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Companion matrices: reducibility, numerical ranges and similarity to contractions ☆

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Abstract

In this paper, we study some unitary-equivalence properties of the companion matrices. We obtain a criterion for a companion matrix to be reducible and show that the numerical range of a companion matrix is a circular disc centered at the origin if and only if the matrix equals the (nilpotent) Jordan block. However, the more general assertion that a companion matrix is determined by its numerical range turns out to be false. We also determine, for an $n \times n$ matrix A with eigenvalues in the open unit disc, the defect index of a contraction to which A is similar.

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For any complex polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$, there is associated an $n \times n$ matrix

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix}, \tag{1}$$

called the *companion matrix* of p and denoted by $M(p)$. A familiar special case is the (nilpotent) *Jordan block* J_n when all the a_j's are zero. Such a matrix has the property that its minimal polynomial and characteristic polynomial are both equal to p. Hence companion matrices are nonderogatory and, in particular, are such that every eigenvalue has geometric multiplicity one. They arise as the building blocks in the *rational form* of general matrices: every square matrix is similar to a direct sum $M(p_1) \oplus \cdots \oplus M(p_k)$ of companion matrices with p_{j+1} dividing p_j for all j.

In this paper, we study some unitary-equivalence properties of the companion matrices. Previous works in this respect are the determination of their singular values [1, pp. 224–225] and an explicit construction of their polar decomposition [10]. Here we consider other properties of these matrices such as their reducibility and their numerical ranges. In Section 1 below, we solve the problem when a companion matrix is reducible, that is, when it is unitarily equivalent to the direct sum of two other matrices. We obtain a complete characterization of reducibility in terms of the eigenvalues. It roughly says that a companion matrix is reducible when its eigenvalues are "equally distributed" on at most two circles with center at the origin and radii reciprocal to each other. It follows as corollaries that a companion matrix unitarily equivalent to a direct sum with one unitary summand or with at least three summands must itself be unitary. We take up the numerical ranges of companion matrices in Section 2. We show that a companion matrix whose numerical range is a circular disc centered at the origin must be equal to the Jordan block. However, the more general assertion that a companion matrix is determined by its numerical range is not true. We give an example of two distinct 3×3 companion matrices whose numerical ranges are the same elliptic disc. On the other hand, we also prove that if two 3×3 companion matrices have the same circular disc as their numerical ranges, then they must equal to each other. Whether this remains true for $n \times n$ companion matrices is not known. Finally, in Section 3, we use the rational form for matrices to prove an improvement over a classical result of Rota on the similarity of a matrix to a contraction.

1. Reducibility

A matrix is *reducible* if it is unitarily equivalent to the direct sum of two other matrices. In this section, we give a criterion in terms of the eigenvalues for a companion matrix to be reducible.

Theorem 1.1. *An $n \times n$ $(n \geqslant 2)$ companion matrix A is reducible if and only if its eigenvalues are of the form*: $a\omega_n^{j_1}, \ldots, a\omega_n^{j_p}, (1/\bar{a})\omega_n^{j_{p+1}}, \ldots, (1/\bar{a})\omega_n^{j_n}$, *where* $a \neq 0$, ω_n *denotes the* nth *primitive root of* 1, $1 \leqslant p \leqslant n-1$, *and* $\{j_1, \ldots, j_p\}$ *and* $\{j_{p+1}, \ldots, j_n\}$ *form a partition of* $\{0, 1, \ldots, n-1\}$. *In this case, A is unitarily equivalent to a direct sum $A_1 \oplus A_2$ with* $\sigma(A_1) = \{a\omega_n^{j_1}, \ldots, a\omega_n^{j_p}\}$ *and* $\sigma(A_2) = \{(1/\bar{a})\omega_n^{j_{p+1}}, \ldots, (1/\bar{a})\omega_n^{j_n}\}$. *In particular, every reducible companion matrix is invertible.*

Here for any matrix B, $\sigma(B)$ denotes the set of its eigenvalues.

Proof of Theorem 1.1. Assume that A is unitarily equivalent to the direct sum $A_1 \oplus A_2$ on $\mathbb{C}^p \oplus \mathbb{C}^{n-p}$ $(1 \leqslant p \leqslant n-1)$: $UA = (A_1 \oplus A_2)U$ for some unitary U. Since A is nonderogatory, A_1 and A_2 have no common eigenvalue. We next show that all eigenvalues of A have algebraic multiplicity one. Indeed, if a is an eigenvalue of A with algebraic multiplicity bigger than one, then $x_1 = (1, a, a^2, \ldots, a^{n-1})^{\mathrm{T}}$ and $x_2 = (0, 1, 2a, \ldots, (n-1)a^{n-2})^{\mathrm{T}}$ are generalized eigenvectors of a. We assume that a is also an eigenvalue of A_1. Let b be any eigenvalue of A_2. Then b is also an eigenvalue of A with the corresponding eigenvector $y = (1, b, b^2, \ldots, b^{n-1})^{\mathrm{T}}$. Since $a \neq b$, we infer that Ux_1 and Ux_2 are in $\mathbb{C}^p \oplus 0$ and Uy is in $0 \oplus \mathbb{C}^{n-p}$. Hence

$$\langle Ux_1, Uy\rangle = \langle x_1, y\rangle = \sum_{j=0}^{n-1} (a\bar{b})^j = 0 \tag{2}$$

and

$$\langle Ux_2, Uy\rangle = \langle x_2, y\rangle = \bar{b}\sum_{j=0}^{n-1} j(a\bar{b})^{j-1} = 0. \tag{3}$$

We obtain $b \neq 0$ from (2) and hence $\sum_{j=0}^{n-1} j(a\bar{b})^{j-1} = 0$ from (3). These imply that $a\bar{b}$ is a multiple zero of the polynomial $\sum_{j=0}^{n-1} z^j$, which is certainly absurd. We conclude that eigenvalues of A can only have algebraic multiplicity one. Moreover, from (2) we also have $(a\bar{b})^n = 1$. Since b is an arbitrary eigenvalue of A_2, we deduce that eigenvalues of A_2 are of the form $(1/\bar{a})\omega_n^{j_k}$, $k = p+1, \ldots, n$, while those of A_1 are of the form $a\omega_n^{j_k}$, $k = 1, \ldots, p$. It is obvious that $\{j_1, \ldots, j_p\}$ and $\{j_{p+1}, \ldots, j_n\}$ form a partition of $\{0, 1, \ldots, n-1\}$.

To prove the converse, we assume that the eigenvalues of A are of the asserted form. If $b = a\omega_n^{j_k}$ and $c = (1/\bar{a})\omega_n^{j_l}$, where $1 \leqslant k \leqslant p$ and $p+1 \leqslant l \leqslant n$, then their corresponding eigenvectors $x = (1, b, b^2, \ldots, b^{n-1})^{\mathrm{T}}$ and $y = (1, c, c^2, \ldots, c^{n-1})^{\mathrm{T}}$ satisfy

$$\langle x, y\rangle = \sum_{j=0}^{n-1} (b\bar{c})^j = \sum_{j=0}^{n-1} \omega_n^{j(j_k - j_l)} = 0.$$

Let H_1 and H_2 be the subspaces of $\mathbb{C}^n$ generated by the eigenvectors of $a\omega_n^{j_1}, \ldots, a\omega_n^{j_p}$ and $(1/\bar{a})\omega_n^{j_{p+1}}, \ldots, (1/\bar{a})\omega_n^{j_n}$, respectively. Then H_1 and H_2 are invariant subspaces of A which are orthogonal to each other. A is obviously unitarily equivalent to the direct sum of the restrictions $A_1 = A|H_1$ and $A_2 = A|H_2$. This completes the proof. □

The next corollary gives conditions for a companion matrix to be unitary. The equivalence of (a) and (e) therein is a consequence of Theorem 1.1.

Corollary 1.2. *The following conditions are equivalent for an $n \times n$ companion matrix A of the form* (1):

(a) *A is unitary*;
(b) *A is normal*;
(c) $a_1 = \cdots = a_{n-1} = 0$ *and* $|a_n| = 1$;
(d) *the eigenvalues of A are of the form $a\omega_n^j$, $j = 0, 1, \ldots, n-1$, where $|a| = 1$ and ω_n is the nth primitive root of* 1;
(e) *A is unitarily equivalent to a direct sum $A_1 \oplus A_2$ with A_1 unitary.*

Proof. (a)⇒(b) is trivial. To prove (b)⇒(c), assume that A is normal. Carrying out the matrix multiplications in $AA^* = A^*A$ and equating the first $n-1$ diagonal entries of the two products reveal that $|a_n| = 1$ and $a_2 = \cdots = a_{n-1} = 0$. Then the equality of the $(n, 1)$ entries ($-a_{n-1} = \overline{a_1}a_n$) yields that $a_1 = 0$.

If (c) holds, then the characteristic polynomial of A is $z^n + a_n$. Hence the eigenvalues of A are of the form asserted in (d).

Next assume that (d) holds. If $b = a\omega_n^k$ and $c = a\omega_n^l$ are two distinct eigenvalues of A, then their corresponding eigenvectors $x = (1, b, b^2, \ldots, b^{n-1})^{\mathrm{T}}$ and $y = (1, c, c^2, \ldots, c^{n-1})^{\mathrm{T}}$ satisfy

$$\langle x, y\rangle = \sum_{j=0}^{n-1} (b\bar{c})^j = \sum_{j=0}^{n-1} \omega_n^{j(k-l)} = 0.$$

Thus A is unitarily equivalent to the diagonal matrix $\operatorname{diag}(a, a\omega_n, \ldots, a\omega_n^{n-1})$, which is obviously unitary. Therefore, A is unitary, that is, (a) holds.

To complete the proof, we need only show that (e) implies (d). Indeed, if (e) holds, then A is reducible. Therefore, the eigenvalues of A_1 and A_2 are of the form asserted in Theorem 1.1. Since A_1 is unitary, we must have $|a| = 1$. Thus the eigenvalues of A are $a\omega_n^j$, $j = 0, 1, \ldots, n-1$, that is, (d) holds. □

Corollary 1.3. *A companion matrix unitarily equivalent to the direct sum of three or more matrices must be unitary.*

Proof. Let A be an $n \times n$ companion matrix unitarily equivalent to $A_1 \oplus \cdots \oplus A_k$, $k \geqslant 3$, and let a, b and c be any eigenvalues of A_1, A_2 and A_3, respectively. We infer from Theorem 1.1 that $|ab| = |bc| = |ca| = 1$ and hence $|a| = |b| = |c| = 1$. This shows that all eigenvalues of A have modulus one. By Theorem 1.1 again, the eigenvalues of A are of the form $a\omega_n^j$, $j = 0, 1, \ldots, n-1$. Therefore, A is unitary by Corollary 1.2. □

2. Numerical ranges

Recall that the *numerical range* of an $n \times n$ matrix A is the subset

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

of the plane. Properties of the numerical range can be found in [5, Chapter 1].

In this section, we consider to what extent a companion matrix is determined by its numerical range. For 2×2 companion matrices, the numerical range provides the complete information: *if A and B are 2×2 companion matrices, then $A = B$ if and only if $W(A) = W(B)$*. This is the consequence of the fact that 2×2 matrices with equal numerical ranges are unitarily equivalent. Unfortunately, the same cannot be said about companion matrices of size three. The next example gives two distinct such matrices with equal numerical ranges.

Example 2.1. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sqrt{3}\mathrm{i} & 4 & (\sqrt{3}/4)\mathrm{i} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}\mathrm{i} & 4 & -(\sqrt{3}/4)\mathrm{i} \end{bmatrix}.$$

We show that $W(A) = W(B)$ via a result of Kippenhahn [7] that the numerical range of any $n \times n$ matrix C equals the convex hull of the real points of the dual curve of $p_C(x, y, z) = 0$, where p_C is the degree-n homogeneous polynomial in x, y and z given by

$$p_C(x, y, z) = \det(x\mathrm{Re}\, C + y\mathrm{Im}\, C + zI_n)$$

with $\operatorname{Re} C = (C + C^*)/2$ and $\operatorname{Im} C = (C - C^*)/(2\mathrm{i})$. In our case, we have

$$\begin{aligned}
p_A(x, y, z) &= \det\left(x \begin{bmatrix} 0 & 1/2 & (\sqrt{3}/2)\mathrm{i} \\ 1/2 & 0 & 5/2 \\ -(\sqrt{3}/2)\mathrm{i} & 5/2 & 0 \end{bmatrix} \right. \\
&\qquad \left. + y \begin{bmatrix} 0 & -\mathrm{i}/2 & -\sqrt{3}/2 \\ \mathrm{i}/2 & 0 & (3/2)\mathrm{i} \\ -\sqrt{3}/2 & -(3/2)\mathrm{i} & \sqrt{3}/4 \end{bmatrix} + z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} z & (x - y\mathrm{i})/2 & \sqrt{3}(-y + x\mathrm{i})/2 \\ (x + y\mathrm{i})/2 & z & (5x + 3y\mathrm{i})/2 \\ -\sqrt{3}(y + x\mathrm{i})/2 & (5x - 3y\mathrm{i})/2 & (\sqrt{3}/4)y + z \end{bmatrix} \\
&= z^3 + \frac{\sqrt{3}}{4} y z^2 - \frac{1}{4}(29x^2 + 13y^2)z - \frac{\sqrt{3}}{16}(29x^2 + 13y^2)y \\
&= \left(z + \frac{\sqrt{3}}{4} y\right)\left(z^2 - \frac{29}{4}x^2 - \frac{13}{4}y^2\right)
\end{aligned}$$

and, similarly,

$$p_B(x, y, z) = \left(z - \frac{\sqrt{3}}{4} y\right)\left(z^2 - \frac{29}{4}x^2 - \frac{13}{4}y^2\right).$$

The dual curve of $p_A = 0$ consists of the point $(0, \sqrt{3}/4)$ and the ellipse

$$\frac{4x^2}{29} + \frac{4y^2}{13} = 1. \tag{4}$$

Since $(0, \sqrt{3}/4)$ lies inside the ellipse, the numerical range $W(A)$ equals the (closed) elliptic disc bounded by (4). In a similar fashion, we obtain that $W(B)$ equals this same elliptic disc.

The (noncircular) elliptic disc turns out to be the only exceptional numerical range for 3×3 companion matrices.

Theorem 2.2. *Let A and B be* 3×3 *companion matrices. If* $W(A) = W(B)$ *is not a noncircular elliptic disc, then* $A = B$.

We start the proof by noting that a classification of numerical ranges of 3×3 matrices A was obtained before by Kippenhahn [7] followed by Keeler et al. [6]. The former is based on the factorability of p_A and has $W(A)$ classified into four classes:

(a) p_A factors into three linear factors:

$$p_A(x, y, z) = \prod_{j=1}^{3} (z + a_j x + b_j y).$$

In this case, A is normal and $W(A)$ is the (closed) triangular region with vertices (a_j, b_j), $j = 1, 2, 3$.

(b) p_A factors into a linear factor and an irreducible quadratic one:

$$p_A(x, y, z) = (z + ax + by)q(x, y, z).$$

Then $W(A)$ is the convex hull of the point (a, b) and the ellipse E which is the dual of $q(x, y, z) = 0$. It is an elliptic disc if (a, b) lies inside E.

(c) p_A is irreducible and the dual curve of $p_A = 0$ has degree four. In this case, $W(A)$ has a line segment on its boundary.

(d) p_A is irreducible and the dual curve of $p_A = 0$ has degree six. Then the dual curve consists of two parts, one inside the other, and $W(A)$ is an ovular region (that is, a region with a strictly convex boundary).

The paper [6] further developes these into criteria in terms of entries of A for the above cases. These we will also use in the following discussions.

The next proposition and its corollaries take care of the cases of irreducible p_A.

Proposition 2.3. *Let A and B be square matrices* (*not necessarily of the same size*). *If $W(A) = W(B)$, then p_A and p_B have a common irreducible factor.*

Proof. Let $p_A = p_1 \cdots p_k$ and $p_B = q_1 \cdots q_l$ be factorizations of p_A and p_B into irreducible factors. Let C_i and D_j denote the curves $p_i = 0$ and $q_j = 0$, respectively, and let C_i^* and D_j^* be their respective duals. If $W(A) = W(B)$, then some C_i^* and D_j^* have a common arc (by Kippenhahn's result) so that they have infinitely many common tangent lines. By duality, the curves C_i and D_j have infinitely many common points. Since p_i and q_j are irreducible, Bézout's theorem [8, Theorem 3.1] implies that $p_i = q_j$ as required. □

Corollary 2.4. *Let A and B be $n \times n$ matrices and assume that p_A is irreducible. Then $W(A) = W(B)$ if and only if $p_A = p_B$.*

Proof. The necessity follows easily from Proposition 2.3 while the sufficiency is a consequence of Kippenhahn's result. □

Corollary 2.5. *Let A and B be $n \times n$ companion matrices and assume that p_A is irreducible. Then $W(A) = W(B)$ if and only if $A = B$.*

Proof. In view of Corollary 2.4 and the fact that a companion matrix is completely determined by its eigenvalues, we need only prove that the equality of p_A and p_B implies that A and B have the same eigenvalues. Indeed, if $p_A(x, y, z) = p_B(x, y, z)$ for all x, y and z, then, letting $x = 1$ and $y = i$, we obtain $\det(\mathrm{Re}\, A + \mathrm{i}\mathrm{Im}\, A + zI_n) = \det(\mathrm{Re}\, B + \mathrm{i}\mathrm{Im}\, B + zI_n)$ or $\det(A + zI_n) = \det(B + zI_n)$ for all z, which implies that A and B have the same eigenvalues. □

We next move to the case of reducible p_A. The following proposition gives the uniqueness result for 3×3 companion matrices when the numerical range is a circular disc.

Proposition 2.6. (a) *For any point a in the plane, there is a 3×3 companion matrix whose numerical range is a circular disc centered at a. The number of such matrices is at most three.*

(b) *Let A and B be 3×3 companion matrices. If $W(A) = W(B)$ is a circular disc, then $A = B$.*

The proof of this proposition involves quite a bit of algebraic computations. It depends on the following criterion for a 3×3 matrix to have a circular numerical range (cf. [6, Corollary 2.5]).

Proposition 2.7. *The numerical range of a 3×3 matrix A is a circular disc if and only if*

(a) *A has a multiple eigenvalue a (so that its eigenvalues are a, a and b),*
(b) $2a\,\mathrm{tr}(A^*A) = \mathrm{tr}(A^*A^2) + 2|a|^2 a + 2(2a - b)|b|^2$, *and*
(c) $4|a - b|^2 + 2|a|^2 + |b|^2 \leqslant \mathrm{tr}(A^*A)$.

In this case, $W(A)$ is the circular disc with center a and radius $(\mathrm{tr}(A^*A) - 2|a|^2 - |b|^2)^{1/2}/2$.

The next lemma simplifies the present situation by allowing us to focus on the companion matrices whose circular numerical ranges are centered on the x-axis.

Lemma 2.8. *If A is a companion matrix, then λA is unitarily equivalent to a companion matrix for any λ with $|\lambda| = 1$.*

Proof. Assume that the companion matrix A is of size n. For any λ with $|\lambda| = 1$, let U be the $n \times n$ unitary matrix diag $(\lambda, 1, \bar{\lambda}, \bar{\lambda}^2, \ldots, \bar{\lambda}^{n-2})$. A little computation shows that $U^*(\lambda A)U$ is a companion matrix. □

We now proceed to prove Proposition 2.6.

Proof of Proposition 2.6. (a) In view of Lemma 2.8, we may assume that $a \geqslant 0$. Since, by Proposition 2.7, a matrix with its numerical range a circular disc has two eigenvalues equal to its center, we need only consider the companion matrix A of the form

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a^2 b & -(a^2 + 2ab) & 2a + b \end{bmatrix},$$

where b is to be determined. Some computations with the above matrix yield that the equality in Proposition 2.7(b) is the same as

$$\begin{aligned}&2a(a^4|b|^2 + |a^2 + 2ab|^2 + |2a + b|^2 + 2)\\&\quad = (2a + b)a^4|b|^2 - a(a + 2\bar{b})[a^2b - a(a + 2b)(2a + b)]\\&\qquad + (2a + b) + (2a + \bar{b})[-a(a + 2b) + (2a + b)^2]\\&\qquad + 2a^3 + (2a - b)|b|^2,\end{aligned}$$

which can be simplified to

$$a^2(a^2 + 4)|b|^2b + 2a(a^2 + 1)b^2 + 2a|b|^2 + b - a^2\bar{b} - 2a = 0. \tag{5}$$

We show that any b satisfying (5) must be real. Indeed, substituting $b = x + \mathrm{i}y$ (x, y real) into (5) and taking the real and imaginary parts of the resulting equality, we obtain

$$\begin{aligned}&a^2(a^2 + 4)(x^2 + y^2)x + 2a(a^2 + 1)(x^2 - y^2)\\&\quad + 2a(x^2 + y^2) + x - a^2x - 2a = 0\end{aligned} \tag{6}$$

and

$$a^2(a^2 + 4)(x^2 + y^2)y + 2a(a^2 + 1)2xy + y + a^2y = 0. \tag{7}$$

If $y \neq 0$, then we derive from (7) that

$$y^2 = -x^2 - \frac{4(a^2 + 1)}{a(a^2 + 4)}x - \frac{a^2 + 1}{a^2(a^2 + 4)}. \tag{8}$$

Plugging (8) into (6) and simplifying the resulting equality yields $a^6x = a^3$. If $a = 0$, then (7) already gives $y = 0$, contradicting our assumption. Thus $a \neq 0$ and hence $x = 1/a^3$. Equality (8) then becomes

$$y^2 = -\frac{1}{a^6} - \frac{4(a^2 + 1)}{a^4(a^2 + 4)} - \frac{a^2 + 1}{a^2(a^2 + 4)} < 0,$$

a contradiction. Thus y must be zero and every b satisfying (5) is real. Consider the cubic polynomial

$$p(z) = a^2(a^2 + 4)z^3 + 2a(a^2 + 2)z^2 + (1 - a^2)z - 2a \tag{9}$$

associated with (5). It has one or three real zeros b. These b's certainly satisfy (5) and hence also the equality in Proposition 2.7(b) for our A. We now check that they satisfy the inequality of Proposition 2.7(c). Assume otherwise that some b is outside the circle with center a and radius $(\mathrm{tr}(A^*A) - 2|a|^2 - |b|^2)^{1/2}/2$. Then b must be a corner of $W(A)$ and hence a reducing eigenvalue of A ($Ax = bx$ and $A^*x = \bar{b}x$ for some nonzero vector x) (cf. [5, Theorems 1.6.3 and 1.6.6]). Thus A is reducible, Theorem 1.1 implies that the eigenvalues of A are distinct, which contradicts our assumption that a is a multiple eigenvalue of A. We conclude from Proposition 2.7 that there exists a companion matrix with numerical range a circular disc centered at a.

(b) Again, by Lemma 2.8 we may assume that $W(A) = W(B)$ is a circular disc centered at a point $a \geqslant 0$. Then A and B are of the forms:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a^2 b & -(a^2+2ab) & 2a+b \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a^2 c & -(a^2+2ac) & 2a+c \end{bmatrix}.$$

We need to prove that $b = c$. As in (a), b and c must both be real. If $a = 0$, then $b = c = 0$ by (5) and hence $A = B$ as desired. For the remaining part of the proof, we assume that $a > 0$. Since the radius of $W(A)$ is given by $(\mathrm{tr}(A^*A) - 2a^2 - b^2)^{1/2}/2$ or

$$\frac{1}{2}(a^4b^2 + (a^2+2ab)^2 + (2a+b)^2 + 2 - 2a^2 - b^2)^{1/2}$$

by Lemma 2.7 and that for $W(B)$ by a similar expression, we have

$$\begin{aligned} &a^4b^2 + (a^2+2ab)^2 + (2a+b)^2 - b^2 \\ &\quad = a^4c^2 + (a^2+2ac)^2 + (2a+c)^2 - c^2. \end{aligned}$$

This can be simplified to

$$[a(a^2+4)(b+c) + 4(a^2+1)](b-c) = 0.$$

Assume contrapositively that $b \neq c$. Then we obtain from above

$$b + c = -\frac{4(a^2+1)}{a(a^2+4)}. \tag{10}$$

On the other hand, consider the cubic polynomial p given in (9). As proved in part (a), we have $p(b) = p(c) = 0$. Let d be the remaining (real) zero of p so that

$$b + c + d = -\frac{2(a^2+2)}{a(a^2+4)} \tag{11}$$

and

$$bcd = \frac{2}{a(a^2+4)} \tag{12}$$

hold. We subtract (10) from (11) to obtain $d = 2a/(a^2+4)$ and then divide (12) by this expression for d to have $bc = 1/a^2$. Using this, we may eliminate c from (10) to obtain

$$a^2(a^2+4)b^2 + 4a(a^2+1)b + (a^2+4) = 0. \tag{13}$$

Moreover, substitute $d = 2a/(a^2+4)$ into

$$p(d) = a^2(a^2+4)d^3 + 2a(a^2+2)d^2 + (1-a^2)d - 2a = 0$$

and simplify the resulting equality to obtain $6a(a^2+4)(2a^4-a^2-4) = 0$ or

$$2a^4 = a^2 + 4. \tag{14}$$

This can be solved for a^2 to give

$$a^2 = \frac{1}{4}(1+\sqrt{33}) < \frac{7}{4}. \tag{15}$$

Using (14), we simplify (13) to

$$a^5b^2 + 2(a^2+1)b + a^3 = 0. \tag{16}$$

Considered as an equation in b, this has discriminant

$$4(a^2+1)^2 - 4a^8 = \frac{3}{2}(a^2-4) < 0$$

by (14) and (15). This shows that solutions b of (16) are not real, a contradiction. We conclude that $b = c$ and hence $A = B$, completing the proof. □

We now wrap up the proof of Theorem 2.2.

Proof of Theorem 2.2. Consider the following three cases:

(a) p_A factors into three linear factors. Then A is normal and hence $W(A)$ is an equilateral triangular region (cf. Corollary 1.2). Thus if $W(A) = W(B)$, then the eigenvalues of A and B are both the vertices of the triangular region. Hence $A = B$.

(b) p_A factors into a linear factor and an irreducible quadratic one. Then $W(A)$ is the convex hull of a point P and an ellipse E. If P is inside E, then $W(A)$ is, by our assumption, a circular disc. Hence the equality of $W(A)$ and $W(B)$ implies $A = B$ by Proposition 2.6(b). On the other hand, if P is outside E, then P is a corner of $W(A)$. In this case, $W(A) = W(B)$ implies that the three eigenvalues (one is the point P and the other two are the foci of E) of A and B coincide. It follows that $A = B$.

(c) p_A is irreducible. Then $A = B$ follows from Corollary 2.5. □

In view of Proposition 2.6(b), we may wonder whether an $n \times n$ companion matrix can be completely determined by its circular numerical range. For this we have some reservation. But in case the circular numerical range is centered at the origin, then this is indeed true.

Theorem 2.9. *If A is an $n \times n$ companion matrix whose numerical range is a circular disc centered at the origin, then A equals the Jordan block J_n.*

The proof is based on two known facts: (a) a finite matrix A has its numerical range equal to a circular disc centered at the origin if and only if the maximum eigenvalue λ of $\mathrm{Re}(wA)$ is independent of w, $|w| = 1$, in which case, λ is the radius of the disc, and (b) $W(J_n)$ is a circular disc with center at the origin and radius $\cos(\pi/(n+1))$ (cf. [3, Proposition 1]).

Proof of Theorem 2.9. Let A be the matrix given by (1) and let λ be the maximum eigenvalue of $\mathrm{Re}(wA)$, $|w| = 1$. Thus $\det(\lambda I_n - \mathrm{Re}(wA)) = 0$ for all w. The expansion of the determinant of

$$\lambda I_n - \mathrm{Re}(wA) = \begin{bmatrix} \lambda & -w/2 & & & & & \overline{a_n}\bar{w}/2 \\ -\bar{w}/2 & \lambda & \cdot & & & & \cdot \\ & \cdot & \cdot & \cdot & & & \cdot \\ & & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \cdot & -w/2 & \overline{a_3}\bar{w}/2 \\ & & & & -\bar{w}/2 & \lambda & (\overline{a_2}\bar{w} - w)/2 \\ a_n w/2 & \cdot & \cdot & \cdot & a_3 w/2 & (a_2 w - \bar{w})/2 & \lambda + \mathrm{Re}(a_1 w) \end{bmatrix} \tag{17}$$

can be considered as a (trigonometric) polynomial in w. Since it has infinitely many zeros, the coefficients of w^j for $j = 0, \pm 1, \ldots, \pm n$ are all zero. Making use of this, we show that all the a_j's are also zero. Indeed, since the coefficient of w^n can be computed to be $(-1)^{n-1}a_n/2^n$, it follows that $a_n = 0$. Assuming that $a_n = \cdots = a_{j+1} = 0$ $(2 \leqslant j \leqslant n-1)$, we prove by induction that $a_j = 0$. Consider the matrix in (17) partitioned as

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix},$$

where A_j, B_j, C_j and D_j are submatrices of sizes $(n-j) \times (n-j)$, $(n-j) \times j$, $j \times (n-j)$ and $j \times j$, respectively. We claim that A_j is invertible. Indeed, if p_j denotes the characteristic polynomial of $\mathrm{Re}\, J_{n-j}$, then $\det A_j = \det(\lambda I_{n-j} - \mathrm{Re}(wJ_{n-j})) = p_j(\lambda)$. Hence we have to show that $p_j(\lambda) \neq 0$. Assume otherwise that $p_j(\lambda) = 0$. Then λ is an eigenvalue of $\mathrm{Re}\, J_{n-j}$ and hence is in $W(\mathrm{Re}\, J_{n-j}) = \mathrm{Re}\, W(J_{n-j})$, which implies that $\lambda \leqslant \cos(\pi/(n-j+1))$. On the other hand, since J_{n-1} is a submatrix of A, we have $W(J_{n-1}) \subseteq W(A)$. These are circular discs with center the origin and radii $\cos(\pi/n)$ and λ, respectively. Thus $\cos(\pi/n) \leqslant \lambda$ and therefore $\cos(\pi/n) \leqslant \cos(\pi/(n-j+1))$. It follows that $j \leqslant 1$, contradicting our assumption. Hence we have $p_j(\lambda) = \det A_j \neq 0$ and therefore A_j is invertible. Then

$$\begin{aligned} \det(\lambda I_n - \mathrm{Re}(wA)) &= \det A_j \cdot \det(D_j - C_j A_j^{-1} B_j) \\ &= p_j(\lambda) \cdot \det(D_j - C_j A_j^{-1} B_j). \end{aligned}$$

Since

$$
\begin{aligned}
&\det(D_j - C_j A_j^{-1} B_j) \\
&= \det\left(D_j - \begin{bmatrix} 0 & \cdots & -\bar{w}/2 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & \det A_{j+i}/\det A_j \end{bmatrix} \right. \\
&\qquad \left. \times \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ -w/2 & \cdots & 0 \end{bmatrix} \right) \\
&= \det \begin{bmatrix} \lambda - (1/4)(p_{j+1}(\lambda)/p_j(\lambda)) & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix},
\end{aligned}
$$

where the remaining entries of this last matrix are exactly the same as those of D_j, the coefficient of w^j in $\det(D_j - C_j A_j^{-1} B_j)$ is $a_j/2^j$. Hence the coefficient of w^j in $\det(\lambda I_n - \operatorname{Re}(wA))$ equals $p_j(\lambda)a_j/2^j$. Since this is zero and $p_j(\lambda) \neq 0$, we obtain $a_j = 0$ as asserted.

Finally, we need to check that $a_1 = 0$. Since

$$
\lambda I_n - \operatorname{Re}(wA) = \begin{bmatrix} \lambda & -w/2 & & & \\ -\bar{w}/2 & \lambda & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \lambda & -w/2 \\ & & & -\bar{w}/2 & \lambda + \operatorname{Re}(a_1 w) \end{bmatrix}
$$

from what was proved above, we obtain

$$
\begin{aligned}
\det(\lambda I_n - \operatorname{Re}(wA)) &= (\lambda + \operatorname{Re}(a_1 w)) p_1(\lambda) - \left(-\frac{w}{2}\right)\left(-\frac{\bar{w}}{2}\right) p_2(\lambda) \qquad (18) \\
&= (\lambda + \operatorname{Re}(a_1 w)) p_1(\lambda) - \frac{1}{4} p_2(\lambda) = 0.
\end{aligned}
$$

The coefficient of w in $\det(\lambda I_n - \operatorname{Re}(wA))$ is $p_1(\lambda)a_1/2$. Hence $p_1(\lambda)a_1 = 0$. We claim that $p_1(\lambda) \neq 0$. Indeed, if $p_1(\lambda) = 0$, then (18) yields $p_2(\lambda) = 0$, which in turn leads to $\lambda \leqslant \cos(\pi/(n-1))$ and hence contradicts $\lambda \geqslant \cos(\pi/n)$ as before. We conclude that $a_1 = 0$ and $A = J_n$ as asserted. □

We end this section with a general question on the numerical ranges of companion matrices.

Problem 2.10. Which nonempty closed convex subset of the plane is the numerical range of some $n \times n$ companion matrix?

We doubt that there will be any easy-to-describe and clean-cut answer. However, some partial ones obtained from the results in these two sections are already very interesting. For example, we have the answer for the 2×2 case: *a closed elliptic disc with foci a and b is the numerical range of some* 2×2 *companion matrix if and only if its minor axis has length* $|1 + a\bar{b}|$. This is the same as saying that the matrix $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ is unitarily equivalent to a companion matrix if and only if $|c| = |1 + a\bar{b}|$. The latter can be proved by the equalities of traces, determinants and Frobenius norms of two 2×2 unitarily equivalent matrices. On the other hand, *a closed polygonal region* (*with at least three sides*) *is the numerical range of some* $n \times n$ *companion matrix if and only if its boundary is a regular* n*-gon which is inscribed in the unit circle*. This is a consequence of Corollaries 1.3 and 1.2. Finally, *a closed circular disc centered at the origin is the numerical range of some* $n \times n$ *companion matrix if and only if its radius equals* $\cos(\pi/(n + 1))$ (cf. Theorem 2.9).

3. Similarity to contractions

A classical result of Rota on similarity to contractions says that if A is an operator with spectrum contained in the open unit disc $\mathbb{D}$, then A is similar to a strict contraction (one with norm strictly less than one) (cf. [4, Corollary 2 to Problem 153]). In this section, we use a slight generalization of his arguments to prove a more precise improvement for finite matrices.

For any $n \times n$ matrix A, let $\mu(A)$ be its *multiplicity*, that is, $\mu(A)$ is the minimum number of vectors $\{x_1, \ldots, x_m\}$ in $\mathbb{C}^n$ for which $\{A^j x_k : j \geqslant 0, 1 \leqslant k \leqslant m\}$ spans $\mathbb{C}^n$. It is well-known that $\mu(A)$ equals the number of companion matrices in the rational form of A. A is *cyclic* if its multiplicity is one. The *defect index* of an $n \times n$ contraction A is $d_A = \text{rank } (I_n - A^*A)$.

Theorem 3.1. *Let A be an $n \times n$ matrix with all its eigenvalues in $\mathbb{D}$. Then A is similar to a contraction with defect index k if and only if $\mu(A) \leqslant k \leqslant n$.*

Since an $n \times n$ contraction has defect index n if and only if it is a strict one, the aforementioned result of Rota (or rather its finite-dimensional version) is a special case of the preceding theorem. The next lemma is another special case. Its proof is inspired by that of [11, Theorem 2].

Lemma 3.2. *If A is an $n \times n$ companion matrix with eigenvalues in $\mathbb{D}$ and k is a natural number less than or equal to n, then A is similar to a contraction with defect index k.*

Proof. Since the eigenvalues of A are in $\mathbb{D}$, the series $\sum_{m=0}^{\infty}\|A^m\|^2$ converges. Let P be the $n\times n$ diagonal matrix diag $(1,\ldots,1,0,\ldots,0)$ with k many 1's and let $X=\sum_{m=0}^{\infty}(A^m)^*PA^m$. This latter series also converges because

$$(A^m)^*PA^m \leqslant (A^m)^*A^m \leqslant \|A^m\|^2 I_n.$$

Since

$$\begin{aligned} X &\geqslant \sum_{m=0}^{n-k} A^{*m}PA^m \\ &= \sum_{m=0}^{n-k} \operatorname{diag}(0,\ldots,0,\underset{(m+1)\text{st}}{1},\ldots,\underset{(m+k)\text{th}}{1},0,\ldots,0) \\ &\geqslant I_n, \end{aligned}$$

we infer that X is invertible. If $B=X^{1/2}AX^{-1/2}$, then, letting $y=X^{-1/2}x$, we have

$$\begin{aligned} \|Bx\|^2 &= \|X^{1/2}AX^{-1/2}x\|^2 \\ &= \langle A^*XAy, y\rangle \\ &= \langle (X-P)y, y\rangle \\ &= \langle Xy, y\rangle - \langle Py, y\rangle \\ &= \langle X^{1/2}x, X^{-1/2}x\rangle - \|Py\|^2 \\ &= \|x\|^2 - \|PX^{-1/2}x\|^2 \leqslant \|x\|^2 \end{aligned}$$

for any x. This shows that B is a contraction. Moreover, since

$$\begin{aligned} \ker(I_n - B^*B) &= \{x\in\mathbb{C}^n : \|Bx\| = \|x\|\} \\ &= \{x\in\mathbb{C}^n : PX^{-1/2}x = 0\} \end{aligned}$$

from above, we infer that $\dim\ker(I_n-B^*B)=\dim\ker P=n-k$ and hence $d_B=k$. This proves that A is similar to the contraction B with defect index k. □

Proof of Theorem 3.1. Assume that A is similar to a contraction B with defect index k. It is known that $\mu(B)\leqslant k$ for any contraction B with eigenvalues in $\mathbb{D}$ (cf. [2, Proposition 5.3]). Hence $\mu(A)=\mu(B)\leqslant k$ as asserted.

Conversely, assume that $\mu(A)\leqslant k\leqslant n$. Since A is similar to a direct sum $A_1\oplus\cdots\oplus A_l$ $(l=\mu(A))$ of companion matrices with eigenvalues all in $\mathbb{D}$ and since Lemma 3.2 implies that each A_j is similar to a contraction B_j with

$$d_{B_j} = \begin{cases} k-l+1 & \text{if } j=1, \\ 1 & \text{if } 2\leqslant j\leqslant l, \end{cases}$$

we obtain that A is similar to the contraction $B = B_1 \oplus \cdots \oplus B_l$ with

$$d_B = \sum_{j=1}^{l} d_{B_j} = (k - l + 1) + \underbrace{1 + \cdots + 1}_{l-1} = k.$$

This completes the proof. □

The next corollary appears in [9, Theorem 3.27].

Corollary 3.3. *A finite matrix is similar to a contraction of class $\mathscr{S}_n$ if and only if it has eigenvalues in $\mathbb{D}$ and is cyclic.*

Recall that an $n \times n$ matrix is said to be of *class* $\mathscr{S}_n$ if it is a contraction, has all its eigenvalues in $\mathbb{D}$ and has its defect index equal to one.

Proof of Corollary 3.3. For the necessity, since contractions of class $\mathscr{S}_n$ have eigenvalues in $\mathbb{D}$ and have multiplicity one, the same is true for any matrix similar to an $\mathscr{S}_n$-contraction. The sufficiency follows from Theorem 3.1. □

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