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Lower confidence bounds for C_{PU} and C_{PL} based on multiple samples with application to production yield assurance

W. L. PEARN†*, M. H. SHU‡ and B. M. HSU§

For stably normal processes with one-sided specification limits, capability indices C_{PU} and C_{PL} have been used to provide numerical measures on production yield assurance. Statistical properties of the estimators of C_{PU} and C_{PL} have been investigated extensively for cases with a single sample. It is shown that for multiple samples, the uniformly minimum-variance unbiased estimators of C_{PU} and C_{PL} are consistent and asymptotically efficient. Based on the uniformly minimum-variance unbiased estimators, an algorithm is developed with an efficient program using a direct search method to compute the lower confidence bounds for C_{PU} and C_{PL}. The lower confidence bounds convey critical information to the minimum capability of a process, providing a necessary yield assurance of production. The lower confidence bounds are tabulated for some commonly used capability requirement so that engineers/practitioners can use them for their in-plant applications. An example of a high-speed buffer amplifier is presented to illustrate the practicality of the approach to data collected from the factories for production yield assurance.

1. Introduction

Process capability indices have been used in the manufacturing industry to measure the capability of a process to reproduce items satisfying the requirement preset by the product designers or customer's specifications. Several capability indices, including C_p, C_{PU}, C_{PL}, C_{pk}, C_{pm} and C_{pmk}, are developed for this purpose (Kane 1986, Chan *et al.* 1988, Cheng 1992, 1994–95, Pearn *et al.* 1992). Those indices essentially compare the predefined product specifications with the actual process distribution characteristics, which have been defined as follows:

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma}, C_{PU} = \frac{\text{USL} - \mu}{3\sigma},$$
$$C_{PL} = \frac{\mu - \text{LSL}}{3\sigma}, C_{pk} = \min\left\{\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$
$$C_{pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}},$$
$$C_{pmk} = \min\left\{\frac{\text{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \text{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\},$$

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where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean, σ is the process standard deviation (overall process variation) and T is the target value.

The indices C_p, C_{pk}, C_{pm} and C_{pmk} are appropriate for a product with two-sided specification limits, where both USL and LSL are required. However, the indices C_{PU} and C_{PL} are designed specifically for a product with a one-sided specification limit, and only one USL and LSL is required in this case. Many quality/reliability and statistics literatures have addressed the statistical properties of the estimators of C_{PU} and C_{PL}, and studied their industrial applications based on a single sample. Examples include Chou and Owen (1989) for obtaining the sampling distributions and other statistical properties, Chou (1994) for developing a procedure for selecting better suppliers, Pearn and Chen (2002) for obtaining the uniformly minimum-variance unbiased estimator (UMVUE) and developing a test based on the UMVUE, Lin and Pearn (2002) for implementing the statistical testing with application to capability determination of the voltage level translator, and Pearn and Shu (2002) for proposing an algorithm to calculate the lower confidence bounds (LCBs) with application to low drop-out regulators. However, their investigations on C_{PU} and C_{PL} are restricted to a single sample.

In practice, manufacturing information about product quality characteristics is often derived from multiple samples rather than from a single sample, particularly when a daily-based production control plan is implemented for monitoring process stability. The purpose of the present paper is to consider the capability estimation and testing of the one-sided capability indices C_{PU} and C_{PL} for multiple samples with variable sample sizes, and to apply the proposed LCB approach to real-world manufacturing applications for production yield assurance.

2. Capability requirements for production processes

In current practice (Kotz and Lovelace 1998), a process is called inadequate if $C_I < 1.00$, where $C_I = C_{PU}$ or C_{PL}; it indicates that the process is not adequate with respect to the production tolerances; either the process variation, σ^2, needs to be reduced or the process mean, μ, needs to be shifted closer to the target value, T. A process is called marginally capable if $1.00 \leq C_I < 1.33$; it indicates that caution needs to be taken about the process distribution and some process control is required. A process is called satisfactory if $1.33 \leq C_I < 1.67$; it indicates that process quality is satisfactory, material substitution may be allowed, and no stringent quality control is required. A process is called excellent if $1.67 \leq C_I < 2.00$; it indicates that process quality exceeds satisfactory. Finally, a process is called super if $C_I \geq 2.00$. Table 1 summarizes the above five conditions and the corresponding C_I values. However, Montgomery (2001) recommended some minimum quality requirements on C_{PU} and C_{PL} (table 2) for specific process types that must run under some designated capability conditions. Therefore, it would be desirable to determine a bound that practitioners would be expected to find the true value of the process capability no less than the bound value with certain level of confidence.

For normally distributed processes with one-sided specification limit USL, the process yield $P(X < \text{USL})$ is:

$$P\left(\frac{X-\mu}{3\sigma} < \frac{\text{USL}-\mu}{3\sigma}\right) = P\left(\frac{1}{3}Z < C_{PU}\right) = P(Z < 3C_{PU}) = \Phi(3C_{PU}),$$

Quality condition	C_I values
Inadequate	$C_I < 1.00$
Marginally capable	$1.00 \leq C_I < 1.33$
Satisfactory	$1.33 \leq C_I < 1.67$
Excellent	$1.67 \leq C_I < 2.00$
Super	$2.00 \leq C_I$

Table 1. Some commonly used capability requirement and quality conditions.

Index value	Production process types
1.25	Existing processes
1.45	New processes, orexisting processes on safety, strength, or critical parameters
1.6	New processes on safety, strength, or critical parameters

Table 2. Some minimum capability requirements of C_{PU} and C_{PL} for new and special processes.

C_{PU} or C_{PL}	NCPPM	C_{PU} or C_{PL}	NCPPM
1.00	1349.90	1.45	6.81
1.15	280.29	1.60	0.7933
1.25	88.42	1.67	0.2722
1.33	33.04	2.00	0.0010

Table 3. Various C_{PU} or C_{PL} values and the corresponding NCPPM.

where Z is the standard normal distribution $N(0, 1)$. Similarly, for normally distributed processes with one-sided specification limit LSL, the process yield $P(X > \text{LSL})$ can be obtained as follows:

$$P\left(\frac{\mu - X}{3\sigma} < \frac{\mu - \text{LSL}}{3\sigma}\right) = P\left(-\frac{1}{3}Z < C_{\text{PL}}\right) = P(Z > -3C_{\text{PL}}) = \Phi(3C_{\text{PL}}).$$

Therefore, the corresponding non-conforming units in parts per million (NCPPM) for a well-controlled normally distributed process can be calculated, exactly, as $\text{NCPPM} = 10^6 \times [1 - \Phi(3C_{\text{I}})]$. Table 3 shows various C_{PU} and C_{PL} values and the corresponding NCPPM. Consequently, the production yield for usual existing processes should target no more than 88 NCPPM, noting that $\text{NCPPM} \leq 200$ is the common standard used in most electronic industries for products with two-sided specifications. The production yield for newly set-up processes on safety, strength or with critical parameters, however, should target no more than 0.8 NCPPM, a more stringent requirement set for possible mean shift or variation change.

3. Estimating C_{PU} and C_{PL} based on multiple samples

To estimate the indices C_{PU} and C_{PL} in the presence of single samples, Chou and Owen (1989) considered the following natural estimators of C_{PU} and C_{PL}:

$$\hat{C}_{\mathrm{PU}} = \frac{\mathrm{USL} - \overline{X}}{3S}, \ \hat{C}_{\mathrm{PL}} = \frac{\overline{X} - \mathrm{LSL}}{3S},$$

where n is the sample size, $\overline{X} = \sum_{i=1}^{n} x_i/n$, and $S = \left[(n-1)^{-1}\sum_{i=1}^{n}(x_i - \overline{X})^2\right]^{1/2}$ are conventional estimators of μ and σ, which may be obtained from a process that is demonstrably stable (in control). Chou and Owen (1989) showed that under normality assumption, the estimators $3\sqrt{n}\,\hat{C}_{\mathrm{PU}}$ and $3\sqrt{n}\,\hat{C}_{\mathrm{PL}}$ are distributed as $t(n-1, \delta_{\mathrm{U}})$ and $t(n-1, \delta_{\mathrm{L}})$ respectively, a non-central t distribution with $n-1$ degrees of freedom and non-centrality parameters $\delta_{\mathrm{U}} = 3\sqrt{n}\,\hat{C}_{\mathrm{PU}}$ and $\delta_{\mathrm{L}} = 3\sqrt{n}\,\hat{C}_{\mathrm{PL}}$. However, both estimators are biased. Pearn and Chen (2002) considered the indices C_{PU} and C_{PL} and obtained their UMVUEs. Lin and Pearn (2002) developed efficient SAS/Maple computer programs for calculating the critical values and the p values using those UMVUEs for capability testing. Pearn and Shu (2002) proposed an algorithm for calculating the exact LCBs for C_{PU} and C_{PL}.

Kirmani *et al.* (1991) indicated that a common practice of the process capability estimation in most manufacturing industries is first to implement a daily-based data collection program for monitoring/controlling the process stability, then to analyse the past 'in control' data. Following the designated sampling plan, multiple samples of m_{s} groups each of with sizes n_i, $(x_{i1}, x_{i2}, \ldots, x_{in_i})$, are chosen randomly from a stable process which follows a normal distribution $N(\mu, \sigma^2)$ for $i = 1, 2, \ldots, m_{\mathrm{s}}$. We consider the following natural estimators of C_{PU} and C_{PL}. Let $\overline{X}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $[S_i = (n_i - 1)^{-1}\sum_{j=1}^{n_i}(x_{ij} - \overline{X}_i)^2]^{1/2}$ be the ith sample mean and the sample standard deviation, respectively. Then, $\overline{\overline{X}} = \sum_{i=1}^{m_{\mathrm{s}}} \overline{X}_i/m_{\mathrm{s}}$ and $S_{\mathrm{p}}^2 = \sum_{i=1}^{m_{\mathrm{s}}}(n_i - 1)\, S_i^2 / \sum_{i=1}^{m_{\mathrm{s}}}(n_i - 1)$ are the unbiased estimators of μ and σ^2, respectively, and the estimators of C_{PU} and C_{PL} can be written as:

$$\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} = \frac{b_{N-m_{\mathrm{s}}}(\mathrm{USL} - \overline{\overline{X}})}{3S_{\mathrm{p}}}, \ \tilde{C}_{\mathrm{PL}}^{M} = \frac{b_{N-m_{\mathrm{s}}}(\overline{\overline{X}} - \mathrm{LSL})}{3S_{\mathrm{p}}}.$$

where $N = \sum_{i=1}^{m_{\mathrm{s}}} n_i$ and $b_{N-m_{\mathrm{s}}}$ is the correction factor defined as:

$$b_{N-m_{\mathrm{s}}} = \sqrt{\frac{2}{N - m_{\mathrm{s}}}}\frac{\Gamma[(N - m_{\mathrm{s}})/2]}{\Gamma[(N - m_{\mathrm{s}} - 1)/2]}.$$

Pearn *et al.* (2002) showed that if the process follows the normal distribution $N(\mu, \sigma^2)$, then the estimators $(3\sqrt{N}/b_{N-m_{\mathrm{s}}})\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}$ and $(3\sqrt{N}/b_{N-m_{\mathrm{s}}})\tilde{C}_{\mathrm{PL}}^{\mathrm{M}}$ are distributed as the non-central t distribution with $N - m_{\mathrm{s}}$ degrees of freedom and non-central parameters $\delta_{\mathrm{U}} = 3\sqrt{N}C_{\mathrm{PU}}$ and $\delta_{\mathrm{L}} = 3\sqrt{N}C_{\mathrm{PL}}$, respectively. The rth moment (about zero) can be obtained as (1). Pearn *et al.* (2003) showed that for a fixed total number of observations, with $(N - m_{\mathrm{s}_1}) \geq 3$, if the number of samples $m_{\mathrm{s}_1} > m_{\mathrm{s}_2}$, then $\mathrm{Var}(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}})_{m_{\mathrm{s}_1}} > \mathrm{Var}(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}})_{m_{\mathrm{s}_2}}$. Further, since both estimators depend only on the sufficient and complete statistics (X, S_{P}^2) of (μ, σ^2), then $\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}$ and $\tilde{C}_{\mathrm{PL}}^{\mathrm{M}}$ are UMVUEs of C_{PU} and C_{PL}, respectively, where $Z_{\mathrm{U}} = \sqrt{N}(\mathrm{USL} - \overline{\overline{X}})/\sigma$.

$$E\left[\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}\right]^{\mathrm{r}} = \frac{(\Gamma[(N - m_{\mathrm{s}})/2])^{r-1}\Gamma[(N - m_{\mathrm{s}} - r)/2]}{\left(3\sqrt{N}\right)^{r}(\Gamma[(N - m_{\mathrm{s}} - 1)/2])^{r}} E(Z_{\mathrm{U}})^{r}. \quad (1)$$

The probability density function (PDF) and cumulative density function (CDF) of UMVUEs of C_{PU} and C_{PL} can be easily attained as (2) and (3), respectively, where $v = N - m_{\mathrm{s}}$:

$$f(x) = \frac{3\sqrt{N}v^{(v/2)}\mathrm{e}^{-\delta^2/2}}{b_v\sqrt{\pi}(v/2)(v + 9b_v^{-2}x^2N)^{(v+1)/2}}\sum_{j=0}^{\infty}\left(\frac{\delta^2\Gamma(v+j+1)/2)}{j!}\right)\left(\frac{2x^2}{b_v^2(9N)^{-1}v + x^2}\right)^{j/2}. \tag{2}$$

$$F(t_0) = \int_{-\infty}^{t_0}\frac{3\sqrt{N}v^{(v/2)}\mathrm{e}^{-\delta^2/2}}{b_v\sqrt{\pi}(v/2)\left(v + 9b_v^{-2}x^2N\right)^{(v+1)/2}} \times \sum_{j=0}^{\infty}\left(\frac{\delta^2\Gamma(v+j+1)/2)}{j!}\right)\left(\frac{2x^2}{b_v^2(9N)^{-1}v + x^2}\right)^{j/2}\mathrm{d}x. \tag{3}$$

In the following, it is also shown that both UMVUEs are consistent and asymptotically efficient. Let $\xrightarrow{\mathrm{L}}$ and $\xrightarrow{\mathrm{P}}$ denote convergence in distribution and convergence in probability, respectively. The proofs of Lemmas 1 and 2 and Theorem are shown in appendix 1.

Lemma 1: Define $M_k = E(x-\mu)^k$ as the kth central moment. If M_4 exists, then as $N \to \infty$, $\sqrt{N}\left(\overline{\overline{X}} - \mu, S_{\mathrm{P}}^2 - \sigma^2\right) \xrightarrow{\mathrm{L}} N(0, \Sigma^*)$, where

$$\Sigma^* = \begin{bmatrix} \sigma^2 & \mathrm{M}_3 \\ \mathrm{M}_3 & \mathrm{M}_4 - \sigma^4 \end{bmatrix}.$$

Lemma 2: Let $\Im(n^*)$ be a sequence of random vectors and b a fixed vector such that $\sqrt{n^*}(\Im(n^*) - b)$ has a limiting distribution $N(0, T)$ as $n^* \to \infty$. Let $f(\Im)$ be a vector-valued function of $\Im$ such that each component $f_j(\Im)$ has a non-zero differential at $\Im = b$, and let $\left(\partial f_j(\Im)/\partial\Im\right)\big|_{\Im=b}$ be the (i,j)th component of Φ_b. Then, $\sqrt{n^*}[f(\Im(n^*)) - f(b)]$ has a limiting distribution $N(0, \Phi_b T \Phi_b')$.

Theorem: If the process characteristic follows a normal distribution, then

(a) $\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}$ is consistent.
(b) $\sqrt{N}\left(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} - C_{\mathrm{PU}}\right)$ converges to $N(0, \frac{1}{9} + C_{\mathrm{PU}}^2/2)$ in distribution.
(c) $\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}$ is asymptotically efficient.

4. Lower confidence bounds for C_{PU} and C_{PL}

For cases with a single sample, Pearn and Shu (2002) established the LCBs on C_{PU} and C_{PL} based on the UMVUEs of C_{PU} and C_{PL}. For cases with multiple samples, $\tilde{C}_{PU}^{\mathrm{M}}$ and $\tilde{C}_{\mathrm{PL}}^{\mathrm{M}}$ are used, the UMVUEs of C_{PU} and C_{PL}, which is consistent and asymptotically efficient, to obtain the LCB on C_{PU} and C_{PL}. Let USL $= \overline{\overline{X}} + k_{\mathrm{U}}S_{\mathrm{P}}$ and LSL $= \overline{\overline{X}} - k_{\mathrm{L}}S_{\mathrm{P}}$ so that $k_{\mathrm{U}} = 3\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}/b_{N-m_{\mathrm{s}}}$ and $k_{\mathrm{L}} = 3\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}/b_{N-m_{\mathrm{s}}}$. A $100\gamma\%$ LCB $C_{\mathrm{U}}^{\mathrm{M}}$ for C_{PU} satisfies $\Pr\left(C_{\mathrm{PU}} \geq C_{\mathrm{U}}^{\mathrm{M}}\right) = \gamma$. It can be written as:

$$\Pr\left(\frac{\mathrm{USL}-\mu}{3\sigma} \geq C_{\mathrm{U}}^{\mathrm{M}}\right) = \Pr\left(\frac{\overline{\overline{X}} + k_{\mathrm{U}}S_{\mathrm{P}} - \mu}{3\sigma} \geq C_{\mathrm{U}}^{\mathrm{M}}\right)\Pr = \left(\frac{Z - 3\sqrt{N}C_{\mathrm{U}}^{\mathrm{M}}}{S_{\mathrm{p}}/\sigma} \geq -k_{\mathrm{U}}\sqrt{N}\right)$$

$$= \Pr\left(\frac{Z - 3\sqrt{N}C_{\mathrm{U}}^{\mathrm{M}}}{S_{\mathrm{p}}/\sigma} \geq -\frac{3\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}}{b_{N-m_{\mathrm{s}}}}\sqrt{N}\right) = \Pr(t(N - m_{\mathrm{s}}, \delta_{\mathrm{U}}) \geq t_{\mathrm{U}}) = \gamma,$$

and $\Pr(t(N-m_s,\delta_U)\le t_U)=1-\gamma$. Similarly, a $100\gamma\%$ LCB C_L^M for C_{PL} satisfies $\Pr(C_{PL}\ge C_L^M)=\gamma$. It can be shown that $\Pr(t(N-m_s,\ \delta_L)\le t_L)=\gamma$, where $Z\sim N(0,1)$, $t_L=k_L\sqrt{N}$, $\delta_U=-3\sqrt{N}C_U^M$ and $\delta_L=3\sqrt{N}C_L^M$. Thus, to obtain the LCB, one can proceed as follows:

4.1. *Algorithm for the LCB*

To compute the LCBs, C_U^M, an algorithm called the LCB is developed. An auxiliary function for evaluating C_U^M, the cumulative distribution function of the non-central chi-square distribution (Lenth 1989) is required. The step sizes for numerical computation are t_1 and t_2, where $0<t_2<t_1\le 0.1$.

Step 1. Read the sample data $(x_1,x_2,\ldots,x_N)$, USL (or LSL), m_s, and γ.
Step 2. Calculate

$$\overline{X}_i=\sum_{j=1}^{n_i}x_{ij}/n_i\ S_i=[(n_i-1)^{-1}\sum_{j=1}^{n_i}(x_{ij}-\overline{X}_i)^2]^{1/2}$$

$$\overline{\overline{X}}_i=\sum_{i=1}^{m_s}\overline{X}_i/m_s,\ S_p^2=\sum_{i=1}^{m_s}(n_i-1)S_i^2\Big/\sum_{i=1}^{m_s}(n_i-1).$$

$$b_{N-m_s}\cong\sqrt{(N-m_s-1)/(N-m_s)}(1-1/(4(N-m_s-1)) + 1/(32(N-m_s-1)^2)+5/(128(N-m_s-1)^3),$$

and $\tilde{C}_{PU}^M=b_{N-m_s}(\text{USL}-\overline{\overline{X}})/3S_P$.

Step 3. Compute an initial guess for C_U^M.
For $i=1,2,\ldots$, evaluate $C_U^M(i)=it_1$, $t_U=3\sqrt{N}\tilde{C}_{PU}^M/b_{N-m_s}$, and $\delta_U=3\sqrt{N}C_U^M(i)$, until $(t_\gamma(N-m_s,\delta_U)-t_U)\ge 0$.
Step 4. Find the LCB C_U^M on C_{PU} through numerical iterations.
For $j=0,1,\ldots$, evaluate $C_U^M(j)=C_U^M(i)-jt_2$ and $\delta_U=3\sqrt{N}C_U^M(j)$ until $(t_\gamma(N-m_s,\delta_U)-t_U)\le 0$. Set $C_U^M=C_U^M(j)$.
Step 5. Output the conclusive message, 'The true value of the process capability C_{PU} is no less than the C_U^M with $100\gamma\%$ level of confidence.'

We implement the algorithm and develop a Fortran program to compute the LCBs (see appendix 2). Tables 4–6 summarize the LCBs C_U^M on C_{PU} and C_{PL} for $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 0.8(0.1)3.0, for the total number of observations $N=100$, 150 and 200 with various m_s and $\gamma=0.95$. The results indicate that the LCB C_U^M decreases as m_s increases and increases as N increases in all cases. Figures 1 and 2 plot the curves of the LCB C_U^M on C_{PU} and C_{PL} with the sample sizes $N=100$ and 200 versus various subgroups m_s, respectively, for $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 0.8, 1.2, 1.5, 2.0, 2.5, 3.0 with confidence level $\gamma=0.95$. For bottom curve 1, $\tilde{C}_{PU}^M$ (or $\tilde{C}_{PL}^M$) = 0.8; for bottom curve 2, $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 1.2; for bottom curve 3, $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 1.5; for top curve 3, $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 2.0; for top curve 2, $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 2.5; and for top curve 1, $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 3.0. For example, if $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 1.5 for $N=100$ with $m_s=25$, then from table 4 and figure 1, the LCB $C_U^M=1.302$, and so one can conclude that C_{PU} (or C_{PL}) > 1.302, with 95% confidence.

4.2. *Sample size determination*

The sample size determination is essential to most factory applications, particularly for those implementing a routine-basis data collection plan for monitoring and controlling process quality. It directly relates to the sampling cost of a data collection plan. Extensive calculations are performed for $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 0.8(0.1)3.0 with the

$m_s/\tilde{C}^{M}_{PU}$	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
1	0.696	0.786	0.877	0.966	1.056	1.146	1.235	1.325	1.414	1.504	1.593	1.682	1.771	1.861	1.950	2.039	2.128	2.217	2.306	2.395	2.484	2.573	2.662
2	0.696	0.786	0.876	0.966	1.056	1.145	1.235	1.324	1.413	1.503	1.592	1.681	1.770	1.860	1.949	2.038	2.127	2.216	2.305	2.394	2.483	2.572	2.661
3	0.695	0.786	0.876	0.965	1.055	1.145	1.234	1.323	1.413	1.502	1.591	1.680	1.769	1.858	1.947	2.036	2.125	2.214	2.303	2.392	2.481	2.570	2.659
4	0.695	0.785	0.875	0.965	1.054	1.144	1.233	1.323	1.412	1.501	1.590	1.679	1.768	1.857	1.946	2.035	2.124	2.213	2.302	2.391	2.480	2.569	2.658
5	0.695	0.785	0.875	0.964	1.054	1.143	1.233	1.322	1.411	1.500	1.589	1.678	1.767	1.856	1.945	2.034	2.123	2.212	2.301	2.389	2.478	2.567	2.656
10	0.693	0.782	0.872	0.961	1.050	1.140	1.229	1.318	1.406	1.495	1.584	1.673	1.761	1.850	1.939	2.027	2.116	2.204	2.293	2.382	2.470	2.559	2.647
15	0.690	0.780	0.869	0.958	1.047	1.136	1.224	1.313	1.401	1.490	1.578	1.667	1.755	1.844	1.932	2.020	2.108	2.197	2.285	2.373	2.461	2.550	2.638
20	0.688	0.777	0.866	0.954	1.043	1.131	1.220	1.308	1.396	1.484	1.572	1.660	1.748	1.836	1.924	2.012	2.100	2.188	2.276	2.364	2.452	2.540	2.627
25	0.685	0.774	0.862	0.951	1.039	1.127	1.215	1.302	1.390	1.478	1.566	1.653	1.741	1.829	1.916	2.004	2.091	2.179	2.266	2.354	2.441	2.529	2.616
30	0.682	0.770	0.858	0.946	1.034	1.122	1.209	1.296	1.384	1.471	1.558	1.646	1.733	1.820	1.907	1.994	2.081	2.168	2.255	2.342	2.429	2.517	2.604
35	0.679	0.767	0.854	0.941	1.029	1.116	1.203	1.290	1.377	1.463	1.550	1.637	1.724	1.810	1.897	1.984	2.070	2.157	2.243	2.330	2.417	2.503	2.590
40	0.675	0.762	0.849	0.936	1.023	1.109	1.196	1.282	1.368	1.455	1.541	1.627	1.713	1.799	1.886	1.972	2.058	2.144	2.230	2.316	2.402	2.488	2.574
45	0.671	0.758	0.844	0.930	1.016	1.102	1.188	1.274	1.359	1.445	1.531	1.616	1.702	1.787	1.873	1.958	2.044	2.129	2.215	2.300	2.386	2.471	2.557
50	0.666	0.752	0.838	0.923	1.008	1.094	1.179	1.264	1.349	1.434	1.519	1.604	1.689	1.773	1.858	1.943	2.028	2.113	2.198	2.282	2.367	2.452	2.537
55	0.660	0.745	0.830	0.915	0.999	1.084	1.168	1.252	1.337	1.421	1.505	1.589	1.673	1.757	1.841	1.926	2.010	2.094	2.178	2.262	2.346	2.430	2.514
60	0.654	0.738	0.822	0.905	0.989	1.072	1.156	1.239	1.322	1.406	1.489	1.572	1.655	1.739	1.822	1.905	1.988	2.071	2.154	2.237	2.320	2.403	2.486
65	0.646	0.728	0.811	0.894	0.976	1.059	1.141	1.223	1.305	1.387	1.470	1.552	1.634	1.716	1.798	1.880	1.962	2.044	2.126	2.208	2.290	2.372	2.454
70	0.636	0.717	0.798	0.880	0.961	1.042	1.123	1.203	1.284	1.365	1.446	1.527	1.607	1.688	1.769	1.849	1.930	2.011	2.091	2.172	2.253	2.333	2.414
75	0.623	0.702	0.782	0.861	0.941	1.020	1.099	1.178	1.257	1.336	1.416	1.495	1.574	1.653	1.732	1.811	1.889	1.968	2.047	2.126	2.205	2.284	2.363
80	0.605	0.683	0.760	0.837	0.914	0.991	1.068	1.145	1.221	1.298	1.375	1.452	1.528	1.605	1.682	1.759	1.835	1.912	1.989	2.065	2.142	2.219	2.295
85	0.580	0.654	0.728	0.802	0.876	0.949	1.023	1.096	1.170	1.243	1.317	1.390	1.464	1.537	1.611	1.684	1.757	1.831	1.904	1.978	2.051	2.124	2.198
90	0.539	0.608	0.676	0.745	0.813	0.881	0.949	1.018	1.086	1.154	1.222	1.290	1.358	1.427	1.495	1.563	1.631	1.699	1.767	1.835	1.903	1.971	2.039
95	0.452	0.509	0.566	0.624	0.681	0.738	0.795	0.852	0.909	0.966	1.023	1.080	1.137	1.194	1.251	1.308	1.365	1.422	1.479	1.536	1.593	1.650	1.707

Table 4. Lower confidence bounds for $N=100$, with $m_s=1(1)5$, $10(5)95$, $\gamma=0.95$, and $\tilde{C}^{M}_{PU}=0.8(0.1)3.0$.

$m_s/\tilde{C}^{M}_{PU}$	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
1	0.715	0.807	0.899	0.990	1.082	1.173	1.265	1.356	1.447	1.539	1.630	1.721	1.812	1.903	1.994	2.085	2.176	2.267	2.358	2.482	2.540	2.631	2.722
2	0.714	0.806	0.898	0.990	1.081	1.173	1.264	1.356	1.447	1.538	1.629	1.720	1.811	1.903	1.994	2.085	2.176	2.267	2.358	2.482	2.540	2.631	2.722
3	0.714	0.806	0.898	0.990	1.081	1.173	1.264	1.355	1.446	1.538	1.629	1.720	1.811	1.902	1.993	2.084	2.175	2.266	2.357	2.481	2.539	2.630	2.721
4	0.714	0.806	0.898	0.989	1.081	1.172	1.264	1.355	1.446	1.537	1.628	1.719	1.810	1.901	1.992	2.083	2.174	2.265	2.356	2.481	2.538	2.629	2.720
5	0.714	0.806	0.897	0.989	1.080	1.172	1.263	1.354	1.445	1.537	1.628	1.719	1.810	1.901	1.992	2.083	2.174	2.264	2.355	2.480	2.537	2.628	2.719
10	0.713	0.804	0.896	0.987	1.079	1.170	1.261	1.352	1.443	1.534	1.625	1.716	1.807	1.897	1.988	2.079	2.170	2.261	2.351	2.477	2.533	2.624	2.714
15	0.712	0.803	0.895	0.986	1.077	1.168	1.259	1.350	1.441	1.531	1.622	1.713	1.803	1.894	1.985	2.075	2.166	2.257	2.347	2.474	2.528	2.619	2.709
20	0.710	0.802	0.893	0.984	1.075	1.166	1.257	1.347	1.438	1.528	1.619	1.710	1.800	1.891	1.981	2.071	2.162	2.252	2.343	2.472	2.523	2.614	2.704
25	0.709	0.800	0.891	0.982	1.073	1.164	1.254	1.345	1.435	1.525	1.616	1.706	1.796	1.887	1.977	2.067	2.157	2.248	2.338	2.469	2.518	2.608	2.699
30	0.708	0.799	0.889	0.980	1.071	1.161	1.251	1.342	1.432	1.522	1.612	1.703	1.793	1.883	1.973	2.063	2.153	2.243	2.333	2.465	2.513	2.603	2.693
35	0.706	0.797	0.888	0.978	1.068	1.159	1.249	1.339	1.429	1.519	1.609	1.699	1.789	1.878	1.968	2.058	2.148	2.238	2.328	2.462	2.507	2.597	2.687
40	0.705	0.795	0.886	0.976	1.066	1.156	1.246	1.336	1.425	1.515	1.605	1.695	1.784	1.874	1.963	2.053	2.143	2.232	2.322	2.459	2.501	2.591	2.680
45	0.703	0.793	0.883	0.973	1.063	1.153	1.243	1.332	1.422	1.511	1.601	1.690	1.780	1.869	1.958	2.048	2.137	2.226	2.316	2.455	2.494	2.584	2.673
50	0.701	0.791	0.881	0.971	1.060	1.150	1.239	1.328	1.418	1.507	1.596	1.685	1.775	1.864	1.953	2.042	2.131	2.220	2.309	2.451	2.487	2.576	2.665
60	0.697	0.787	0.876	0.965	1.054	1.143	1.231	1.320	1.409	1.498	1.586	1.675	1.763	1.852	1.941	2.029	2.118	2.206	2.295	2.443	2.472	2.560	2.649
70	0.692	0.781	0.869	0.958	1.046	1.134	1.222	1.310	1.399	1.486	1.574	1.662	1.750	1.838	1.926	2.014	2.102	2.190	2.277	2.434	2.453	2.541	2.629
80	0.686	0.774	0.862	0.949	1.037	1.124	1.212	1.299	1.386	1.473	1.560	1.647	1.735	1.822	1.909	1.996	2.083	2.170	2.257	2.424	2.431	2.518	2.605
90	0.679	0.766	0.853	0.939	1.025	1.112	1.198	1.284	1.370	1.457	1.543	1.629	1.715	1.801	1.887	1.973	2.059	2.145	2.231	2.412	2.403	2.489	2.575
100	0.670	0.755	0.841	0.926	1.011	1.096	1.181	1.266	1.351	1.436	1.520	1.605	1.690	1.775	1.860	1.945	2.029	2.114	2.199	2.399	2.368	2.453	2.538
110	0.657	0.741	0.824	0.908	0.991	1.074	1.158	1.241	1.324	1.407	1.490	1.574	1.657	1.740	1.823	1.906	1.989	2.072	2.155	2.384	2.321	2.404	2.487
120	0.638	0.720	0.801	0.882	0.963	1.043	1.124	1.205	1.286	1.366	1.447	1.528	1.609	1.689	1.770	1.850	1.931	2.012	2.092	2.366	2.254	2.334	2.415
130	0.608	0.685	0.762	0.839	0.916	0.992	1.069	1.146	1.223	1.299	1.376	1.453	1.529	1.606	1.683	1.759	1.836	1.913	1.989	2.066	2.143	2.219	2.296
140	0.541	0.609	0.678	0.746	0.814	0.882	0.950	1.019	1.087	1.155	1.223	1.291	1.359	1.427	1.495	1.563	1.632	1.700	1.768	1.836	1.904	1.972	2.040

Table 5. Lower confidence bounds for $N = 150$, with $m_s = 1(1)5$, $10(5)50$, $60(10)140$, $\gamma = 0.95$, and $\tilde{C}^{M}_{PU} = 0.8(0.1)3.0$.

$m_s/\tilde{C}^{M}_{PU}$	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
1	0.726	0.819	0.912	1.005	1.097	1.190	1.282	1.375	1.467	1.560	1.652	1.744	1.837	1.929	2.021	2.113	2.205	2.298	2.390	2.482	2.574	2.666	2.759
2	0.726	0.819	0.912	1.004	1.097	1.190	1.282	1.375	1.467	1.559	1.652	1.744	1.836	1.928	2.021	2.113	2.205	2.297	2.389	2.482	2.574	2.666	2.758
3	0.725	0.819	0.911	1.004	1.097	1.189	1.282	1.374	1.467	1.559	1.651	1.744	1.836	1.928	2.020	2.112	2.205	2.297	2.389	2.481	2.573	2.665	2.757
4	0.725	0.818	0.911	1.004	1.097	1.189	1.282	1.374	1.466	1.559	1.651	1.743	1.835	1.928	2.020	2.112	2.204	2.296	2.388	2.481	2.573	2.665	2.757
5	0.725	0.818	0.911	1.004	1.096	1.189	1.281	1.374	1.466	1.558	1.651	1.743	1.835	1.927	2.019	2.111	2.204	2.296	2.388	2.480	2.572	2.664	2.756
10	0.725	0.817	0.910	1.003	1.095	1.188	1.280	1.372	1.464	1.557	1.649	1.741	1.833	1.925	2.017	2.109	2.201	2.293	2.385	2.477	2.569	2.661	2.753
15	0.724	0.817	0.909	1.002	1.094	1.186	1.279	1.371	1.463	1.555	1.647	1.739	1.831	1.923	2.015	2.107	2.199	2.291	2.383	2.474	2.566	2.658	2.750
20	0.723	0.816	0.908	1.001	1.093	1.185	1.277	1.369	1.461	1.553	1.645	1.737	1.829	1.921	2.013	2.104	2.196	2.288	2.380	2.472	2.563	2.655	2.747
25	0.722	0.815	0.907	0.999	1.092	1.184	1.276	1.368	1.459	1.551	1.643	1.735	1.827	1.918	2.010	2.102	2.194	2.285	2.377	2.469	2.560	2.652	2.744
30	0.721	0.814	0.906	0.998	1.090	1.182	1.274	1.366	1.458	1.549	1.641	1.733	1.824	1.916	2.008	2.099	2.191	2.282	2.374	2.465	2.557	2.648	2.740
35	0.721	0.813	0.905	0.997	1.089	1.181	1.272	1.364	1.456	1.547	1.639	1.730	1.822	1.913	2.005	2.096	2.188	2.279	2.371	2.462	2.554	2.645	2.736
40	0.720	0.812	0.904	0.996	1.087	1.179	1.271	1.362	1.454	1.545	1.637	1.728	1.819	1.911	2.002	2.093	2.185	2.276	2.367	2.459	2.550	2.641	2.732
45	0.719	0.811	0.903	0.994	1.086	1.177	1.269	1.360	1.452	1.543	1.634	1.725	1.817	1.908	1.999	2.090	2.182	2.273	2.364	2.455	2.546	2.637	2.728
50	0.718	0.810	0.901	0.993	1.084	1.176	1.267	1.358	1.449	1.541	1.632	1.723	1.814	1.905	1.996	2.087	2.178	2.269	2.360	2.451	2.542	2.633	2.724
60	0.716	0.807	0.898	0.990	1.081	1.172	1.263	1.354	1.445	1.536	1.626	1.717	1.808	1.899	1.989	2.080	2.171	2.262	2.352	2.443	2.534	2.624	2.715
70	0.713	0.804	0.895	0.986	1.077	1.168	1.258	1.349	1.439	1.530	1.620	1.711	1.801	1.892	1.982	2.072	2.163	2.253	2.344	2.434	2.524	2.615	2.705
80	0.710	0.801	0.892	0.982	1.073	1.163	1.253	1.343	1.433	1.524	1.614	1.704	1.794	1.884	1.974	2.064	2.154	2.244	2.334	2.424	2.514	2.604	2.694
90	0.707	0.798	0.888	0.978	1.068	1.157	1.247	1.337	1.427	1.516	1.606	1.696	1.785	1.875	1.965	2.054	2.144	2.233	2.323	2.412	2.502	2.591	2.681
100	0.704	0.793	0.883	0.973	1.062	1.151	1.241	1.330	1.419	1.508	1.597	1.687	1.776	1.865	1.954	2.043	2.132	2.221	2.310	2.399	2.488	2.577	2.666
110	0.700	0.789	0.878	0.967	1.055	1.144	1.233	1.322	1.410	1.499	1.587	1.676	1.764	1.853	1.941	2.030	2.118	2.207	2.295	2.384	2.472	2.561	2.649
120	0.695	0.783	0.871	0.960	1.048	1.136	1.224	1.312	1.400	1.488	1.576	1.663	1.751	1.839	1.927	2.015	2.103	2.190	2.278	2.366	2.454	2.542	2.629
150	0.672	0.757	0.842	0.927	1.012	1.097	1.182	1.267	1.352	1.437	1.521	1.606	1.691	1.776	1.860	1.945	2.030	2.115	2.199	2.284	2.369	2.454	2.538
180	0.609	0.686	0.763	0.840	0.916	0.993	1.070	1.147	1.223	1.300	1.377	1.453	1.530	1.607	1.683	1.760	1.836	1.913	1.990	2.066	2.143	2.220	2.296

Table 6. Lower confidence bounds for $N = 200$, with $m_s = 1(1)5$, 10(5)50, 60(10)120, 150, 180, $\gamma = 0.95$, and $\tilde{C}^{M}_{PU} = 0.8(0.1)3.0$.

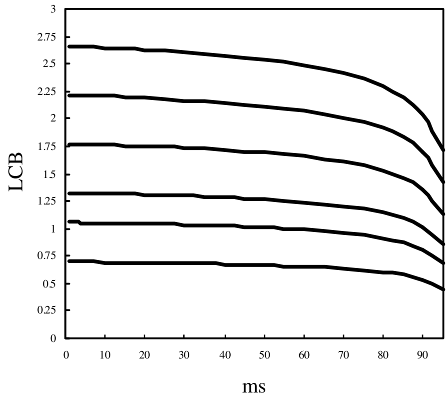

Figure 1. Lower confidence bound C_U^M with $N = 100$ versus subgroup m_s for $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 0.8, 1.2, 1.5, 2.0, 2.5, 3.0 (from bottom to top).

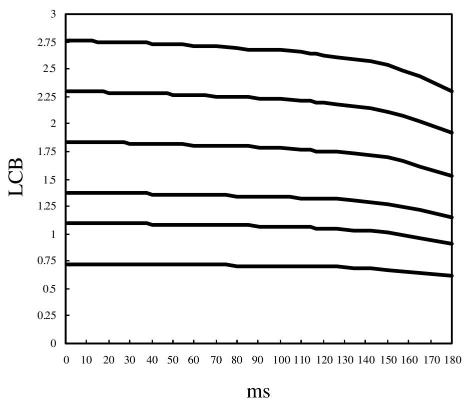

Figure 2. Lower confidence bound C_U^M with $N = 200$ versus subgroup m_s for $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) = 0.8, 1.2, 1.5, 2.0, 2.5, 3.0 (from bottom to top).

sample sizes $N = 20(10)220$, $m_s = 10(10)120$ with confidence level $\gamma = 0.90$, 0.95, 0.975 and 0.99, and analyse the estimation precision $R_{PU} = C_U^M / \tilde{C}_{PU}^M$. The parameter values investigated cover a wide range of applications. The results indicate that in all cases investigated, the estimating precision R_{PU} decreases as m_s increases, and increases as N and $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) increases. Hence, for practical purpose, one can take the minimum among those to obtain a quick reference on the minimum C_{PU} and C_{PL} without further calculations.

Table 7 shows the sample size, N, and number of samples, m_s, required and the corresponding minimal (conservative) precision of the estimation R_{PU}. For example, $\gamma = 0.95$, $N = 150$, $m_s = 30$ gives $R_{PU} = 0.885$. Thus, the true value of C_{PU} is no less than $\tilde{C}_{PU}^M$(or $\tilde{C}_{PL}^M$) $\times$ 0.885. However, if $R_{PU} = 0.885$ is chosen, then one can determine $N = 140$ with $m_s = 10$, or $N = 150$ with $m_s = 30$, or $N = 160$ with $m_s = 40$. Similarly, if $R_{PU} = 0.9$ is chosen, then one can determine $N = 180$ with $m_s = 10$, or $N = 190$ with $m_s = 20$, or $N = 200$ with $m_s = 40$, or $N = 210$ with $m_s = 50$, or $N = 220$ with $m_s = 60$, depending on which sampling plan is more appropriate to the application.

N/m_s	10	20	30	40	50	60	70	80	90	100	110	120
20	0.649	—	—	—	—	—	—	—	—	—	—	—
30	0.737	0.659	—	—	—	—	—	—	—	—	—	—
40	0.779	0.744	0.664	—	—	—	—	—	—	—	—	—
50	0.805	0.784	0.748	0.668	—	—	—	—	—	—	—	—
60	0.824	0.809	0.787	0.751	0.670	—	—	—	—	—	—	—
70	0.838	0.827	0.812	0.790	0.753	0.671	—	—	—	—	—	—
80	0.849	0.840	0.829	0.814	0.792	0.754	0.672	—	—	—	—	—
90	0.858	0.851	0.842	0.831	0.816	0.793	0.756	0.673	—	—	—	—
100	0.866	0.860	0.853	0.844	0.833	0.817	0.794	0.757	0.674	—	—	—
110	0.872	0.867	0.861	0.854	0.845	0.834	0.818	0.795	0.757	0.674	—	—
120	0.878	0.873	0.868	0.863	0.855	0.846	0.835	0.819	0.796	0.758	0.675	—
130	0.883	0.879	0.875	0.870	0.864	0.856	0.847	0.836	0.820	0.797	0.759	0.675
140	0.887	0.884	0.880	0.876	0.871	0.864	0.857	0.848	0.837	0.821	0.798	0.759
150	0.891	0.888	0.885	0.881	0.877	0.871	0.865	0.858	0.849	0.837	0.821	0.798
160	0.894	0.892	0.889	0.885	0.882	0.877	0.872	0.866	0.859	0.850	0.838	0.822
170	0.898	0.895	0.893	0.890	0.886	0.882	0.878	0.873	0.867	0.859	0.850	0.838
180	0.901	0.898	0.896	0.893	0.890	0.887	0.883	0.879	0.873	0.867	0.860	0.851
190	0.903	0.901	0.899	0.897	0.894	0.891	0.887	0.884	0.879	0.874	0.868	0.860
200	0.906	0.904	0.902	0.900	0.897	0.894	0.891	0.888	0.884	0.880	0.874	0.868
210	0.908	0.906	0.904	0.902	0.900	0.898	0.895	0.892	0.888	0.885	0.880	0.875
220	0.910	0.909	0.907	0.905	0.903	0.901	0.898	0.895	0.892	0.889	0.885	0.880

Table 7. Total number of sample observations, N (left), number of samples, m_s (top), and precision of estimation with $\gamma = 0.95$.

5. HSBA production yield assurance

The product investigated is a monolithic open-loop unity-gain buffer amplifier with a high symmetrical slew rate of up to 3600 V/μs and a very wide bandwidth of 320 MHz at 5 Vp–p output swing called the high-speed buffer amplifier (HSBA). A complementary bipolar IC process is used that incorporates pn-junction-isolated high-frequency NPN (a layer of P-doped semiconductor between two N-doped layers) and PNP (a layer of N-doped semiconductor between two P-doped layers) transistors to achieve high-frequency performance previously unattainable with conventional integrated circuit technology. The unique design offers a high-performance alternative to expensive discrete or hybrid solutions. The HSBA features low quiescent currents, low input bias current, small signal delay time and phase shift, and low differential gain and phase errors.

The two types of HSBA with a 3 or 6 mA quiescent current are well suited for the operation between high-frequency processing stages. The HSBA demonstrates outstanding performance even in feedback loops of wide-band amplifiers or phase-locked loop systems. The type II HSBA, with 6 mA quiescent current and, therefore, a lower output impedance, can easily drive 50 inputs or 75 systems and cables. The broad range of analogue and digital applications extends from the decoupling of signal processing stages, impedance transformation and input amplifiers for radio frequency (RF) equipment and automatic test equipment (ATE) systems to video systems, distribution fields, communications systems and output drivers for graphic cards. The HSBA is available in an industry standard pin-out SO–8 package (figure 3). The simplified circuit diagram and quiescent current versus temperature for HSBA are shown in figure 4.

The quiescent current is an essential product characteristic, which has a significant impact on product quality. For the particular type II HSBA, the USL placed on quiescent current is set to 6 mA. A previous study verified that the measurement

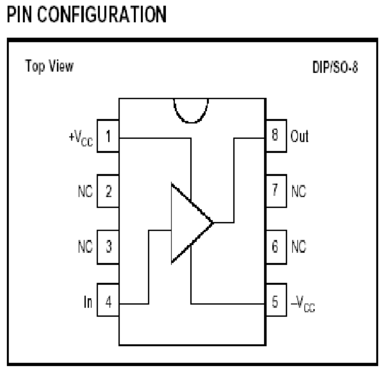

Figure 3. High-speed buffer amplifier (HSBA).

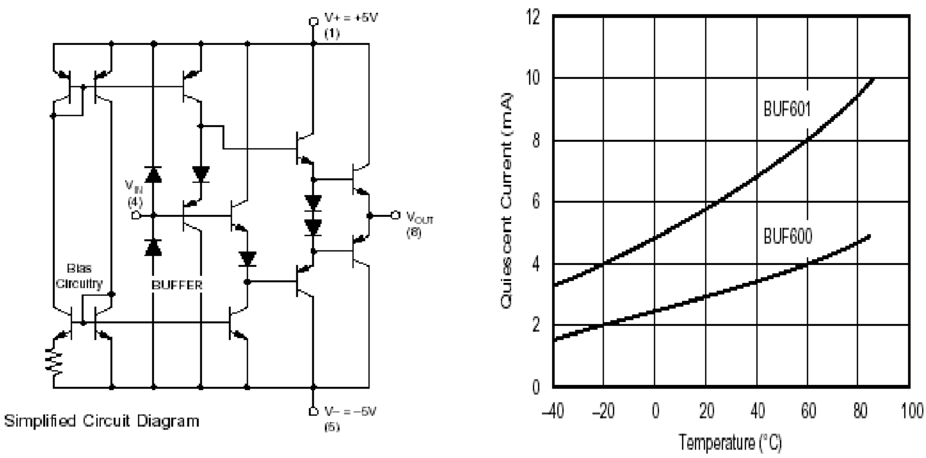

Figure 4. Simplified circuit diagram and quiescent current versus temperature for the HSBA.

system used introduces an ignorable measurement error. Sample data are collected from 20 subgroups of five observations each by measuring quiescent currents of HSBAs (table 8). Figures 5 and 6 show the histogram and normal probability plot of the 100 HSBA data with no observations outside the USL, and both figures show that the sample data appear to be approximately normal. A Shapiro–Wilk test is also applied to verify the normality assumption. Figure 7 shows the corresponding X and S control charts and indicates the process under 'in control'. Thus, it is concluded that the sample data can be regarded as taken from a stably normal process. To obtain the LCB on C_{PU}, the Fortran program shown in appendix 1 is executed. The program reads the sample data file and the input of the sample size $N = 100$, $USL = 6$ mA, $m_s = 20$ and confidence level $\gamma = 0.95$, then outputs the overall sample mean $X = 5.610$, and the pooled sample standard deviation, $S_P = 0.082$, the estimator $\tilde{C}^{M}_{PU} = 1.5712$ and the LCB $C^{M}_{U} = 1.3707$. The actual program execution output is shown in appendix 3. It is therefore concluded that the true process capability C_{PU} is no less than 1.3707 with a 95% level of confidence. Hence, the number of product items conforming to the manufacturing specifications

Subgroups	Observations (mA)					$\overline{X}$	S
1	5.90	5.73	5.72	5.54	5.64	5.706	0.13259
2	5.68	5.51	5.62	5.66	5.49	5.592	0.087006
3	5.52	5.67	5.60	5.64	5.72	5.63	0.075498
4	5.64	5.53	5.66	5.59	5.53	5.59	0.060415
5	5.78	5.61	5.52	5.65	5.74	5.66	0.103682
6	5.65	5.67	5.61	5.59	5.67	5.638	0.036332
7	5.58	5.61	5.56	5.54	5.58	5.574	0.026077
8	5.50	5.62	5.61	5.60	5.41	5.548	0.09094
9	5.56	5.62	5.63	5.62	5.60	5.606	0.027928
10	5.46	5.63	5.59	5.64	5.65	5.594	0.078294
11	5.59	5.63	5.57	5.57	5.57	5.586	0.026077
12	5.52	5.62	5.71	5.55	5.56	5.592	0.075299
13	5.66	5.60	5.73	5.46	5.51	5.592	0.109407
14	5.54	5.79	5.64	5.56	5.60	5.626	0.099398
15	5.57	5.68	5.60	5.67	5.66	5.636	0.04827
16	5.58	5.59	5.61	5.53	5.58	5.578	0.029496
17	5.59	5.51	5.68	5.50	5.54	5.564	0.073689
18	5.50	5.69	5.54	5.45	5.67	5.57	0.105594
19	5.72	5.66	5.61	5.53	5.54	5.612	0.080436
20	5.70	5.70	5.55	5.67	5.92	5.708	0.133679

Table 8. Twenty groups of five observations (100 sample data).

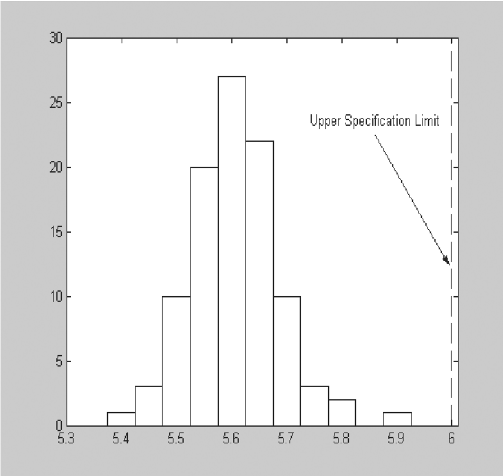

Figure 5. Histogram of 100 HSBA data.

are assured to be no greater than 20 parts per million. Equivalently, the production yield is assured to be no less than 99.9980%. These product items conformed to the manufacturing specifications (with the quiescent current, USL, not exceeding 6 mA, which obviously satisfies the preset quality requirements) and are considered as reliable products.

6. Conclusions

Process capability indices C_{PU} and C_{PL} have been widely used in the manufacturing industry to provide quantitative measures on process performance,

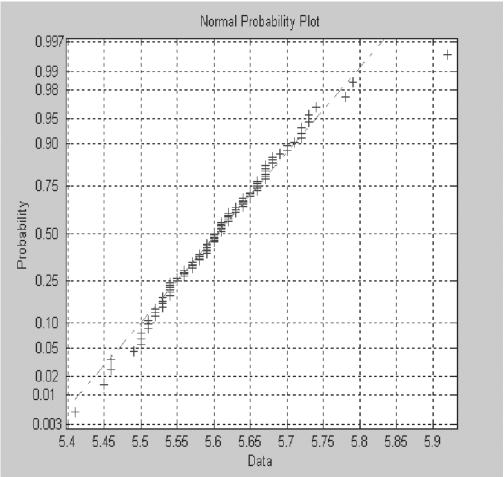

Figure 6. Normal probability plot for 100 HSBA data.

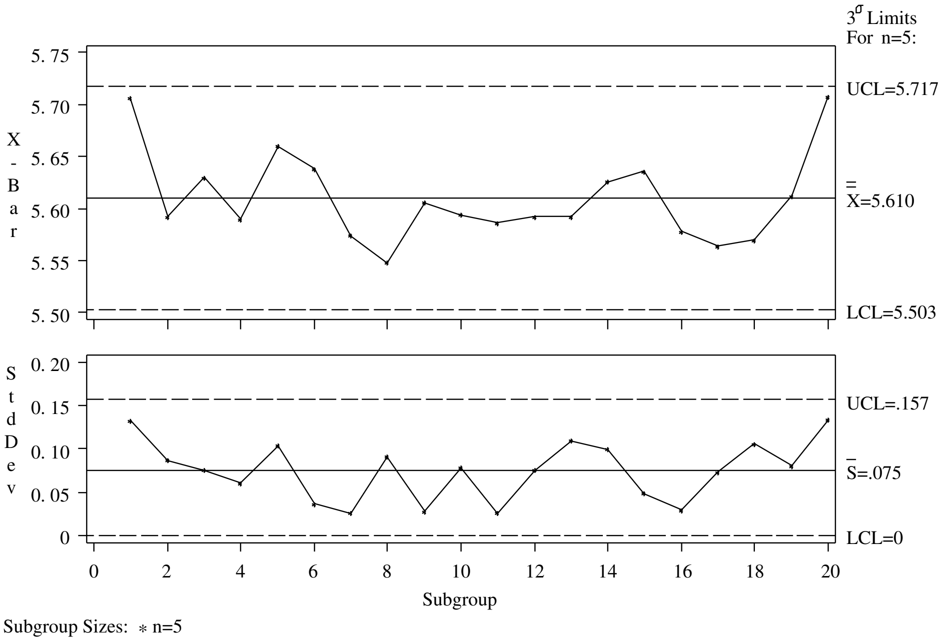

Figure 7. $\overline{X}$ and S control charts for HSBA data.

particularly for processes with one-sided specification limits. Statistical properties of the estimators of C_{PU} and C_{PL} have been investigated extensively but are restricted to cases with single samples. The indices C_{PU} and C_{PL} provide yield assurance of production. The present paper considered the UMVUE of C_{PU} and C_{PL} for cases of multiple samples and showed that this UMVUE is consistent and asymptotically efficient. An efficient algorithm/program is presented to compute the LCBs on C_{PU} and C_{PL}, which presents a measure on the minimum capability of the process. The paper also provided tables for engineers/practitioners to use for in-plant factory applications. An example of HSBA is presented to illustrate the practicality of the LCB approach to actual data collected from the real-world applications. The implementation of the existing statistical theory for the process yield assessment makes it possible for the production industry to apply the complicated theoretical results to the factory actual productions.

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Appendix 1

A.1. *Proof of Lemma 1*

See Serfling (1980, p. 72). For the process following the normal distribution, $M_3 = 0$ and $M_4 = 3\sigma^4$, hence:

$$\Sigma^* = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}.$$

A.2. *Proof of Lemma 2*

See Anderson (1984, Theorem 4.2.3). An estimator $\tilde{\theta}_N$ of θ is said to be consistent if for all $\varepsilon > 0$. $p(|\tilde{\theta}_N - \theta| > \varepsilon) \to 0$ as $N \to \infty$ for all θ. A sufficient condition for the consistency is that $E(\tilde{\theta}_N) \to \theta$ and $\mathrm{Var}(\tilde{\theta}_N) \to 0$. Under the regular conditions, the estimator $\tilde{\theta}_N$ is said to be asymptotically efficient if $\tilde{\theta}_N$ is asymptotically normal, $\sqrt{N}(\tilde{\theta}_N - E(\tilde{\theta}_N)) \to 0$, and $N\,\mathrm{Var}(\tilde{\theta}_N - E(\tilde{\theta}_N))$ converges to the Cramer–Rao lower bound (CRLB).

A.3. *Proof of Theorem*

(a) Using the Chebyshev inequality:

$$p\left(\left|\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} - C_{\mathrm{PU}}\right| > \varepsilon\right) < \frac{E(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} - C_{\mathrm{PU}})^2}{\varepsilon^2}.$$

Since $E(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} - C_{\mathrm{PU}})^2 = \mathrm{Var}(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}) = E(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}})^2 - C_{\mathrm{PU}}^2$, then by Stirling's formula, one obtains $E(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}})^2$, which converges to C_{PU}^2. Hence, $E(\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} - C_{\mathrm{PU}})^2$ converges to zero. Therefore, $\tilde{C}_{\mathrm{PU}}^{\mathrm{M}} \xrightarrow{\mathrm{P}} C_{\mathrm{PU}}$ and so $\tilde{C}_{\mathrm{PU}}^{\mathrm{M}}$ is consistent.

(b) Note that the following function is a real-valued function, which is differentiable for μ and σ^2, thus:

$$C_{\mathrm{PU}}(\mu, \sigma^2) = \frac{\mathrm{USL} - \mu}{3(\sigma^2)^{1/2}} \quad \text{with}$$

$$\frac{\partial C_{\mathrm{PU}}}{\partial \mu} = -\frac{1}{3(\sigma^2)^{1/2}}, \quad \frac{\partial C_{\mathrm{PU}}}{\partial \sigma^2} = -\frac{\mathrm{USL} - \mu}{6(\sigma^2)^{3/2}},$$

If $\Phi = (\partial C_{\mathrm{PU}}/\partial \mu \quad \partial C_{\mathrm{PU}}/\partial \sigma^2)$ is defined, then by Lemmas 1 and 2, one has the following. From (a) $\tilde{C}^{\mathrm{M}}_{\mathrm{PU}} \xrightarrow{\mathrm{P}} C_{\mathrm{PU}}$ and Slutsky's theorem (Bain and Engehardt 1992), it can be shown that $\sqrt{N}\left(\tilde{C}^{\mathrm{M}}_{\mathrm{PU}} - C_{\mathrm{PU}}\right) \xrightarrow{\mathrm{L}} N(0, \sigma^2_{C_{\mathrm{PU}}})$:

$$\sqrt{N}\left(\tilde{C}^{\mathrm{M}}_{\mathrm{PU}} - C_{\mathrm{PU}}\right) = \sqrt{N}\left[\tilde{C}^{\mathrm{M}}_{\mathrm{PU}}(\overline{\overline{X}}, S^2_{\mathrm{P}}) - C_{\mathrm{PU}}(u, \sigma^2)\right] \xrightarrow{\mathrm{L}} N(0, \sigma^2_{C_{\mathrm{PU}}})$$

$$\sigma^2_{C_{\mathrm{PU}}} = \Phi \Sigma \Phi' = \frac{1}{9}\left(1 + \frac{(\mathrm{USL} - \mu)^2}{2\sigma^2}\right) = \frac{1}{9} + \frac{C^2_{\mathrm{PU}}}{2}$$

(c) For a normal distribution, the log-maximum likelihood function is:

$$L = \ln\{f(x; u, \sigma^2)\} = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}.$$

The information matrix is:

$$I(\mu, \sigma^2) = -E\begin{bmatrix} \partial^2 L/\partial\mu^2 & \partial^2 L/\partial\mu\partial\sigma^2 \\ \partial^2 L/\partial\mu\partial\sigma^2 & \partial^2 L/\partial(\sigma^2)^2 \end{bmatrix} = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^2 \end{bmatrix}.$$

Hence,

$$\mathrm{CRLB} = \frac{1}{N}\Phi I^{-1}(u, \sigma^2)\Phi' = \frac{1}{9N}\left(1 + \frac{((\mathrm{USL} - \mu)^2)}{2\sigma^2}\right) = \frac{\sigma^2_{C_{\mathrm{PU}}}}{N}.$$

From (b), $\sqrt{N}\left(\tilde{C}^{\mathrm{M}}_{\mathrm{PU}} - C_{\mathrm{PU}}\right) \xrightarrow{\mathrm{L}} N(0, \sigma^2_{C_{\mathrm{PU}}})$, and therefore $\tilde{C}^{\mathrm{M}}_{\mathrm{PU}}$ is asymptotically efficient.

Appendix 2

```
! Fortran 90 Program for lower confidence bounds based on multiple samples
! Read the sample data (x1, x2, . . . , xN), USL (or LSL), ms, and γ.
    REAL, DIMENSION(1:5, 1:20) :: A
    INTEGER IDF, NOUT
    REAL DELTA, P, T, TNIN, b, bf, y, y1, Icpu, Tcpu, mean(20), var(20), std(20)
    EXTERNAL TNIN, UMACH
    A = reshape((/5.8971, 5.7319, 5.7176, 5.5356, 5.6376, &
    &5.6774, 5.5087, 5.6171, 5.6648, 5.4905, 5.519, 5.6661, 5.5995, 5.6446, 5.716, &
    &5.641, 5.5331, 5.6639, 5.5858, 5.5306, 5.778, 5.6068, 5.5203, 5.654, 5.743, &
    &5.6505, 5.6672, 5.6089, 5.5865, 5.674, 5.5834, 5.6064, 5.5552, 5.5417, 5.5772, &
    &5.499, 5.6211, 5.6087, 5.5965, 5.409, 5.5566, 5.6212, 5.6314, 5.6152, 5.5968, &
    &5.4637, 5.6288, 5.5889, 5.6412, 5.6509, 5.5927, 5.6276, 5.5715, 5.5726, 5.5675, &
    &5.519, 5.6199, 5.705, 5.5508, 5.5574, 5.6614, 5.6034, 5.731, 5.4554, 5.5121, &
    &5.5351, 5.785, 5.6367, 5.5623, 5.6025, 5.5732, 5.6831, 5.6016, 5.6695, 5.6597, &
    &5.5803, 5.5903, 5.6109, 5.5321, 5.5797, 5.5924, 5.513, 5.6797, 5.5038, 5.5396, &
    &5.5006, 5.6906, 5.5422, 5.4533, 5.6691, 5.7226, 5.6571, 5.6118, 5.5269, 5.5361, &
    &5.7045, 5.6978, 5.5483, 5.6685, 5.9185/), (/5, 20/))
    PRINT*, 'Please Enter: Overall Sample Observations, ⊕ of Subsamples, &
    &USL, alpha – risk (1 – r).'
    READ*, N, m, USL, alpha
```

! Calculate $\overline{X}_i$ S_i, X, S_p^2, b_{n-m_s}, and $\tilde{C}_{PU}^M$.

```
b = 0; bf = 0
b = sqrt((N − m − 1)/2.0)*(1 − 1.0/(4*(N − m − 1)) + &
&1.0/(32*(N − m − 1)**2) + 5.0/(128*(N − m − 1)**3))
bf = sqrt(2.0/(N − m))*b
DO e = 1, 20
mean(e) = sum(A(:, e))/5
DO ee = 1, 5
var(e) = var(e) + ((A(ee, e) − mean(e))**2)
ENDDO
std(e) = sqrt(var(e)/(5.0 − 1.0))
ENDDO
amean = sum(mean)/20.0
pstd = sqrt(sum(var)/(100.0 − 20.0))
PRINT*, 'The Overall Sample Mean = ', amean
PRINT*, 'The Pooled Sample Standard Deviation = ', pstd
ecpu = bf*(USL − amean)/(3*pstd)
print*, 'The Estimated Cpu Based on Multiple Samples = ', ecpu
```

! Compute an initial guess for C_U^M.

```
Do i = 1, 100
Cpu = i*0.1
IDF = 0; DELTA = 0; P = 0; T = 0; y = 0; y1 = 0; Icpu = 0; Tcpu = 0
CALL UMACH (2, NOUT)
IDF = N − m
DELTA = 3*sqrt(N/1.0)*Cpu
P = 1 − alpha
T = 3*sqrt(N/1.0)*eCpu/bf
y = TNIN(P, IDF, DELTA)
IF ((yc).GE. 0) Then
Icpu = DELTA/(3*sqrt(N/1.0))
EXIT
ENDIF
ENDDO
```

! Find the lower confidence bound C_U^M on C_{PU} through numerical iterations.

```
Do J = 1, 10000
Tcpu = Icpu − 0.0001*J
DELTA1 = 3*sqrt(N/1.0)*TCpu
y1 = TNIN(P, IDF, DELTA1)
IF ((y1 − T).LE. 0) Then
EXIT
ENDIF
ENDDO
```

! Output the Lower Confidence Bounds

```
PRINT*, 'The Lower Confidence Bounds on Cpu Based on Multiple &
& Samples = ', Tcpu
END
```

! End

Appendix 3

Input:

Please enter: Overall sample observations, ♯ of subsamples, USL, alpha – risk (1 – r).
100, 20, 6.0, 0.05.

Output:

Overall sample mean = 5.609857.
Pooled sample standard deviation = 8.198889E – 02.
Estimated Cpu-based on multiple samples = 1.571239.
Lower confidence bounds on Cpu-based on multiple samples = 1.370700.

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