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Numerical range circumscribed by two polygons ☆

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Abstract

We show that, for any $2n+2$ distinct points $a_1, a'_1, a_2, a'_2, \ldots, a_{n+1}, a'_{n+1}$ (in this order) on the unit circle, there is an n-by-n matrix A, unique up to unitary equivalence, which has norm one and satisfies the conditions that it has all its eigenvalues in the open unit disc, $I_n - A^*A$ has rank one and its numerical range is circumscribed by the two $(n+1)$-gons $a_1a_2\cdots a_{n+1}$ and $a'_1a'_2\cdots a'_{n+1}$. This generalizes the classical result of the existence of a conical curve circumscribed by two triangles which are already inscribed on another conical curve.

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1. Introduction

An n-by-n complex matrix A is said to be of *class* $\mathscr{S}_n$ if (1) A is a contraction, that is, the operator norm of A is at most one, (2) the eigenvalues of A are all in the open unit disc $\mathbb{D}$, and (3) A satisfies $\operatorname{rank}(I_n - A^*A) = 1$. In recent years, properties of the numerical ranges of $\mathscr{S}_n$-matrices have been intensely studied (cf. [2–4,8–10,13]).

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Among other things, it was obtained that the boundary of the numerical range $W(A)$ of an $\mathscr{S}_n$-matrix A has the $(n+1)$-Poncelet property. This means that there are infinitely many $(n+1)$-gons interscribing between the unit circle $\partial\mathbb{D}$ and the boundary $\partial W(A)$ or, put more precisely, for any point a on $\partial\mathbb{D}$ there is a (unique) $(n+1)$-gon with a as one of its vertices such that all its $n+1$ vertices are in $\partial\mathbb{D}$ and all its $n+1$ edges are tangent to $\partial W(A)$ (cf. [2, Theorem 2.1] or [8, Theorem 1]). This amazing property links the field of numerical range with that of the classical projective geometry. For an introductory survey on this connection, the reader is referred to [13]; for a more updated one, see [4].

In this paper, we prove another result on the numerical ranges of $\mathscr{S}_n$-matrices which is inspired by the projective geometry theorem that *if two triangles $\triangle ABC$ and $\triangle A'B'C'$ in the plane have their six (distinct) vertices on a conic alternatively, then there is another conic which is tangent to their six edges* (cf. Fig. 1). To obtain a polygon generalization of this, we first normalize the outer conic, that is, the one on which the vertices lie, as the unit circle $\partial\mathbb{D}$ via some projective and affine transformations. Then we consider $2n+2$ distinct points $a_1, a'_1, a_2, a'_2, \ldots, a_{n+1}, a'_{n+1}$ (in this order) on $\partial\mathbb{D}$. The conclusion is that there is a matrix A of class $\mathscr{S}_n$, which is unique up to unitary equivalence, with the boundary of its numerical range $W(A)$ tangent to the $n+1$ edges of each of the two $(n+1)$-gons $a_1a_2\cdots a_{n+1}$ and $a'_1a'_2\cdots a'_{n+1}$. When $n=2$, this gives essentially the previous classical case for triangles since the numerical range of an $\mathscr{S}_2$-matrix is a closed elliptic disc. For the polygon case, the algebraic curve $\partial W(A)$ plays the role of the ellipse. We will prove this generalization in Section 3. Note that the dual of the triangle theorem, namely, *if two triangles $\triangle ABC$ and $\triangle A'B'C'$ have their six edges all tangent to a conic, then the vertices lie on another conic* (cf. Fig. 2), is proved in [12, p. 184] via the use of cross ratios.

In Section 2, we collect, for easy reference, a list of necessary and sufficient conditions for the numerical range of an $\mathscr{S}_n$-matrix to be circumscribed by a given $(n+1)$-gon inscribing on the unit circle. Some of the conditions here are already known. Their equivalence, which involves the geometry, operator, analyticity, algebra and convexity aspects of the problem, indicates the fertility of this subject, worthy of further exploration.

Recall that the *numerical range* of an n-by-n matrix A is by definition the set

$$W(A)=\{\langle Ax,x\rangle\in\mathbb{C}: x\in\mathbb{C}^n, \|x\|=1\},$$

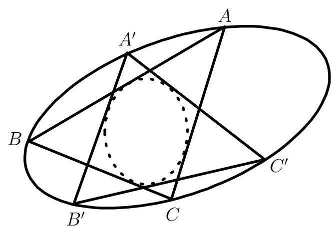

Fig. 1.

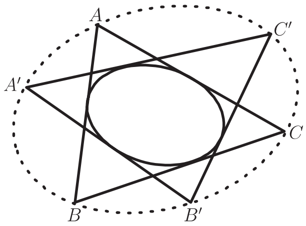

Fig. 2.

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and the Euclidean norm in $\mathbb{C}^n$, respectively. It is always a nonempty compact and convex subset of the complex plane. The suggested references on the numerical range of matrices are [6] and [7, Chapter 1].

2. One polygon

The following theorem summarizes what we know about the circumscribing property of the numerical range of an $\mathscr{S}_n$-matrix. The major part of it has been proved in [2,3,5]. We present it here for easy reference.

Theorem 2.1. *Let* $U = \mathrm{diag}(a_1, \ldots, a_{n+1})$ *be a diagonal unitary matrix whose eigenvalues* a_j*'s are distinct and arranged consecutively in the order of* $a_1, \ldots, a_{n+1}$ *around the unit circle, let A be an* $\mathscr{S}_n$*-matrix with eigenvalues* $b_1, \ldots, b_n$, *and let* $\phi(z) = \prod_{j=1}^{n}(z - b_j)/(1 - \overline{b_j}z)$ *be the corresponding finite Blaschke product. Then the following conditions are equivalent*:

(1) $W(A)$ *is circumscribed by the* $(n+1)$*-gon* $a_1 \cdots a_{n+1}$;

(2) *A dilates to* U;

(3) *there exist* $m_1, \ldots, m_{n+1} > 0$ *with* $\sum_j m_j = 1$ *such that A is unitarily equivalent to* V^*UV, *where V is an* $(n+1)$*-by-n matrix of the inclusion map from the orthogonal complement in* $\mathbb{C}^{n+1}$ *of the one-dimensional subspace generated by* $\left[\sqrt{m_1}\ \cdots\ \sqrt{m_{n+1}}\right]^{\mathrm{T}}$ *into* $\mathbb{C}^{n+1}$;

(4) *there exist* $m_1, \ldots, m_{n+1} > 0$ *with* $\sum_j m_j = 1$ *such that*

$$\prod_{k=1}^{n}(z - b_k) = \sum_{j=1}^{n+1} m_j (z - a_1)\cdots\widehat{(z - a_j)}\cdots(z - a_{n+1})$$

for all z, where the hat "∧" over $z - a_j$ *denotes that this term is absent from the product*;

(5) *there exist* $m_1, \ldots, m_{n+1} > 0$ *with* $\sum_j m_j = 1$ *such that*

$$\sum_{j=1}^{n+1} \frac{m_j}{b_k - a_j} = 0$$

for all k, $1 \leqslant k \leqslant n$;

(6) *there exist* $m_1, \ldots, m_{n+1} > 0$ *with* $\sum_j m_j = 1$ *such that*

$$\frac{\phi(z)}{z\phi(z) - (-1)^n \prod_l a_l} = \sum_{j=1}^{n+1} \frac{m_j}{z - a_j}$$

for all $z \neq a_1, \ldots, a_{n+1}$;

(7) $z \prod_k (z - b_k) - (-1)^n \big(\prod_l a_l\big)\big(\prod_k (1 - \overline{b_k} z)\big) = \prod_j (z - a_j)$ *for all* z;

(8) $\phi(a_j) = (-1)^n a_1 \cdots \widehat{a_j} \cdots a_{n+1}$ *for all* j, $1 \leqslant j \leqslant n+1$;

(9) $\alpha_j = \beta_j + \alpha_{n+1}\overline{\beta_{n-j+1}}$ *for all* j, $1 \leqslant j \leqslant n$, *where*

$$\alpha_j = \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n+1} a_{k_1} \cdots a_{k_j} \quad \text{and} \quad \beta_j = \sum_{1 \leqslant k_1 < \cdots < k_j \leqslant n} b_{k_1} \cdots b_{k_j}$$

are the jth *elementary symmetric functions of the* a_k*'s and* b_k*'s*, *respectively*;

(10) $\alpha_j = \beta_j + \alpha_{n+1}\overline{\beta_{n-j+1}}$ *for all* j, $1 \leqslant j \leqslant \lceil n/2 \rceil$.

In this case, *the* m_j*'s in* (3)–(6) *are unique and given by*

$$m_j = \frac{\prod_k (a_j - b_k)}{\prod_{l \neq j} (a_j - a_l)}, \quad 1 \leqslant j \leqslant n+1,$$

and the tangent point of the edge $[a_j, a_{j+1}]$ *of the* $(n+1)$*-gon* $a_1 \cdots a_{n+1}$ *with* $\partial W(A)$ *is*

$$\frac{m_{j+1}a_j + m_j a_{j+1}}{m_j + m_{j+1}}, \quad 1 \leqslant j \leqslant n+1.$$

Recall that an n-by-n matrix A is said to *dilate* to the m-by-m matrix B ($n \leqslant m$) or, equivalently, A is a *compression* of B if there is an m-by-n matrix V with $V^*V = I_n$ such that $A = V^*BV$. This is the same as saying that B is unitarily equivalent to a matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$.

For the proof of certain parts of the preceding theorem, it is convenient to have the following alternative description of the matrices in $\mathscr{S}_n$.

Lemma 2.2. *An* n*-by-*n *matrix* A *lies in* $\mathscr{S}_n$ *if and only if it is unitarily equivalent to* V^*UV, *where* $U = \text{diag}(a_1, \ldots, a_{n+1})$ *is a diagonal unitary matrix with distinct eigenvalues* a_j*'s and* V *is an* $(n+1)$*-by-*n *matrix with* $V^*V = I_n$ *such that the orthogonal complement in* $\mathbb{C}^{n+1}$ *of its range space is generated by a vector with all its entries nonzero.*

Proof. If A lies in $\mathscr{S}_n$, then it dilates to a unitary matrix $U = \text{diag}(a_1, \ldots, a_{n+1})$ with distinct a_j's (cf. [2, Lemma 2.2]). Hence $A = V_1^*UV_1$ for some $(n+1)$-by-n

matrix V_1 with $V_1^*V_1 = I_n$. In this case, the dimension of the range space K of V_1 is equal to n. We express V_1 as a product VW, where V is some $(n+1)$-by-n matrix of the inclusion map from K to $\mathbb{C}^{n+1}$ and W is the corresponding n-by-n unitary matrix from $\mathbb{C}^n$ onto K. Obviously, V satisfies $V^*V = I_n$ and has range space equal to K. Let $x = [x_1 \ \cdots \ x_{n+1}]^{\mathrm{T}}$ be any nonzero vector orthogonal to K. We claim that $x_j \neq 0$ for all j. Indeed, if $x_j = 0$ for some j, then the vector $e_j = [0 \ \cdots \ 0 \ \underset{j\text{th}}{1} \ 0 \ \cdots \ 0]^{\mathrm{T}}$ is in K and hence

$$(WAW^*)e_j = V^*UVe_j = V^*Ue_j = V^*(a_je_j) = a_je_j.$$

This shows that a_j is an eigenvalue of WAW^*, contradicting the fact that all eigenvalues of A lie in the open unit disc $\mathbb{D}$. Thus $A = W^*(V^*UV)W$ is unitarily equivalent to V^*UV with the asserted properties.

Conversely, let $B = V^*UV$ be an n-by-n compression of U of the asserted form. Then obviously $\|B\| \leqslant 1$, and by [5, Theorem 3], we have $W(B) \subseteq \mathbb{D}$. Consequently, all the eigenvalues of B lie in $\mathbb{D}$. Moreover, since

$$I_n - B^*B = I_n - (V^*U^*V)(V^*UV) = V^*U^*(I_{n+1} - VV^*)UV$$

and $I_{n+1} - VV^*$ has rank one, the same is true for $I_n - B^*B$. We infer that B, and hence A, is of class $\mathscr{S}_n$. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. The equivalence of (1) and (2) is established in [2, Theorem 2.1], that of (2) and (3) follows from Lemma 2.2, and that of (2) and (8) from [3, Lemma 2.4]. That (3) implies (4) is a consequence of [5, Theorem 1]. We now prove that (4) implies (3). Indeed, if (4) holds, then let $x = \left[\sqrt{m_1} \ \cdots \ \sqrt{m_{n+1}}\right]^{\mathrm{T}}$, let V be some $(n+1)$-by-n matrix of the inclusion map from the orthogonal complement of $x \in \mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$, and let $B = V^*UV$. Then B is an $\mathscr{S}_n$-matrix by Lemma 2.2 and has the same eigenvalues as A by [5, Theorem 1] and condition (4). Since matrices in $\mathscr{S}_n$ are determined up to unitary equivalence by their eigenvalues, it follows that A is unitarily equivalent to B. This proves (4) ⇒ (3).

Next we assume that (4) is true. By dividing both sides of the equality in (4) by $\prod_{l=1}^{n+1}(z - a_l)$ and setting $z = b_k$, we obtain (5). On the other hand, if (5) holds and $p(z)$ denotes the polynomial on the right-hand side of the equality in (4), then, since $p(z)$ is of degree n, has leading coefficient one and has b_k's as its zeros, we obtain $p(z) = \prod_k (z - b_k)$, that is, (4) holds.

If (7) is true, then by dividing both sides of the equality in (7) by $\prod_k (1 - \overline{b_k}z)$ and letting $z = a_j$, $1 \leqslant j \leqslant n+1$, we obtain (8). Conversely, if (8) holds, then the two $(n+1)$st degree polynomials $z\prod_k(z - b_k) - (-1)^n\left(\prod_l a_l\right)\left(\prod_k(1 - \overline{b_k}z)\right)$ and $\prod_j (z - a_j)$, being equal at the $n+2$ points $a_1, \ldots, a_{n+1}$ and 0, are equal for all z, that is, (7) holds.

We now prove (6) $\Rightarrow$ (8). Assuming that (6) is true, we have

$$\frac{\phi(z)}{z\phi(z)-(-1)^n\prod_l a_l}=\frac{\sum_j m_j(z-a_1)\cdots\widehat{(z-a_j)}\cdots(z-a_{n+1})}{\prod_k(z-a_k)}. \tag{a}$$

Since the a_k's are distinct and the m_j's are nonzero, the a_k's are not zeros of the numerator of the right-hand side of (a). So they are poles of (a) and hence are zeros of $z\phi(z)-(-1)^n\prod_l a_l$. This shows that (8) holds. To prove (8) $\Rightarrow$ (6), assume that (8) holds. Then

$$\begin{aligned}\frac{\phi(z)}{z\phi(z)-(-1)^n\prod_l a_l}&=\frac{\prod_k(z-b_k)}{z\prod_k(z-b_k)-(-1)^n\left(\prod_l a_l\right)\left(\prod_k(1-\overline{b_k}z)\right)}\\&=\frac{\prod_k(z-b_k)}{\prod_k(z-a_k)} \qquad \text{(by (7))}\\&=\frac{\sum_j m_j(z-a_1)\cdots\widehat{(z-a_j)}\cdots(z-a_{n+1})}{\prod_k(z-a_k)} \qquad \text{(by (4))}\\&=\sum_j\frac{m_j}{z-a_j},\end{aligned}$$

which proves (6).

The equivalence of (1) and (9) is proved in [3, Theorem 2.5]. Hence the above discussion yields the equivalence of conditions (1)–(9). Here we give a direct proof of (8) $\Leftrightarrow$ (9). Since

$$\prod_{j=1}^{n+1}(z-a_j)=\sum_{k=0}^{n+1}(-1)^k\alpha_k z^{n-k+1}$$

and

$$\prod_{j=1}^{n}(z-b_j)=\sum_{k=0}^{n}(-1)^k\beta_k z^{n-k}$$

($\alpha_0=\beta_0\equiv 1$), condition (9) is equivalent to

$$\begin{aligned}\prod_j(z-a_j)&=\sum_k(-1)^k(\beta_k+\alpha_{n+1}\overline{\beta_{n-k+1}})z^{n-k+1}\\&=\sum_k(-1)^k\beta_k z^{n-k+1}+\alpha_{n+1}\sum_k(-1)^k\overline{\beta_{n-k+1}}z^{n-k+1}\\&=z\prod_j(z-b_j)-(-1)^n\alpha_{n+1}\sum_k(-1)^{n-k+1}\overline{\beta_{n-k+1}}z^{n-k+1}\\&=z\prod_j(z-b_j)-(-1)^n\alpha_{n+1}\prod_j(1-\overline{b_j}z)\end{aligned}$$

($\beta_{n+1}\equiv 0$), which is easily seen to be the same as (8).

That (9) and (10) are equivalent follows from the fact that, for each j, the equality $\alpha_j = \beta_j + \alpha_{n+1}\overline{\beta_{n-j+1}}$ holds if and only if $\alpha_{n-j+1} = \beta_{n-j+1} + \alpha_{n+1}\overline{\beta_j}$ does. Indeed, if the former is the case, then we multiply $\overline{\alpha_{n+1}} = \prod_k \overline{a_k}$ on its both sides to obtain $\overline{\alpha_{n-j+1}} = \overline{\alpha_{n+1}}\beta_j + \overline{\beta_{n-j+1}}$, which is the same as the latter equality. The converse follows by symmetry.

Finally, the expression $m_j = \prod_k (a_j - b_k)/\prod_{l\neq j}(a_j - a_l)$, $1 \leqslant j \leqslant n+1$, follows by setting $z = a_j$ in the equality of (4). On the other hand, the expression $(m_{j+1}a_j + m_j a_{j+1})/(m_j + m_{j+1})$ for the tangent point of $\partial W(A)$ with $[a_j, a_{j+1}]$ follows as in the proof of [5, Theorem 3]. This completes the proof. □

To conclude this section, we give two remarks. Firstly, condition (6), which is a hybrid of (4) and (7), was originally in [1]. In particular, [1, Lemma 4] proves (8) $\Rightarrow$ (6) by taking the partial fraction decomposition $\sum_j m_j/(z - a_j)$ of the rational function $\phi(z)/\big(z\phi(z) - (-1)^n \prod_l a_l\big)$ and showing that the coefficients m_j's satisfy $m_j > 0$ for all j and $\sum_j m_j = 1$ via some analytic arguments. Secondly, the equivalence of (3)–(5) carries over to the more general normal compression case [5].

3. Two polygons

As stated in the introduction, the main result of this section is motivated by the classical result of two triangles interscribing between two conics. The following theorem is a natural generalization to the polygon case.

Theorem 3.1. *If $a_1, a_1', a_2, a_2', \ldots, a_{n+1}, a_{n+1}'$ (in this order) are $2n+2$ distinct points on the unit circle, then there is an $\mathscr{S}_n$-matrix A, unique up to unitary equivalence, with numerical range $W(A)$ circumscribed by the two $(n+1)$-gons $a_1 a_2 \cdots a_{n+1}$ and $a_1' a_2' \cdots a_{n+1}'$.*

The proof is lengthy; it involves some complicated algebraic derivations. However, the basic idea is pretty simple. If A is the asserted $\mathscr{S}_n$-matrix with eigenvalues $b_1, \ldots, b_n$ and $\phi(z) = \prod_{j=1}^n (z - b_j)/(1 - \overline{b_j}z)$ is the corresponding Blaschke product, then since $W(A)$ is circumscribed by $a_1 \cdots a_{n+1}$, by Theorem 2.1(8), we have that $a_j\phi(a_j) = (-1)^n \prod_k a_k$ for every $j = 1, 2, \ldots, n+1$. Furthermore, since $W(A)$ is also circumscribed by $a_1' \cdots a_{n+1}'$, by Theorem 2.1(6), there are $m_1, \ldots, m_{n+1} > 0$ with $\sum_k m_k = 1$ such that

$$\frac{\phi(z)}{z\phi(z) - (-1)^n \prod_l a_l'} = \sum_{k=1}^{n+1} \frac{m_k}{z - a_k'} \quad (z \neq a_1', \ldots, a_{n+1}').$$

Setting $z = a_j$ yields

$$\frac{\phi(a_j)}{a_j\phi(a_j) - (-1)^n \prod_l a_l'} = \sum_{k=1}^{n+1} \frac{m_k}{a_j - a_k'}, \quad 1 \leqslant j \leqslant n+1.$$

Combining these two statements, we obtain

$$\sum_{k=1}^{n+1} \frac{a_j m_k}{a_j - a_k'} = \frac{(-1)^n \prod_l a_l}{(-1)^n \left(\prod_l a_l\right) - (-1)^n \left(\prod_l a_l'\right)}$$
$$= \frac{1}{1 - \prod_l (\overline{a_l} a_l')}, \quad 1 \leqslant j \leqslant n+1. \tag{b}$$

Thus, to prove Theorem 3.1, it is enough to show that the solutions m_k's of the linear system (b) satisfy $m_k > 0$ and $\sum_k m_k = 1$. This is what we do in the following three lemmas.

Lemma 3.2. *Let* $a_1, a_1', \ldots, a_{n+1}, a_{n+1}'$ *be as in Theorem* 3.1, *let* $M = \left[a_j/(a_j - a_k')\right]_{j,k=1}^{n+1}$, *and for any* k_0, $1 \leqslant k_0 \leqslant n+1$, *let* M_{k_0} *be the matrix obtained from* M *by replacing its* k_0th *column by* $[1 \ \cdots \ 1]^{\mathrm{T}}$. *Then*

(1) $\prod_j a_j \neq \prod_k a_k'$,

(2) $\det M = \frac{\left(\prod_j a_j\right) \prod_{j<k} [(a_j - a_k)(a_k' - a_j')]}{\prod_{j,k} (a_j - a_k')}$, *and*

(3) $\det M_{k_0} = \frac{(-1)^{k_0 - 1} \left(\prod_{j<k} (a_j - a_k)\right) \left(\prod_{j \neq k_0} a_j'\right) \left(\prod_{j<k,\, j,k \neq k_0} (a_k' - a_j')\right)}{\prod_j \prod_{k \neq k_0} (a_j - a_k')}$.

Proof. (1) We may assume that $a_j = \exp(\mathrm{i}\theta_j)$ and $a_k' = \exp(\mathrm{i}\theta_k')$ with $0 \leqslant \theta_1 < \theta_1' < \cdots < \theta_{n+1} < \theta_{n+1}' < 2\pi$. Our assertion is then equivalent to

$$0 < \sum_{k=1}^{n+1} \theta_k' - \sum_{j=1}^{n+1} \theta_j < 2\pi.$$

The first inequality is obvious. To prove the second, we have

$$\sum_{k=1}^{n+1} \theta_k' - \sum_{j=1}^{n+1} \theta_j \leqslant \sum_{k=1}^{n+1} \theta_k' - \sum_{j=2}^{n+1} \theta_j$$
$$= \theta_{n+1}' - \sum_{j=1}^{n} (\theta_{j+1} - \theta_j')$$
$$< \theta_{n+1}' < 2\pi$$

as required.

(2) This part of the lemma should have been known to Cauchy before 1841. However, for clarity, we give a proof different from the one in [11, Part VII, Problem 3].

To evaluate $\det M$, we first take out $a_j/\prod_k(a_j - a'_k)$ from the jth row of M to obtain

$$\det M = \frac{\prod_j a_j}{\prod_{j,k}(a_j - a'_k)} \det\left[\prod_{l\neq k}(a_j - a'_l)\right]_{j,k=1}^{n+1}.$$

If $a_j = a_k$ or $a'_j = a'_k$ for any $j < k$, the latter determinant becomes zero. Since it is a homogeneous polynomial of degree $n(n+1)$, this implies that each of $a_j - a_k$ and $a'_j - a'_k$ is its factor. By comparing the coefficients, we conclude that this determinant is equal to $\prod_{j<k}[(a_j - a_k)(a'_k - a'_j)]$. When multiplied by $\left(\prod_j a_j\right)/\left(\prod_{j,k}(a_j - a'_k)\right)$, this yields the asserted expression for $\det M$.

(3) If we let

$$a''_k = \begin{cases} a'_k & \text{if } k \neq k_0, \\ 0 & \text{otherwise,} \end{cases}$$

then $M_{k_0} = \left[a_j/(a_j - a''_k)\right]_{j,k=1}^{n+1}$ is a matrix of the same type as M. Hence, by (2),

$$\begin{aligned}
&\det M_{k_0} \\
&= \frac{\left(\prod_j a_j\right)\prod_{j<k}[(a_j - a_k)(a''_k - a''_j)]}{\prod_{j,k}(a_j - a''_k)} \\
&= \frac{\left(\prod_j a_j\right)\left(\prod_{j<k}(a_j - a_k)\right)\left(\prod_{k_0<k} a'_k\right)\left(\prod_{j<k_0}(-a'_j)\right)\left(\prod_{j<k,\, j,k\neq k_0}(a'_k - a'_j)\right)}{\left(\prod_j a_j\right)\left(\prod_j \prod_{k\neq k_0}(a_j - a'_k)\right)} \\
&= \frac{(-1)^{k_0-1}\left(\prod_{j<k}(a_j - a_k)\right)\left(\prod_{j\neq k_0} a'_j\right)\left(\prod_{j<k,\, j,k\neq k_0}(a'_k - a'_j)\right)}{\prod_j \prod_{k\neq k_0}(a_j - a'_k)},
\end{aligned}$$

and the proof is completed. □

To proceed further, we denote

$$m_k = \frac{\det M_k}{\left(1 - \prod_j(\overline{a_j}a'_j)\right)\det M}, \quad 1 \leqslant k \leqslant n+1. \tag{c}$$

Note that here the denominator is nonzero because of Lemma 3.2(1) and (2). The m_k's are the solution of the linear system (b). As mentioned above, we need to show that $m_k > 0$ for all k and $\sum_k m_k = 1$. The next lemma implies the positivity of the m_k's.

Lemma 3.3. *The m_k's in* (c) *are all strictly positive.*

Proof. Using the expressions for the determinants in Lemma 3.2(2) and (3), we have that for any k_0, $1 \leqslant k_0 \leqslant n+1$,

$$
\begin{aligned}
m_{k_0} &= \frac{\dfrac{(-1)^{k_0-1}\left(\prod_{j<k}(a_j-a_k)\right)\left(\prod_{j\neq k_0} a'_j\right)\left(\prod_{j<k,\, j,k\neq k_0}(a'_k-a'_j)\right)}{\prod_j \prod_{k\neq k_0}(a_j-a'_k)}}{\left(1-\prod_j(\overline{a_j}a'_j)\right)\dfrac{\left(\prod_j a_j\right)\prod_{j<k}[(a_j-a_k)(a'_k-a'_j)]}{\prod_{j,k}(a_j-a'_k)}} \\
&= \frac{(-1)^{k_0-1}\left(\prod_{j\neq k_0} a'_j\right)\left(\prod_j (a_j-a'_{k_0})\right)}{\left(\prod_j a_j-\prod_j a'_j\right)\left(\prod_{k_0<k}(a'_k-a'_{k_0})\right)\left(\prod_{j<k_0}(a'_{k_0}-a'_j)\right)} \\
&= \frac{a_{k_0}-a'_{k_0}}{\left(\prod_j a_j\right)\left(\prod_{j\neq k_0}\overline{a'_j}\right)-a'_{k_0}} \prod_{k\neq k_0} \frac{a_k-a'_{k_0}}{a'_k-a'_{k_0}}.
\end{aligned}
$$

Note that, for any a, b and c on the unit circle with $a, b \neq c$, we have $\arg(a-c) = (\pi - \arg a + \arg c)/2$ and $\arg(b-c) = (\pi - \arg b + \arg c)/2$, and hence $\arg((a-c)/(b-c)) = (\arg b - \arg a)/2$. Applied to our present situation, this yields

$$
\begin{aligned}
\arg m_{k_0} &= \frac{1}{2}\left(\sum_j \arg a_j + \sum_{j\neq k_0} \arg \overline{a'_j} - \arg a_{k_0}\right) + \sum_{k\neq k_0} \frac{1}{2}(\arg a'_k - \arg a_k) \\
&= 0.
\end{aligned}
$$

Hence $m_{k_0} > 0$ as asserted. □

Next we prove that $\sum_{k=1}^{n+1} m_k = 1$.

Lemma 3.4. *The sum of $m_1, \ldots, m_{n+1}$ equals one.*

Proof. In view of the expressions for m_k's in the proof of Lemma 3.3, we need only to show that

$$
\begin{aligned}
&\sum_{k=1}^{n+1}\left[(-1)^{k-1}\left(\prod_{j\neq k} a'_j\right)\left(\prod_j (a_j-a'_k)\right)\left(\prod_{\substack{j<l \\ j,l\neq k}}(a'_l-a'_j)\right)\right] \\
&\quad = \left(\prod_j a_j - \prod_j a'_j\right)\prod_{j<l}(a'_l-a'_j). \qquad \text{(d)}
\end{aligned}
$$

Let

$$
p(x) = \prod_{j=1}^{n+1}(a_j - x) = \sum_{l=0}^{n+1} \alpha_l (-x)^{n-l+1},
$$

where α_l is the lth elementary symmetric function of the a_j's. Then the left-hand side of (d) becomes

$$\sum_{k=1}^{n+1}\left[(-1)^{k-1}\left(\prod_{j\neq k} a'_j\right)\left(\prod_{\substack{j<l\\ j,l\neq k}}(a'_l-a'_j)\right)p(a'_k)\right]$$

or

$$\beta_{n+1}+\alpha_1\beta_n+\cdots+\alpha_n\beta_1+\alpha_{n+1}\beta_0,$$

where

$$\beta_m=\sum_{k=1}^{n+1}\left[(-1)^{k-1}\left(\prod_{j\neq k} a'_j\right)\left(\prod_{\substack{j<l\\ j,l\neq k}}(a'_l-a'_j)\right)(-a'_k)^m\right],\quad 0\leqslant m\leqslant n+1.$$

We will check that

$$\beta_m=\begin{cases}\prod_{j<l}(a'_l-a'_j) & \text{if } m=0,\\ 0 & \text{if } 1\leqslant m\leqslant n,\\ -\big(\prod_j a'_j\big)\big(\prod_{j<l}(a'_l-a'_j)\big) & \text{if } m=n+1.\end{cases}\tag{e}$$

If this is indeed true, then the left-hand side of (d) equals

$$-\left(\prod_j a'_j\right)\left(\prod_{j<l}(a'_l-a'_j)\right)+\alpha_{n+1}\prod_{j<l}(a'_l-a'_j),$$

which is the same as the right-hand side of (d).

To prove (e), let

$$q_m=\sum_{k=1}^{n+1}(-1)^{k+m-1}{a'_k}^{m-1}\prod_{\substack{j<l\\ j,l\neq k}}(a'_l-a'_j),\quad 0\leqslant m\leqslant n+1.$$

We claim that if $a'_{l_0}=a'_{j_0}$ for any $j_0<l_0$, then $q_m=0$. Indeed, in this case

$$q_m=(-1)^{j_0+m-1}{a'_{j_0}}^{m-1}\prod_{\substack{j<l\\ j,l\neq j_0}}(a'_l-a'_j)+(-1)^{l_0+m-1}{a'_{l_0}}^{m-1}\prod_{\substack{j<l\\ j,l\neq l_0}}(a'_l-a'_j).$$

The two products in the above expression can be simplified as

$$\begin{aligned}\prod_{\substack{j<l\\ j,l\neq j_0}}(a'_l-a'_j)&=\left(\prod_{\substack{j<l\\ j,l\neq j_0,l_0}}(a'_l-a'_j)\right)\left(\prod_{l_0<l}(a'_l-a'_{l_0})\right)\left(\prod_{\substack{j<l_0\\ j\neq j_0}}(a'_{l_0}-a'_j)\right)\\ &=\left(\prod_{\substack{j<l\\ j,l\neq j_0,l_0}}(a'_l-a'_j)\right)(-1)^{l_0-2}\left(\prod_{l\neq j_0,l_0}(a'_l-a'_{l_0})\right)\end{aligned}$$

and, similarly,

$$\prod_{\substack{j<l\\ j,l\neq l_0}}(a'_l-a'_j)=\left(\prod_{\substack{j<l\\ j,l\neq j_0,l_0}}(a'_l-a'_j)\right)(-1)^{j_0-1}\left(\prod_{l\neq j_0,l_0}(a'_l-a'_{j_0})\right).$$

Plugging these into the expression of q_m yields

$$\begin{aligned}q_m&=\left[(-1)^{j_0+m-1+l_0-2}+(-1)^{l_0+m-1+j_0-1}\right]{a'_{j_0}}^{m-1}\\&\quad\times\left(\prod_{\substack{j<l\\ j,l\neq j_0,l_0}}(a'_l-a'_j)\right)\left(\prod_{l\neq j_0,l_0}(a'_l-a'_{j_0})\right)=0.\end{aligned}$$

Let

$$r_m=\begin{cases}\left(\prod_j a'_j\right)q_0 & \text{if } m=0,\\ q_m & \text{otherwise.}\end{cases}$$

Then r_m is a homogeneous polynomial in $a'_1,\dots,a'_{n+1}$ of degree $(n+1)-1+\binom{n}{2}=n(n+1)/2$ if $m=0$ and of degree $m-1+\binom{n}{2}$ if $m\geqslant 1$. We deduce from above that $q\equiv\prod_{j<l}(a'_l-a'_j)$ is a factor of r_m. Since q is a polynomial of degree $n(n+1)/2$, it follows that r_0 and r_{n+1} are constant multiples of q and $r_m=0$ for $1\leqslant m\leqslant n$. Say, $r_0=c_1q$ and $r_{n+1}=c_2q$ for some constants c_1 and c_2. Letting $a'_1=0$ in the former, we obtain

$$\begin{aligned}c_1q(0,a'_2,\dots,a'_{n+1})&=r_0(0,a'_2,\dots,a'_{n+1})\\&=\left(\prod_{j\neq 1}a'_j\right)\left(\prod_{\substack{j<l\\ j,l\neq 1}}(a'_l-a'_j)\right)\\&=q(0,a'_2,\dots,a'_{n+1}),\end{aligned}$$

which shows that $c_1=1$. On the other hand, observing that the coefficients of a'^{n}_{n+1} in r_{n+1} and q, considered as polynomials in a'_{n+1}, are $-\prod_{j<l<n+1}(a'_l-a'_j)$ and $\prod_{j<l<n+1}(a'_l-a'_j)$, respectively, we obtain $c_2=-1$. In conclusion, we have

$$\begin{aligned}\beta_m=\left(\prod_j a'_j\right)q_m&=\begin{cases}r_0 & \text{if } m=0,\\ \left(\prod_j a'_j\right)r_m & \text{otherwise,}\end{cases}\\&=\begin{cases}\prod_{j<l}(a'_l-a'_j) & \text{if } m=0,\\ 0 & \text{if } 1\leqslant m\leqslant n,\\ -\left(\prod_j a'_j\right)\left(\prod_{j<l}(a'_l-a'_j)\right) & \text{if } m=n+1.\end{cases}\end{aligned}$$

This proves (e) and thus completes the proof of this lemma. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For the existence of the $\mathscr{S}_n$-matrix A, note that the m_k's in (c) satisfy the equalities in (b) and are such that $m_k > 0$ and $\sum_k m_k = 1$ by Lemmas 3.3 and 3.4, respectively. By Theorem 2.1(3), we obtain the $\mathscr{S}_n$-matrix $A = V^*UV$, where U is the unitary matrix $\operatorname{diag}(a'_1, \ldots, a'_{n+1})$ and V is an $(n+1)$-by-n matrix determined by the m_k's as therein. Then $W(A)$ is circumscribed by the $(n+1)$-gon $a'_1 \cdots a'_{n+1}$. To prove the circumscribing by $a_1 \cdots a_{n+1}$, we form the finite Blaschke product $\phi(z) = \prod_{j=1}^{n}(z - b_j)/(1 - \overline{b_j}z)$, where the b_j's are the eigenvalues of A. Since by Theorem 2.1(6)

$$\frac{\phi(z)}{z\phi(z) - (-1)^n \prod_l a'_l} = \sum_{k=1}^{n+1} \frac{m_k}{z - a'_k}$$

for all $z \neq a'_1, \ldots, a'_{n+1}$, letting $z = a_j$ and combining with (b) yield

$$\frac{\phi(a_j)}{a_j\phi(a_j) - (-1)^n \prod_l a'_l} = \sum_{k=1}^{n+1} \frac{m_k}{a_j - a'_k} = \frac{1}{a_j\left(1 - \prod_l(\overline{a_l}a'_l)\right)},$$
$$1 \leqslant j \leqslant n+1.$$

Simplifying the above, we obtain $a_j\phi(a_j) = (-1)^n \prod_l a_l$ for all j. By Theorem 2.1(8), this implies that $W(A)$ is circumscribed by $a_1 \cdots a_{n+1}$.

We now prove the uniqueness of A. If A' is another $\mathscr{S}_n$-matrix whose numerical range is circumscribed by $a_1 \cdots a_{n+1}$ and $a'_1 \cdots a'_{n+1}$, then, as was pointed out after the statement of Theorem 3.1, we have the linear system

$$\sum_{k=1}^{n+1} \frac{a_j m'_k}{a_j - a'_k} = \frac{1}{1 - \prod_l(\overline{a_l}a'_l)}, \quad 1 \leqslant j \leqslant n+1, \qquad \text{(b}'\text{)}$$

for some m'_k's satisfying $m'_k > 0$ and $\sum_k m'_k = 1$. Since the matrix M of the coefficients of (b′) (or (b)) is nonsingular by Lemma 3.2(2), it follows that the m'_k's coincide with the m_k's defined in (c). Therefore, the eigenvalues of A and A' are the same by Theorem 2.1(4), and hence A and A' are unitarily equivalent, completing the proof. □

Note that we can transform the unit circle to an arbitrary ellipse via some affine transformation and thus obtain from Theorem 3.1 that if two $(n+1)$-gons are inscribed on an ellipse with interlacing vertices, then there is an algebraic convex closed curve of degree at most $n(n-1)$ which is tangent to the $n+1$ edges of each of the two polygons.

We now formulate some equivalent conditions to the conclusion in Theorem 3.1. These are obtained by invoking Theorem 2.1(2), (4), (8) and (9), respectively.

Proposition 3.5. *Let $a_1, a'_1, \ldots, a_{n+1}, a'_{n+1}$ (in this order) be $2n+2$ distinct points on the unit circle. Then*

(1) *there exists an* $\mathscr{S}_n$*-matrix, unique up to unitary equivalence, which dilates to* $\mathrm{diag}(a_1, \ldots, a_{n+1})$ *and* $\mathrm{diag}(a'_1, \ldots, a'_{n+1})$;
(2) *there exist unique* $m_1, \ldots, m_{n+1}, m'_1, \ldots, m'_{n+1} > 0$ *with* $\sum_j m_j = \sum_k m'_k = 1$ *such that*

$$\sum_j m_j(z-a_1)\cdots\widehat{(z-a_j)}\cdots(z-a_{n+1}) = \sum_k m'_k(z-a'_1)\cdots\widehat{(z-a'_k)}\cdots(z-a'_{n+1})$$

for all z;
(3) *there exists a unique finite Blaschke product* ϕ *with n zeros such that*

$$\phi(a_j) = (-1)^n a_1\cdots\widehat{a_j}\cdots a_{n+1}$$

and

$$\phi(a'_j) = (-1)^n a'_1\cdots\widehat{a'_j}\cdots a'_{n+1}$$

for all j, $1 \leqslant j \leqslant n+1$;
(4) *there exist unique n points* $b_1, \ldots, b_n$ *in* $\mathbb{D}$ *such that*

$$\beta_j = \frac{\overline{\alpha_{n-j+1}} - \overline{\alpha'_{n-j+1}}}{\overline{\alpha_{n+1}} - \overline{\alpha'_{n+1}}}$$

for all j, $1 \leqslant j \leqslant n$, *where* α_j, α'_j *and* β_j *denote the* j*th elementary symmetric functions of the* a_k*'s,* a'_k*'s and* b_k*'s, respectively.*

To illustrate our main result, we end this paper by giving an example of two 4-gons inscribed on the unit circle both of which circumscribe the numerical range of some $\mathscr{S}_3$-matrix.

Example 3.6. Let $a_1 = 1$, $a_2 = (-1+\sqrt{255}\mathrm{i})/16$, $a_3 = -1$ and $a_4 = (-1-\sqrt{255}\mathrm{i})/16$ [resp., $a'_1 = \omega(-1-\sqrt{255}\mathrm{i})/16$, $a'_2 = \omega$, $a'_3 = \omega(-1+\sqrt{255}\mathrm{i})/16$ and $a'_4 = -\omega$, where $\omega = (-1+\sqrt{3}\mathrm{i})/2$] be the vertices of a 4-gon inscribed on the unit circle. Then

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1/8 & 0 \end{bmatrix}$$

is the unique (up to unitary equivalence) $\mathscr{S}_3$-matrix whose numerical range is circumscribed by the 4-gons $a_1a_2a_3a_4$ and $a'_1a'_2a'_3a'_4$. This is because, for any real θ, A has the 4-by-4 unitary dilation

$$U_\theta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/8 & 0 & -\sqrt{63}/8 \\ 0 & \mathrm{e}^{\mathrm{i}\theta}\sqrt{63}/8 & 0 & \mathrm{e}^{\mathrm{i}\theta}/8 \end{bmatrix},$$

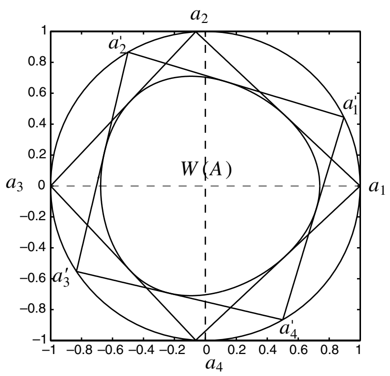

Fig. 3.

and hence $W(A)$ is circumscribed by the 4-gon whose vertices are the eigenvalues of U_θ. For θ equal to π and $-\pi/3$, this gives $a_1a_2a_3a_4$ and $a_1'a_2'a_3'a_4'$, respectively (cf. Fig. 3). In this case, since the eigenvalues of A are $b_1 = 1/2$, $b_2 = \omega/2$ and $b_3 = \omega^2/2$, the corresponding m_j's and m_j''s can be calculated to be $m_1 = 7/34$, $m_2 = 21/85$, $m_3 = 3/10$, $m_4 = 21/85$, and $m_1' = 21/85$, $m_2' = 7/34$, $m_3' = 21/85$, $m_4' = 3/10$. The calculation can be simplified by observing that A, ωA and $\omega^2 A$ are mutually unitarily equivalent. Hence, in particular, $W(A)$ is symmetric with respect to the x-axis and therefore we have $m_2 = m_4$ and $m_1' = m_4$, $m_2' = m_1$, $m_3' = m_2$ and $m_4' = m_3$. Thus we need only to compute m_1 and m_3 from the formula in Theorem 2.1 and obtain $m_2 = m_4 = (1 - m_1 - m_3)/2$ and the m_j''s from the relations above. The tangent points of $\partial W(A)$ with the edges of $a_1a_2a_3a_4$ and $a_1'a_2'a_3'a_4'$ can then be computed easily from Theorem 2.1.

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