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Journal of Sound and Vibration 272 (2004) 1079–1085
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Authors' reply ☆

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Received 11 July 2003

1. Introduction

First of all, the comments of Van Dooren are appreciated [1] who discusses the application of modified interpolated cell mapping (MICM) method to locate all attractors of a non-linear system [2]. The helpful comments may allow the algorithm of the MICM to be adjusted to analyze new types of systems. For the Duffing system governed by the following equation, Dooren mentions that two chaotic attractors are overlooked by MICM for $\omega = 0.479$ in the region $(0, 2.7) \times (-3.6, 2.7)$,

$$\ddot{x} + 0.10\dot{x} - x + x^3 = 3.2\cos(\omega t). \tag{1}$$

In this reply, the Duffing system is carefully studied again. The properties of the Duffing system with $\omega = 0.479$ are first studied by both the method of integration of a grid of points (IGP) and the MICM. When assigning various values of parameters, the behaviour of the system found in these analyses is somewhat different. It is hoped to find appropriate values of parameters for a correct global analysis of this system. Then the causes of overlooking the two chaotic attractors are discussed. Finally, the parametric analysis of the Duffing system is studied again by the MICM.

2. Parameters used and accuracies obtained in a MICM analysis

In this section, the parameters used and accuracies obtained in a MICM global analysis are debated. First the Duffing system is studied with $\omega = 0.479$ by IGP for 1000 periods, using 303^2 points in the region $(0, 2.7) \times (-3.6, 2.7)$. The mapping time step duration, i.e., the forced period $T = 2\pi/\omega$, is divided into 200 integration steps for the fourth order Runge–Kutta numerical integration algorithm. A criterion of 10^{-10} is used to find periodic attractors. Two periodic and

☆ Refers to doi:10.1016/j.jsv.2003.09.001.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.09.002

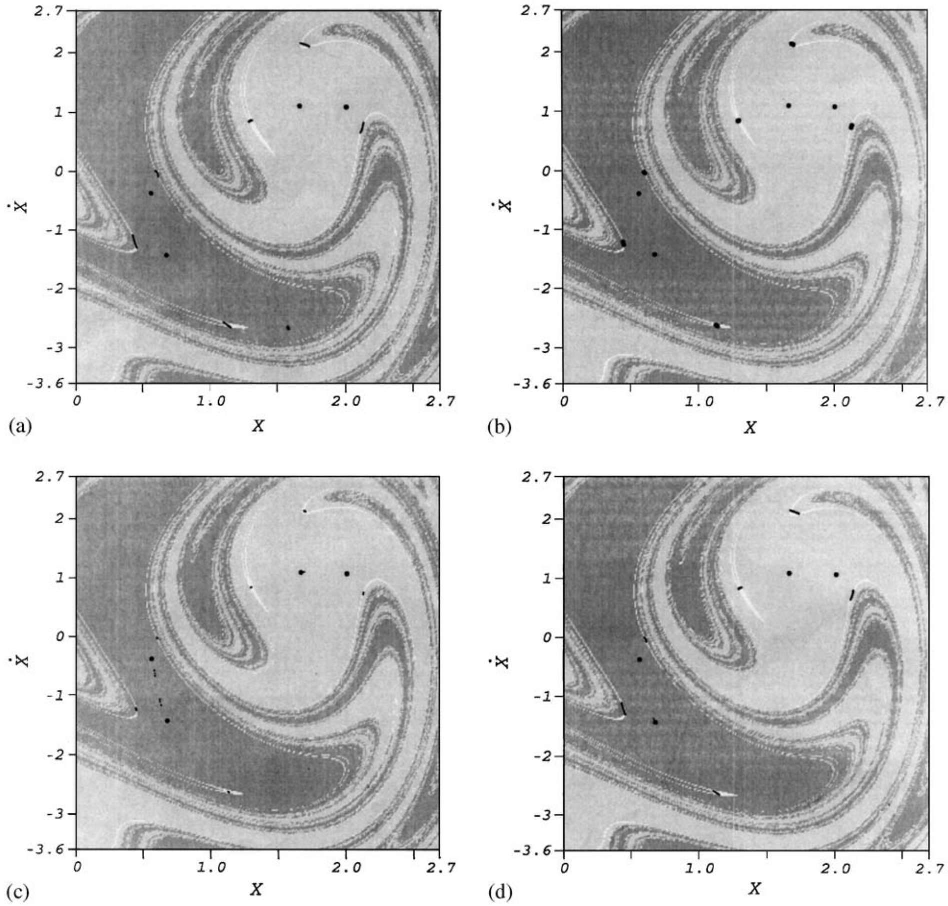

Fig. 1. Global analysis of the Duffing system using IGP and MICM with various values of parameters assigned in analyses: (a–b) IGP, with the forced period divided into 200 and 100 integration steps for the RK4 algorithm; (c) IGP, applied to this system within only 100 periods; (d) MICM global analysis.

two chaotic attractors with their basins of attraction are located as shown in Fig. 1(a), and which are similar to those located by Dooren. When the criterion used to locate periodic attractors is assigned by a value of 10^{-6} or 10^{-8}, several spurious periodic attractors having long periods are located on the two chaotic attractors. This should be due to the fact that the two chaotic attractors only reside in a small region. A point will be mapped near one of the previous mappings within many periods. Therefore, the criterion to check periodic attractors for this system should not be too large.

We then study the system with each period divided into 100 integration steps. Even with a criterion of 10^{-12}, the two chaotic attractors are alternately found as two periodic attractors both having period $6T$, in whose basins of attraction all the points are finally mapped among only the corresponding six positions. This phenomenon seemingly indicates that the chaotic behaviours of the system caused by the non-linearity are eliminated by the large truncation error [3] resulting from the numerical integration algorithm. Besides, the located two $2T$ periodic attractors settle to (2.0079, 1.0745), (1.6684, 1.0937) and (0.5639, −0.3836), (0.6785, −1.4300), but (2.0106, 1.0739), (1.6653, 1.0925) and (0.5628, −0.3728) and (0.6791, −1.4377) in the above analysis. The positions are somewhat different with various integration steps. The four attractors and their basins of attraction are shown in Fig. 1(b). In our previous global analysis using MICM, all the forced periods of the systems studied are just simply divided into 100 integration steps. For this Duffing system, division of each period into 200 integration steps should be a better choice to locate the chaotic attractors correctly.

IGP is further applied to this system within only 100 periods, with the final mappings of points shown in Fig. 1(c). There are some points still not eventually leading to periodic attractors. It will be *possibly confused* to locate a chaotic attractor in the region of study, although the MICM can indicate that they are in the basins of attraction of periodic attractors due to the criteria of attraction using the basin cells of attractors [4]. In the parametric analysis of a system using the MICM, however, the located attracting cells are iterated further by numerical integration to find the accurate attractors. Hence, 200 periods will be used to identify chaotic attractors.

Finally, the MICM is applied to the global analysis of this system with 303^2 cells studied and each period divided into 200 integration steps. In the MICM analysis, there are two chaotic attractors located due to division of the attracting cells into two sets. The four attractors and their basins of attraction are shown in Fig. 1(d), and which are similar as those located by IGP. The basins of attraction of the two chaotic attractors occupy only 2.8% of the region of study. This should be the major cause of MICM not locating the chaotic attractors. In our previous study, it is indicated that the error of global analysis using a cell mapping method increases when a large cell size is used. In addition, we use 101^2 cells to analyze an attractive region of a chaotic attractor and only 31^2 cells for an attractive region of a periodic attractor [2]. The latter *cell size should be too hard to locate* the two chaotic attractors with such small basins of attraction, and which is due to the limitation of the computer used. In a global analysis of a system, the MICM must use more memory than the IGP to register the first mappings of cells used to establish the following interpolated mappings, especially for the study of a large region. The computer used in the development of the MICM to locate all attractors has only 16MB RAM. Therefore, it is difficult to assign possibly enough cells to study an unknown system. Another cause is that the MICM algorithm is developed step by step. The MICM is first applied to the system with smooth basin boundaries, and then to the system with fractal basin boundaries. When applied to such systems with only small basins of attraction, up to now, only the number of cells can be increased to get enough accuracy of global analysis, and the algorithm of MICM should be adjusted after. Here we thank Dooren again for providing us such a good example. On the other hand, by comparing the above analyses, we can conclude that the MICM should be capable of locating precisely the attracting cells of the attractors in a region of study as IGP, if the same values of parameters and enough number of cells are used.

3. Parametric analysis of the Duffing system using MICM

Based on the appropriate values of parameters found above, MICM is once more applied to the parametric analysis of the system governed by Eq. (1) with $\omega \in [0.472, 0.483]$. Here we have a criterion of 10^{-10} to find periodic attractors. Additionally, 303^2 cells are used by the MICM in the global analysis of each attractive region. The mapping time step duration, i.e., the forced period $T = 2\pi/\omega$, is divided into 200 integration steps. Each attracting cell is first further iterated over 200 periods, in which the former 100 periods are regarded as transient states. Sometimes, the located chaotic attractors of the system with specific values of ω are doubtful, i.e., the chaotic attractors located have no fractal-like structures. In these cases, the system is further studied with the attracting cells further iterated within more than 200 periods, such as 500, 1500 or 2500 periods. In most cases the attracting cells are iterated forward before 4000 periods to identify the periodic attractors of the system correctly, and specifically iterated forward until 8000 periods for $\omega = 0.4756$ and 0.4807.

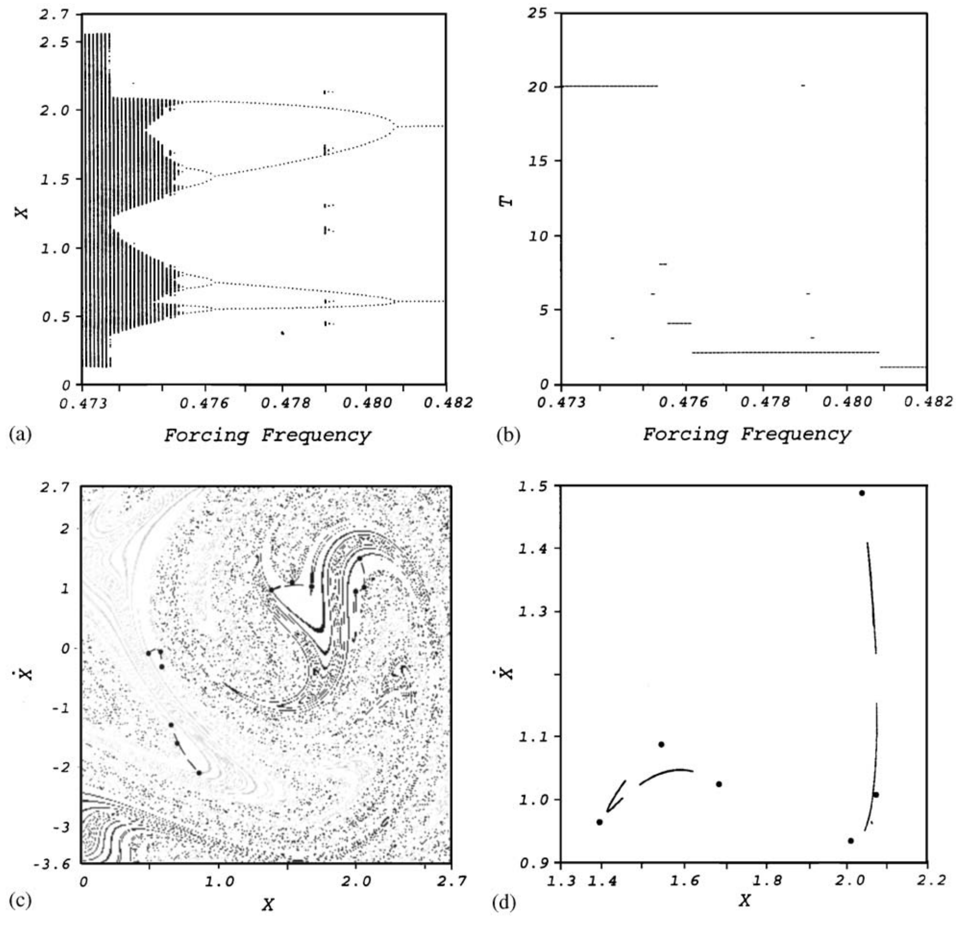

Fig. 2. Parametric analysis of the system using MICM, (a–b) positions and periods of the attractors located with $\omega \in [0.472, 0.483]$, (c) the attractors and basins located with $\omega = 0.4753$, (d) the two attractors in a small region with $\omega = 0.4753$. (e) the four attractors and the basins of attraction of the two periodic attractors located, (f–h) the two periodic attractors and the basins of attraction located in three small regions of study.

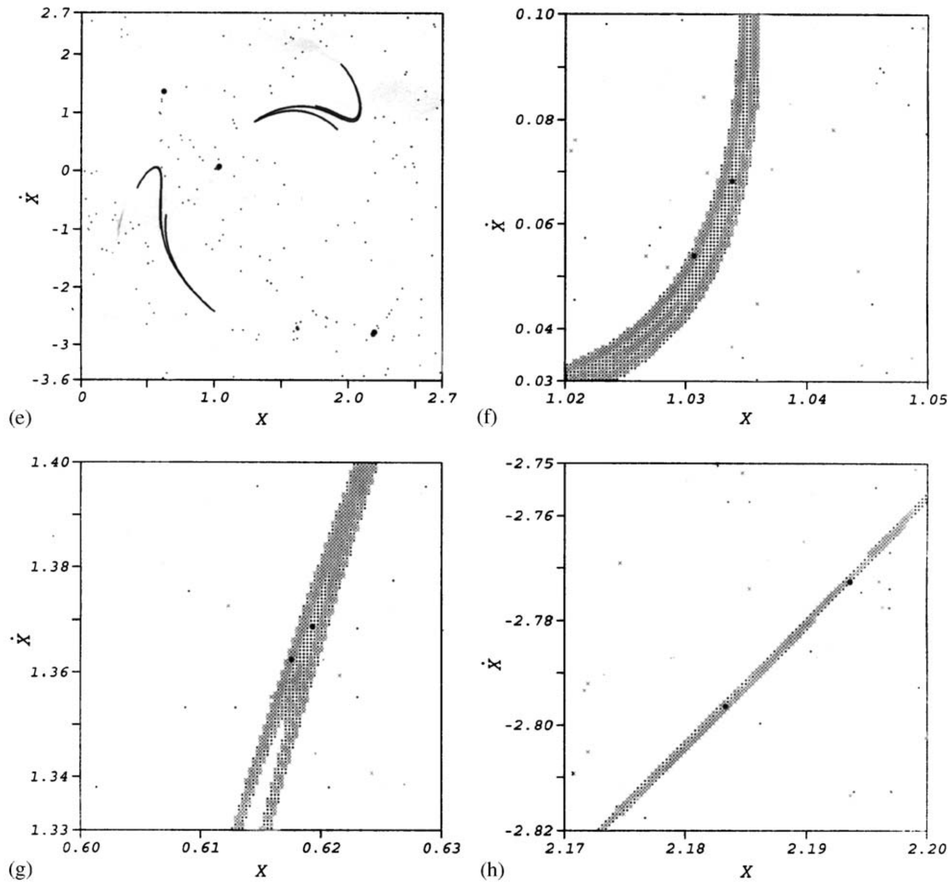

Fig. 2 (*continued*).

Figs. 2(a) and (b) show the positions and periods of the attractors located at different values of the forcing frequency, ω. Here the period of a chaotic attractor is indicated as $20T$. This parametric analysis is more precise and complete than that provided in Ref. [2]. As discussed by Dooren, the chaotic attractors for $\omega = 0.4790$ and periodic attractors having the period $3T$ for $\omega = 0.4792$ are all located. Between the two cases, there are also two periodic attractors having the period $6T$ located for $\omega = 0.4791$. In addition, there are also periodic attractors newly located for two other values of ω.

Firstly, for $\omega = 0.4753$, two periodic attractors having the period $6T$ are located near the two chaotic attractors located previously [2], and whose basins of attraction are shown in Fig. 2(c). The two periodic attractors are first located by the MICM, and then the attracting cells are further studied by numerical integration. The basins of attraction of two periodic attractors are represented in dark grey and black areas, and which occupy 15.4% of the region of study. These two periodic attractors are further checked with each period divided into 1000 integration steps, and which still exist in the analysis. Furthermore, two initial conditions corresponding to a periodic attractor and a chaotic attractor located are iterated further up to 500,000 periods, and they are indeed identified as two individual attractors. The two attractors in the region $(1.3, 2.2) \times (0.9, 1.5)$ are shown in Fig. 2(d), with the big and small points indicating the periodic and chaotic attractor individually. The basin boundaries of the system are much more fractal than those ever studied by us.

Secondly, for $\omega = 0.4743$, the two chaotic attractors and two newly located adjacent periodic attractors having the period $3T$ in the region $(0, 2.7) \times (-3.6, 2.7)$ are shown in Fig. 2(e), with the basins of attraction of the two periodic attractors represented in black areas. These two periodic attractors are not directly located by MICM due to the properties of the system, and on the other hand are located by numerical integration of the sink cells located by MICM. These basins only occupy 0.2% of the region of study, and which means that the studies of 500 point should be generally necessary to locate a periodic attractor. Therefore, the two periodic attractors are hard to be located. The two periodic attractors are very adjacent and their basins of attraction are little. We then check the basins of attractions in the following three small regions of study around each mapping of the two attractors, $(1.02, 1.05) \times (0.03, 0.10)$, $(0.60, 0.63) \times (1.33, 1.40)$ and $(2.17, 2.20) \times (2.82, 2.75)$, in which each period is divided into 200 integration steps and 101^2 points are studied. These regions of study are about 10^2 original cells of the region $(0, 2.7) \times (-3.6, 2.7)$. The basins of the two periodic attractors are shown in Figs. 2(f)–(h), respectively, in which they occupy 10.3%, 8.4%, and 3.6% of the regions of study. In this case, the two periodic attractors are also checked with each period divided into 1000 integration steps, and which still exist in the analysis. The mappings of the two periodic attractors are (2.193823, −2.772058), (1.033894, 0.068658), (0.619358, 1.369127) and (2.182904, −2.797229), (1.030557, 0.053531), (0.617504, 1.362491) with each period divided into 1000 integration steps, on the other hand, (2.193591, −2.772640), (1.033809, 0.068222), (0.619286, 1.368794) and (2.183323,−2.796319), (1.030670, 0.053993), (0.617543, 1.362552) with each period divided into 200 integration steps. Each of these initial conditions can be used to examine the existence of the two periodic attractors with each period divided into 200 or 1000 integration steps.

4. Conclusions

In this reply to the comments of Van Dooren, the parameters use by the MICM and the associative accuracies of global analysis are discussed and compared with IGP method. At the beginning we want to find appropriate values of parameters used in a global analysis to reflect the features of the system. With more points studied in a region, the global behaviour of a system can be discovered detailedly. If the same values of parameters and enough number of cells are used the MICM should be capable of locating precisely the attracting cells in a region of study as IGP with computation efficiency much improved. Therefore, the enormous computation time necessary for a parametric analysis can be reduced apparently.

In addition to the confirmation of the chaotic attractors provided by Dooren, we also locate some periodic attractors in the parametric analysis of the Duffing system using the MICM. This promotes the discovery of properties of the Duffing system. Finally, we thank the Editor for giving us an opportunity to improve our previous study.

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