and consider* $U_\tau = U_0 + H(r, U_0)\tau$, *that is,*

$$E_\tau = E_0 - \frac{2}{r} F_0 \tau, \quad F_\tau = F_0 - \frac{2}{r} F_0 u_0 \tau,$$

where $\tau \geqslant 0$. *Then there are positive numbers* h_1 *and* ε_1 *depending only upon B such that if* $0 \leqslant \tau \leqslant h_1$ *and if* $1/c^2 \leqslant \varepsilon_1$ *then* $U_\tau = U(w_\tau, z_\tau)$ *with* $(w_\tau, z_\tau) \in \Sigma(0, B)$.

Proof. $0 \leqslant z_0 \leqslant w_0 \leqslant B$ means $0 \leqslant y_0 \leqslant x_0, x_0 + y_0 \leqslant B$. Therefore

$$0 \leqslant u_0 \leqslant (1 - \delta)c = C_1, \quad 0 \leqslant \rho_0 \leqslant C_2,$$

where $\delta \in (0, 1), C_1, C_2$ are constants determined by B. In this proof C_j stand for constants depending only upon B. First of all we must show $E_\tau \geqslant 0$. We see

$$\begin{aligned}
E_\tau &= \frac{\rho_0 + P_0/c^2}{1 - u_0^2/c^2}\left(1 - \frac{2}{r} u_0 \tau\right) - \frac{P_0}{c^2} \\
&\geqslant \frac{\rho_0 + P_0/c^2}{1 - u_0^2/c^2}(1 - 2C_1\tau) - \frac{P_0}{c^2} \\
&= \frac{(1 - 2C_1\tau)\rho_0}{1 - u_0^2/c^2}\left(1 - \frac{2C_1\tau}{1 - 2C_1\tau}\frac{P_0}{\rho_0 c^2} + \frac{1}{1 - 2C_1\tau}\frac{P_0 u_0^2}{\rho_0 c^4}\right) \\
&\geqslant \frac{(1 - 2C_1\tau)\rho_0}{1 - u_0^2/c^2}\left(1 - \frac{2C_1\tau}{1 - 2C_1\tau}\right),
\end{aligned}$$

since $P \leqslant c^2\rho$, provided that $4C_1\tau \leqslant 4C_1 h < 1$. Thus we have

$$E_\tau \geqslant \frac{1}{C_3}\rho_0 \tag{3.1}$$

and (ρ_τ, u_τ) is well defined for $\rho_0 > 0$. (If $\rho_0 = 0$, then $E = 0 = F_0 = E_\tau = F_\tau = 0$.) Next we consider $u_\tau = u(E_\tau, F_\tau)$. We have

$$\begin{aligned}
\frac{d}{d\tau} u_\tau &= -\frac{2}{r} F_0 \left(\frac{\partial u}{\partial E}\bigg|_\tau + u_0 \frac{\partial u}{\partial F}\bigg|_\tau\right) \\
&= -\frac{2}{r} F_0 \left(\frac{\partial u}{\partial E} + u \frac{\partial u}{\partial F}\right)\bigg|_\tau + \frac{2}{r} F_0 (u_\tau - u_0) \frac{\partial u}{\partial F}\bigg|_\tau.
\end{aligned}$$

But

$$\frac{\partial u}{\partial E} + u\frac{\partial u}{\partial F} = -\frac{P'/c^2(1-u^2/c^2)^2}{(\rho + P/c^2)(1-P'u^2/c^4)}u$$

and

$$\frac{\partial u}{\partial F} = \frac{(1-u^2/c^2)(1+P'u^2/c^4)}{(\rho + P/c^2)(1-P'u^2/c^4)}.$$

Therefore

$$\frac{d}{d\tau}u_\tau = A(\tau)u_\tau + B(\tau)(u_\tau - u_0)$$

with $A(\tau) \geqslant 0$ and $B(\tau) \geqslant 0$. Thus by the comparison theorem we have $u_\tau - u_0 \geqslant 0$ and $du_\tau/d\tau \geqslant 0$. Hence $0 \leqslant u_0 \leqslant u_\tau$. Moreover, we have

$$\frac{F_\tau}{E_\tau} = \frac{(\rho_\tau + P_\tau/c^2)u_\tau}{\rho_\tau + P_\tau u_\tau^2/c^4} \geqslant u_\tau,$$

since $u_\tau^2/c^2 < 1$. On the other hand

$$\frac{F_\tau}{E_\tau} = \frac{F_0}{E_0}\frac{1-\frac{2}{r}u_0\tau}{1-\frac{2}{r}\frac{F_0}{E_0}\tau} \leqslant \frac{F_0}{E_0}\frac{1}{1-2\frac{F_0}{E_0}\tau}$$

and

$$\frac{F_0}{E_0} = \frac{1+P_0/\rho_0 c^2}{1+P_0 u_0^2/\rho_0 c^4}u_0 \leqslant (1+C_4/c^2)u_0 \leqslant C_5,$$

where $P_0/\rho_0 \leqslant C_4$. Thus

$$u_\tau \leqslant \frac{(1+C_4/c^2)}{1-2C_5\tau}u_0 \leqslant C_6,$$

where

$$C_6 = \frac{1+C_4/c^2}{1-2C_5 h}(1-\delta)c < c,$$

provided that $2C_5 h \leqslant \delta/2$ and $C_4/c^2 \leqslant \delta/2$. Summing up,

$$0 \leqslant u_0 \leqslant u_\tau \leqslant C_6 < c. \tag{3.2}$$

Let us observe ρ_τ. Since $F_\tau \leqslant F_0$ and $u_0 \leqslant u_\tau$,

$$\frac{\rho_\tau + P_\tau/c^2}{1-u_0^2/c^2}u_0 \leqslant \frac{\rho_\tau + P_\tau/c^2}{1-u_\tau^2/c^2}u_\tau \leqslant \frac{\rho_0 + P_0/c^2}{1-u_0^2/c^2}u_0.$$

Hence $\rho_\tau \leqslant \rho_0$. On the other hand, from (3.1),

$$\begin{aligned}\frac{1}{C_3}\rho_0 \leqslant E_\tau &= \frac{\rho_\tau + P_\tau u_\tau^2/c^4}{1-u_\tau^2/c^2}\\ &\leqslant \frac{\rho_\tau(1+P_\tau u_\tau^2/\rho_\tau c^2)}{1-u_\tau^2/c^2}\\ &\leqslant \frac{2\rho_\tau}{1-C_6^2/c^2},\end{aligned}$$

since $P_\tau/\rho_\tau \leqslant c^2$. Thus we have

$$\frac{1}{C_7}\rho_0 \leqslant \rho_\tau \leqslant \rho_0. \tag{3.3}$$

Let us go back to u_τ. From (3.3) we see

$$\frac{d}{d\tau}u_\tau \leqslant C_8\rho_0^{\gamma-1}/c^2 + C_9(u_\tau - u_0).$$

Hence we get

$$u_\tau \leqslant u_0 + C_{10}\rho_0^{\gamma-1}\tau/c^2. \tag{3.4}$$

Now we are ready to prove $w_\tau \leqslant w_0$. We look at

$$\frac{d}{d\tau}w_\tau = -\frac{2}{r}F_0\left(\left(\frac{\partial w}{\partial E} + u\frac{\partial w}{\partial F}\right)\bigg|_\tau - (u_\tau - u_0)\frac{\partial w}{\partial F}\bigg|_\tau\right).$$

Here from (3.3)

$$\frac{\partial w}{\partial E} + u\frac{\partial w}{\partial F} = \frac{\sqrt{P'}(1-\sqrt{P'}u/c^2)(1-u^2/c^2)}{(\rho+P/c^2)(1-P'u^2/c^4)} \geqslant \frac{1}{C_{11}}\rho_0^{\frac{\gamma-3}{2}}$$

and from (3.4)

$$0 \leqslant (u_\tau - u_0)\frac{\partial w}{\partial F}\bigg|_\tau \leqslant C_{12}\rho_0^{\gamma-2}\tau/c^2,$$

since

$$\frac{\partial w}{\partial F} = \frac{(1-\sqrt{P'}u/c^2)^2}{(\rho+P/c^2)(1-P'u^2/c^4)}.$$

Hence

$$\left(\frac{\partial w}{\partial E}+u\frac{\partial w}{\partial F}\right)\Big|_{\tau}-(u_\tau-u_0)\frac{\partial w}{\partial F}\Big|_{\tau}\geqslant\frac{1}{C_{11}}\rho_0^{\frac{\gamma-3}{2}}-C_{12}\rho_0^{\gamma-2}h/c^2$$
$$=\frac{1}{C_{11}}\rho_0^{\frac{\gamma-3}{2}}(1-C_{11}C_{12}\rho_0^{\frac{\gamma-1}{2}}h/c^2)$$
$$>0,$$

provided that $C_{11}C_{12}C_2^{\frac{\gamma-1}{2}}h/c^2<1$. Then $dw_\tau/d\tau\leqslant 0$. Next we look at z_τ. We have

$$\frac{d}{d\tau}z_\tau=\frac{2}{r}F_0\left(-\left(\frac{\partial z}{\partial E}+u\frac{\partial z}{\partial F}\right)\Big|_{\tau}+(u_\tau-u_0)\frac{\partial z}{\partial F}\Big|_{\tau}\right).$$

But

$$\frac{\partial z}{\partial E}+u\frac{\partial z}{\partial F}=-\frac{\sqrt{P'}(1+\sqrt{P'}u/c^2)(1-u^2/c^2)}{(\rho+P/c^2)(1-P'u^2/c^4)}<0$$

and

$$\frac{\partial z}{\partial F}=\frac{(1+u\sqrt{P'}/c^2)^2}{(\rho+P/c^2)(1-P'u^2/c^4)}>0.$$

Hence $dz_\tau/d\tau\geqslant 0$ and $z_\tau\geqslant z_0\geqslant 0$. This completes the proof. □

4. Proof of the main theorem

Let us construct approximate solutions by the Lax–Friedrichs scheme. Since the initial data U^0 are supposed to satisfy (1.7), we can find a sufficiently large B such that $U^0(r)\in\Sigma(0,B)$ for $r\geqslant 1$. Take the mesh lengths $\Delta=\Delta r$ and Δt such that $\Delta r/\Delta t=2\Lambda$, where

$$\Lambda>\sup\{|\lambda_1(U)|,|\lambda_2(U)|:U\in\Sigma(0,B)\}.$$

Here λ_1,λ_2 are characteristic values

$$\lambda_1=\frac{u-\sqrt{P'}}{1-\sqrt{P'}u/c^2},\quad \lambda_2=\frac{u+\sqrt{P'}}{1+\sqrt{P'}u/c^2}.$$

So the Courant–Friedrichs–Lewy condition will be satisfied.

We put

$$U^{\Delta}_{0,j} = \frac{1}{2\Delta}\int_{1+(j-1)\Delta}^{1+(j+1)\Delta} U^0(r)\,dr$$

for $j = 1, 2, \dots$. Then $U^{\Delta}_{0,j} \in \Sigma(0, B)$. Let $k \geqslant 2$ be even. Then we define $U^{\Delta}_0(r,t)$ for $1 + (k-1)\Delta \leqslant r < 1 + (k+1)\Delta, 0 \leqslant t < \Delta t$ as the solution of the Riemann problem to (2.1) with center $1 + k\Delta$ and data $U_L = U^{\Delta}_{0,k-1}, U_R = U^{\Delta}_{0,k+1}$. For $1 \leqslant r < 1 + \Delta, 0 \leqslant t < \Delta t$, we define $U^{\Delta}_0(r,t)$ as the solution of the Riemann problem to (2.1) with center 1 in the following manner. There is U^* on $\rho = 0$ from which $U^{\Delta}_{0,1}$ is connected by a 2-rarefaction wave ($z =$ *Constant*), and U^* and $U^{\Delta}_{0,0} = (0,0)$ is connected through the vacuum. Since $\Sigma(0,B)$ is an invariant region, we have $U^{\Delta}_0(r,t) \in \Sigma(0,B)$ for $1 \leqslant r, 0 \leqslant t < \Delta t$.

We put for $0 \leqslant t < \Delta t$

$$U^{\Delta}(r,t) = U^{\Delta}_0(r,t) + H(r, U^{\Delta}_0(r,t))t.$$

By Proposition 3.1 we have $U^{\Delta}(r,t) \in \Sigma(0,B)$, provided that $\Delta t \leqslant h_1$ and $1/c^2 \leqslant \varepsilon_1$.

Suppose that the approximate solution $U^{\Delta}(r,t)$ has been constructed for $1 \leqslant r, 0 \leqslant t < (n-1)\Delta t$ so that $U^{\Delta}(r,t) \in \Sigma(0,B)$. Then we put

$$U^{\Delta}_{n-1,j} = \frac{1}{2\Delta}\int_{1+(j-1)\Delta}^{1+(j+1)\Delta} U^{\Delta}(r,(n-1)\Delta t - 0)\,dr$$

for $j = 1, 2, \dots$. But if n is even we put

$$U^{\Delta}_{n-1,1} = U^{\Delta}_{n-1,2} = \frac{1}{3\Delta}\int_{1}^{1+3\Delta} U^{\Delta}(r,(n-1)\Delta t - 0)\,dr.$$

We have $U^{\Delta}_{n-1,j} \in \Sigma(0,B)$.

Given $k \geqslant 2$ such that $n + k$ is odd, we define $U^{\Delta}_0(r,t)$ for $1 + (k-1)\Delta \leqslant r < 1 + (k+1)\Delta, (n-1)\Delta t \leqslant t < n\Delta t$ as the solution of the Riemann problem to (2.1) with center $1 + k\Delta$ and data $U_L = U^{\Delta}_{n-1,k-1}, U_R = U^{\Delta}_{n-1,k+1}$. For $1 \leqslant r < 1 + \Delta$, $(n-1)\Delta t \leqslant t < n\Delta t$, $U^{\Delta}_0(r,t)$ is defined as $n = 1$. Then $U^{\Delta}_0(r,t) \in \Sigma(0,B)$. We put for $(n-1)\Delta t \leqslant t < n\Delta t$

$$U^{\Delta}(r,t) = U^{\Delta}_0(r,t) + H(r, U^{\Delta}_0(r,t))(t - (n-1)\Delta t).$$

By Proposition 3.1 we have $U^{\Delta}(r,t) \in \Sigma(0,B)$ provided that $\Delta t \leqslant h_1, 1/c^2 \leqslant \varepsilon_1$ as long as $(n-1)\Delta t \leqslant t < n\Delta t$.

Thus we can construct the approximate solution $U^{\Delta}(r,t)$ confined to $\Sigma(0,B)$.

Consider $n \leqslant T/\Delta t$ for arbitrarily fixed T. The following properties can be proved in the same manner to Makino–Takeno [5, Propositions 1,3], in which we regard $H = \tilde{H}$.

Proposition 4.1.

$$\sum_n \int_1^R |U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 \, dr \leqslant C.$$

To show this we note $|H(r, U)| \leqslant C$ and $|D\eta^*(U)| \leqslant C$ for $U \in \Sigma(0, B)$.

Proposition 4.2 (Makino and Takeno [5, Proposition 3]).

$$\sum_n \int_{(n-1)\Delta t}^{n\Delta t} \int_1^R |U_0^\Delta(r, t) - U_0^\Delta(r, n\Delta t - 0)|^2 \, dr \, dt = O(\Delta).$$

Proposition 4.3. *For any test function $\Phi = (\phi_1, \phi_2)^T \in C_0^\infty([1, R) \times [0, T))$ such that $\phi_2|_{r=1} = 0$ we have*

$$\int_0^T \int_1^\infty (\Phi_t U^\Delta + \Phi_r f(U^\Delta) + \Phi H(r, U^\Delta)) \, dr \, dt + \int_1^\infty \Phi(r, 0) U_0^\Delta(r) \, dr = O(\Delta^{1/2}).$$

To show this we note $|D_U H| \leqslant C$, which is easy to see since

$$\begin{aligned}
&\frac{\partial H_1}{\partial E} = 0, \quad \frac{\partial H_1}{\partial F} = -\frac{2}{r}, \\
&\frac{\partial H_2}{\partial E} = -\frac{2}{r} F \frac{\partial u}{\partial E} = \frac{2}{r} \frac{(1 - P'/c^2)u^2}{1 - P'u^2/c^4}, \\
&\frac{\partial H_2}{\partial F} = -\frac{2}{r} u - \frac{2}{r} F \frac{\partial u}{\partial F} = -\frac{4}{r} \frac{u}{1 - P'u^2/c^4}.
\end{aligned}$$

Proposition 4.4. *For any Darboux entropy–entropy flux (η, q) the divergence $\eta(U^\Delta)_t + q(U^\Delta)_r$ is relatively compact in $H^{-1}_{\text{loc}}(\Omega)$, Ω being a bounded open set of $[1, \infty) \times [0, T)$.*

Proof. The proof is almost the same as that of [5, Proposition 1] and sketched as follows. Suppose that *supp* $U \in [1, R) \times [0, T]$ $(R > \Lambda T + R(0))$. Let ϕ be a test function on Ω. Then we can write

$$\int_0^T \int_1^R (\eta(U^\Delta)\phi_t + q(U^\Delta)\phi_r) \, dr \, dt = (L_0 + L_1 + L_2 + L_3 + L_4)\phi,$$

where

$$L_0\phi = \int_0^T \int_1^R ((\eta(U^\Delta) - \eta(U_0^\Delta))\phi_t + (q(U^\Delta) - q(U_0^\Delta))\phi_r)\, dr\, dt$$

$$L_1\phi = \int_1^R \phi(r,T)\eta(U_0^\Delta(r,T))\, dr - \int_1^R \phi(r,0)\eta(U_0^\Delta(r,0))\, dr,$$

$$L_2\phi = \sum_n \int_1^R \phi(r,n\Delta t)(\eta(U_0^\Delta(r,n\Delta t-0)) - \eta(U_0^\Delta(r,n\Delta t+0)))\, dr,$$

$$L_3\phi = \int_0^T \sum_{shock} (\sigma[\eta]_0 - [q]_0)\phi\, dt,$$

$$L_4\phi = -\int_0^T q(U_0^\Delta(1,t))\phi(1,t)\, dt.$$

Moreover we put $L_2 = L_{21} + L_{22} + L_{23}$ with

$$L_{21}\phi = \sum_{n,k} \phi_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\eta(U^\Delta(r,n\Delta t-0)) - \eta(U_0^\Delta(r,n\Delta t+0)))\, dr,$$

$$L_{22}\phi = \sum_n \int_1^R (\eta(U_0^\Delta(r,n\Delta t-0)) - \eta(U^\Delta(r,n\Delta t-0)))\phi(r,n\Delta t)\, dr,$$

$$L_{23}\phi = \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\eta(U^\Delta(r,n\Delta t-0)) - \eta(U_0^\Delta(r,n\Delta t+0)))(\phi(r,n\Delta t) - \phi_{n,k})\, dr,$$

where $\phi_{n,k} = \phi(1+k\Delta, n\Delta t)$. The summation is taken over n and k such that $n+k$ is odd. (When n is odd and $k=2$, then $1+(k-1)\Delta$ stands for 1.) Substituting $\eta = \eta^*$, $q = q^*$ and $\phi = 1$, direct computation gives

$$\sum_n \int_1^R \int_0^1 |D\eta^*| d\theta H \Delta t\, dr \leqslant C \quad \text{and} \quad \int_0^T \sum_{shock} (\sigma[\eta^*]_0 - [q^*]_0)\, dt \leqslant C.$$

According to Proposition 4.1, we obtain the following estimates:

$$|L_1\phi| \leqslant C\|\phi\|_{C(\Omega)}, \quad |L_3\phi| \leqslant C'\|\phi\|_{C(\Omega)},$$

$$|L_{21}\phi| \leqslant C'\|\phi\|_{C(\Omega)}, \quad |L_{22}\phi| \leqslant C'\|\phi\|_{C(\Omega)},$$

$$|L_{23}\phi| \leqslant C''\Delta^{\alpha-1/2}\|\phi\|_{C(\Omega)}, \quad \text{for } \tfrac{1}{2} < \alpha < 1.$$

and

$$|L_4\phi| \leqslant C\|\phi\|_{C(\Omega)}.$$

On the other hand, since $0\leqslant\rho\leqslant C$ and $|u|\leqslant C$, $L_1+L_2+L_3+L_4$ is bounded in $W^{-1,\beta}(\Omega)$ $(\beta>1)$. Hence $L_1+L_2+L_3+L_4$ is relatively compact in $H^{-1}_{\rm loc}(\Omega)$ by the argument of Ding et al. [3] and

$$|L_0\phi|\leqslant C\Delta||\phi||_{H^1(\Omega)}.$$

Therefore, $L_0+L_1+L_2+L_3+L_4$ is relatively compact in $H^{-1}_{\rm loc}(\Omega)$. The proof is complete. □

Therefore by using the Darboux entropies $\eta_1,\eta_2,,\dots,\eta_6$ defined in Hsu et al. [4] we can show that there is a sequence $\Delta_v\to 0$ such that U^{Δ_v} converge almost everywhere (r,t). The proof is same to that of one dimensional problem. By Proposition 4.3, the limit is a weak solution. This completes the proof of Theorem 1.

The weak solution we have constructed enjoys the entropy condition in the following sense.

Theorem 2. *If (η,q) is a Darboux entropy–entropy flux such that η is convex and if ϕ is a non-negative test function in $C_0^\infty((1,\infty)\times(0,\infty))$, then*

$$\int_0\int_1(\phi_t\eta(U)+\phi_r q(U)+\phi D\eta(U)H(r,U))\,dr\,dt\geqslant 0.$$

Proof. We consider

$$I^\Delta=\int_0\int_1(\phi_t\eta(U^\Delta)+\phi_r q(U^\Delta)+\phi D\eta(U^\Delta)H(r,U^\Delta))\,dr\,dt.$$

We must show $\lim_{\Delta\to 0}I^\Delta\geqslant 0$. We have $I^\Delta=I_1+I_2$, where

$$I_1=\int\int(\phi_t\eta(U_0^\Delta)+\phi_r q(U_0^\Delta)+\phi D\eta(U_0^\Delta)H(r,U_0^\Delta))\,dr\,dt,$$

$$\begin{aligned}I_2=\int\int\ &\phi_t(\eta(U^\Delta)-\eta(U_0^\Delta))+\phi_r(q(U^\Delta)-q(U_0^\Delta))\\&+\phi(D\eta(U^\Delta)H(r,U^\Delta)-D\eta(U_0^\Delta)H(r,U_0^\Delta))\,dr\,dt.\end{aligned}$$

Since $U^\Delta-U_0^\Delta=O(\Delta t)$, we have $I_2\to 0$ as $\Delta\to 0$ by the Lebesgue's dominated convergence theorem. So we consider I_1. We have $I_1=I_{11}+I_{12}+I_{13}$, where

$$I_{11}=\sum_n\int_1^R\phi(r,n\Delta t)(\eta(U_0^\Delta(r,n\Delta t-0))-\eta(U_0^\Delta(r,n\Delta t+0)))\,dr,$$

$$I_{12}=\int_0^T\sum_{shock}\phi(\sigma[\eta]-[q])\,dt,$$

$$I_{13}=\int_0^T\int_1^R\phi D\eta(U_0^\Delta)H(r,U_0^\Delta)\,dr\,dt.$$

Since η is convex, $I_{12} \geqslant 0$ by Hsu et al. [4, Proposition 4.3]. We have $I_{11} = I_{111} + I_{112}$, where

$$I_{111} = \sum_{n,j} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} (\phi(r, n\Delta t) - \phi_{jn})(\eta(U_0^\Delta(n\Delta t - 0)) - \eta(U_0^\Delta(n\Delta t + 0)))\, dr,$$

$$I_{112} = \sum_{n,j} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} \phi_{jn}(\eta(U_0^\Delta(n\Delta t - 0)) - \eta(U_0^\Delta(n\Delta t + 0)))\, dr,$$

$$\phi_{jn} = \phi(1 + j\Delta, n\Delta t).$$

The summation is taken over n, j such that $n + j$ is odd. We see $I_{111} = O(\Delta^{1/2})$ from $|D\eta| \leqslant C$ and Proposition 4.1. So we must study $J = I_{112} + I_{13}$. Since η is convex, we see

$$I_{112} \geqslant I_{112}' = \sum_{n,j} \phi_{jn} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} D\eta(U_0^\Delta(n\Delta t + 0))(U_0^\Delta(n\Delta t - 0) - U_0^\Delta(n\Delta t + 0))\, dr.$$

But for $1 + (j-1)\Delta \leqslant r < 1 + (j+1)\Delta$, we have

$$U_0^\Delta(r, n\Delta t + 0) = \frac{1}{2\Delta} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} U_0^\Delta(s, n\Delta t - 0)\, ds \\ - \frac{\Delta t}{2\Delta} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} H(s, U_0^\Delta(s, n\Delta t - 0))\, ds.$$

Hence

$$I_{112}' = -\sum_{n,j} \phi_{jn} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} D\eta(U_0^\Delta(r, n\Delta t + 0) H(r, U_0^\Delta(r, n\Delta t - 0))\, dr \Delta t.$$

Thus

$$I_{112}' + I_{13} = \sum_{n,j} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} (\phi D\eta(U_0^\Delta) H(r, U_0^\Delta) \\ - \phi_{jn} D\eta(U_0^\Delta(r, n\Delta t + 0)) H(r, U_0^\Delta(r, n\Delta t - 0)))\, dr\, dt.$$

Now we have $I_{112}{}' + I_{13} = J_1 + J_2$, where

$$J_1 = \sum_{n,j} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} (\phi - \phi_{jn})(D\eta(U_0^\Delta)H(r, U_0^\Delta) - D\eta(U_0^\Delta(r, n\Delta t + 0))H(r, U_0^\Delta(r, n\Delta t - 0)))\, dr\, dt,$$

$$J_2 = \sum_{n,j} \phi_{jn} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(j-1)\Delta}^{1+(j+1)\Delta} (D\eta(U_0^\Delta)H(r, U_0^\Delta) - D\eta(U_0^\Delta(r, n\Delta t + 0))H(r, U_0^\Delta(r, n\Delta t - 0)))\, dr\, dt.$$

We see $J_1 = O(\Delta)$ and $|J_2| \leqslant CJ_3$, where

$$J_3 = \sum_{n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1}^{R} |D\eta(U_0^\Delta)H(r, U_0^\Delta) - D\eta(U_0^\Delta(r, n\Delta t + 0))H(r, U_0^\Delta(r, n\Delta t - 0))|\, dr\, dt.$$

We see $J_3 \leqslant J_{31} + J_{32}$, where

$$J_{31} = \sum_{n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1}^{R} |D\eta(U_0^\Delta(r, n\Delta t + 0)) \times (H(r, U_0^\Delta(r, n\Delta t - 0)) - H(r, U_0^\Delta(r, n\Delta t + 0)))|\, dr\, dt,$$

$$J_{32} = \sum_{n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1}^{R} |Q(r, U_0^\Delta(r, t)) - Q(r, U_0^\Delta(r, n\Delta t + 0))|\, dr\, dt,$$

$$Q(r, U) = D\eta(U)H(r, U).$$

Since $|D\eta| \leqslant C, |D_U H| \leqslant C$, we see

$$J_{31} \leqslant C \sum_{n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1}^{R} |U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|\, dr\, dt = O(\Delta^{1/2})$$

by Proposition 4.1. On the other hand, since

$$|D_U H| \leqslant C, \quad |D\eta| \leqslant C, \quad |H| \leqslant C\rho, \quad |D^2\eta| \leqslant C/\rho$$

(see [4, Section 6]), we have $|D_U Q| \leqslant C$. Hence

$$J_{32} \leqslant C \sum_{n} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1}^{R} |U_0^\Delta(r, t) - U_0^\Delta(r, n\Delta t + 0)|\, dr\, dt = O(\Delta^{1/2})$$

by Propositions 4.1 and 4.2. This completes the proof. □

Remark 4.1. Let us apply the entropy condition to the standard entropy–entropy flux (η^*, q^*). Then a direct calculation leads us to

$$D\eta^* H = -\frac{2}{r} q^*.$$

Hence the entropy condition reads

$$\eta^*_t + q^*_r + \frac{2}{r} q^* \leqslant 0 \quad \text{or} \quad \eta^*_t + \frac{1}{r^2}(r^2 q^*)_r \leqslant 0.$$

Of course the equality holds for smooth solutions.

5. Problem including the co-ordinate origin

In this section we consider Eq. (1.1) on $r > 0$ with the initial condition

$$\rho|_{t=0} = \rho^0(r), \quad u|_{t=0} = u^0(r) \tag{5.1}$$

given for $r > 0$ and without boundary conditions. Our goal is

Theorem 3. *For any C_0 there is $\varepsilon_1(C_0) > 0$ such that if $\rho^0(r) \geqslant 0$, $0 \leqslant u^0(r) < c$ and*

$$\int_0^{\rho^0(r)} \frac{\sqrt{P'}}{\rho + P/c^2}\, d\rho \leqslant \frac{c}{2} \log \frac{c + u^0(r)}{c - u^0(r)} \leqslant C_0, \tag{5.2}$$

and if $1/c^2 \leqslant \varepsilon_1(C_0)$, then there exists a global weak solution of (1.1) *and* (5.1).

Here a weak solution $U(r,t)$ means a function which satisfies

$$\int_0^T \int_0^\infty (\Phi_t U + \Phi_r f(U) + \Phi H(r,U))\, dr\, dt + \int_0^\infty \Phi(r,0) U^0(r)\, dr = 0,$$

for any test function $\Phi \in C_0^\infty((0,\infty) \times [0,T))$.

In order to prove this theorem the key lemma Proposition 3.1 is replaced by the following

Proposition 5.1. *Suppose $(w_0, z_0) \in \Sigma(0, B)$. Let $E_0 = E(w_0, z_0), F_0 = F(w_0, z_0)$ and consider $U_\tau = U_0 + H(r, U_0)\tau$, that is,*

$$E_\tau = E_0 - \frac{2}{r} F_0 \tau, \quad F_\tau = F_0 - \frac{2}{r} F_0 u_0 \tau,$$

where $\tau \geqslant 0$. Then there are positive numbers h_2 and ε_2 and a sufficiently large integer J depending only upon B such that if $J\Delta \leqslant r, 0 \leqslant \tau \leqslant \Delta/2\Lambda \leqslant h_2$, and if $1/c^2 \leqslant \varepsilon_2$, then

$U_\tau = U(w_\tau, z_\tau)$ with $(w_\tau, z_\tau) \in \Sigma(0, B)$. Here

$$\Lambda = 1 + \sup\{|\lambda_1(U)|, |\lambda_2(U)| : U \in \Sigma(0, B)\}.$$

Proof. The proof is similar to that of Proposition 3.1. The major change is to check that $E_\tau/E_0 = 1 - \frac{2}{r}\frac{F_0}{E_0}\tau, F_\tau/F_0 = 1 - \frac{2}{r}u_0\tau$ are estimated from below by a positive number. This can be done as follows. Since

$$\frac{F_0}{E_0} = \frac{\rho + P/c^2}{\rho + Pu^2/c^4}u_0 \leqslant \frac{3}{2}u_0$$

provided that c is sufficiently large, it is sufficient to estimate $\frac{\tau}{r}u_0$. But

$$x_0 = \frac{c}{2}\log\frac{c + u_0}{c - u_0} = \frac{1}{2}(w_0 + z_0) \leqslant B$$

implies $0 \leqslant u_0 \leqslant B_c$. Hence, when $r \geqslant J\Delta$ and $\tau \leqslant \Delta/2\Lambda$, we have

$$\frac{\tau}{r}u_0 \leqslant \frac{B_c}{2\Lambda J} \leqslant \frac{1}{4},$$

provided that J is sufficiently large. The $1 - \frac{2}{r}\frac{F_0}{E_0}\tau \geqslant \frac{1}{4}$ and $1 - \frac{2}{r}u_0\tau \geqslant \frac{1}{2}$. This completes the proof.

Now we construct approximate solutions. Put

$$U^\Delta_{j,0} = \frac{1}{2\Delta}\int_{(j-1)\Delta}^{(j+1)\Delta} \chi(r)U^0(r)\,dr$$

for $j = 1, 2, \ldots$, where

$$\chi(r) = \begin{cases} 0 & \text{if } 0 \leqslant r \leqslant 2J\Delta, \\ 1 & \text{if } 2J\Delta < r. \end{cases}$$

Then $U^\Delta_{j,0} \in \Sigma(0, B)$ and $U^\Delta_{j,0} = 0$ for $j \leqslant 2J - 1$.

Let $k \geqslant 2$ be even. Then we define $U^\Delta_0(r, t)$ for $(k-1)\Delta \leqslant r \leqslant (k+1)\Delta, 0 \leqslant t < \Delta t$ as the solution of the Riemann problem to (2.1) with center $r = k\Delta$ and data $U_L = U^\Delta_{k-1,0}, U_R = U^\Delta_{k+1,0}$. Note $U^\Delta_0(r, t) = 0$ for $r \leqslant (2J-2)\Delta$. Since $\Sigma(0, B)$ is an invariant region, we have $U^\Delta_0(r, t) \in \Sigma(0, B)$ for $0 < r, 0 \leqslant t < \Delta t$.

We put for $0 \leqslant t < \Delta t$

$$U^\Delta(r, t) = U^\Delta_0(r,) + H(r, U^\Delta_0(r, t))t.$$

By Proposition 5.1 we have $U^\Delta(r, t) \in \Sigma(0, B)$ for $0 < r, 0 \leqslant t < \Delta t$. In fact $U^\Delta(r, t) = U^\Delta_0(r, t) = 0$ for $r \leqslant (2J-2)\Delta$ and $2J - 2 \geqslant J$.

Suppose that the approximate solutions $U^{\Delta}(r,t)$ has been constructed for $0<r, 0\leqslant t<(n-1)\Delta t$ so that $U^{\Delta}(r,t)\in\Sigma(0,B)$. Then we put

$$U^{\Delta}_{j,n-1}=\frac{1}{2\Delta}\int_{(j-1)\Delta}^{(j+1)\Delta}\chi(r)U^{\Delta}(r,(n-1)\Delta t-0)\,dr$$

for $j=1,2,\ldots$. Of course $U^{\Delta}_{j,n-1}\in\Sigma(0,B)$. Given $k\geqslant 2$ such that $n+k$ is odd, we define $U^{\Delta}_0(r,t)$ for $(k-1)\Delta\leqslant r<(k+1)\Delta, (n-1)\Delta t\leqslant t<n\Delta t$ as the solution of the Riemann problem to (2.1) with center $r=k\Delta$ and data $U_L=U^{\Delta}_{k-1,n-1}, U_R=U^{\Delta}_{k+1,n-1}$. Then $U^{\Delta}_0(r,t)\in\Sigma(0,B)$ and $U^{\Delta}_0(r,t)=0$ for $0<r\leqslant(2J-2)\Delta$. We put for $(n-1)\Delta t\leqslant t<n\Delta t$

$$U^{\Delta}(r,t)=U^{\Delta}_0(r,t)+H(r,U^{\Delta}_0(r,t))(t-(n-1)\Delta t).$$

By Proposition 5.1 we have $U^{\Delta}(r,t)\in\Sigma(0,B)$ while $U^{\Delta}(r,t)=U^{\Delta}_0(r,t)=0$ for $r\leqslant(2J-2)\Delta$. Thus we can construct approximate solutions.

Consider $n\leqslant T/\Delta t$ for fixed T. Although we do not know whether $-H_i(r,U^{\Delta}_0)$ $(\geqslant 0), i=1,2$ are bounded uniformly with respect to Δ or not, we have the following:

Proposition 5.2. *For $i=1,2$, we have*

$$-\sum_n\int_0^R H_i(r,U^{\Delta}_0(r,n\Delta t-0))\,dr\Delta t\leqslant C,$$

therefore

$$\sum_n\int_0^R|H(r,U^{\Delta}_0(r,n\Delta t-0))|\,dr\Delta t\leqslant C.$$

Proof. For $\phi=1$, we have

$$\begin{aligned}0&=\int_0^T\int_0^R(E^{\Delta}_0\phi_t+F^{\Delta}_0\phi_r)\,dr\,dt\\&=L_1+L_2+L_3,\end{aligned}$$

where

$$\begin{aligned}L_1&=\int_0^R E^{\Delta}_0(r,T)\,dr-\int_0^R E^{\Delta}_0(r,0)\,dr,\\L_2&=\sum_n\int_0^R(E^{\Delta}_0(r,n\Delta t-0)-E^{\Delta}_0(r,n\Delta t+0))\,dr,\\L_3&=\int_0^T\sum_{shock}(\sigma[E^{\Delta}_0]-[F^{\Delta}_0])\,dt.\end{aligned}$$

By the Rankine–Hugoniot condition we have $L_3 = 0$. Of course $|L_1| \leqslant C$. Here

$$\begin{aligned} L_2 &= \sum_n \sum_{j \leqslant 2J} \int_{(j-1)\Delta}^{(j+1)\Delta} E_0^\Delta(r, n\Delta t - 0)\, dr \\ &\quad - \sum_n \int_0^R H_1(r, U_0^\Delta(r, n\Delta t - 0)\Delta t\, dr \\ &= O(1) - \sum_n \int_0^R H_1(r, U_0^\Delta(r, n\Delta t - 0)\Delta t\, dr. \end{aligned}$$

This completes the proof for $i = 1$. The proof for $i = 2$ is similar by starting with

$$0 = \int \int (F_0^\Delta 1_t + G_0^\Delta 1_r)\, dr\, dt.$$

This completes the proof. □

Using this estimate, the following properties can be proved in the same manner to Makino–Takeno [5].

Proposition 5.3.

$$\sum_n \int_0^R |U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2\, dr \leqslant C.$$

Proposition 5.4.

$$\sum_n \int_{(n-1)\Delta t}^{n\Delta t} \int_0^R |U_0^\Delta(r, t) - U_0^\Delta(r, n\Delta t - 0)|^2\, dr\, dt = O(\Delta).$$

Proposition 5.5. *For any test function* $\Phi = (\phi_1, \phi_2)^T \in C_0^\infty((0, R) \times [0, T))$ *we have*

$$\int_0^T \int_0^\infty (\Phi_t U^\Delta + \Phi_r f(U^\Delta) + \Phi H(r, U^\Delta))\, dr\, dt + \int_0^\infty \Phi(r, 0) U_0^\Delta(r)\, dr = O(\Delta^{1/2}).$$

Note that we assume that the support of the test function does not touch $r = 0$. The remaining proof of Theorem 3 is just parallel to that of Theorem 1.

Remark 5.1. It is difficult to remove the restriction that $1/c^2$ is sufficiently small even if we consider the one-dimensional motion, because this restriction is needed to guarantee the required properties of Darboux entropies used to apply the compensated compactness theory. There is no telling what will happen if the initial data are large and c is small. The question is open for future studies.

Remark 5.2. Also it is difficult to remove the assumption (1.7) or (5.2). There is no telling what happen if the initial velocity is large and negative, that is, the initial flow is inward coming to the origin. We are not sure but solutions could blow up after a finite time.

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