Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 293 (2004) 258–268
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Lagrangian stability of a nonlinear quasi-periodic system ☆

Song-Sun Lin [a] and Yiqian Wang [b,*]

[a] *Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan, ROC*
[b] *Department of Mathematics, Nanjing University, Nanjing 210093, PR China*

Received 19 April 2001
Submitted by R. Manasevich

Abstract

In this paper, we prove the Lagrangian stability of the quasi-periodic system $d^2x/dt^2 + G_x(x,t) = 0$, where G is quasi-periodic in both x and t, respectively.

1. Introduction

In this paper, we consider the existence of quasi-periodic solutions and the Lagrangian stability of the following quasi-periodic system:

$$\frac{d^2x}{dt^2} + G_x(x,t) = 0, \tag{1.1}$$

where $G(x,t)$ is quasi-periodic in x and t with basic frequencies $\omega_1, \dots, \omega_m$ and $\omega_{m+1}, \dots, \omega_{m+n}$, respectively, i.e.,

$$G(x,t) = \sum_{(k,l)\in\mathbb{Z}^m\times\mathbb{Z}^n} G_{kl} e^{i\langle k,\Omega^1\rangle x + i\langle l,\Omega^2\rangle t} \tag{1.2}$$

with $\Omega^1 = (\omega_1, \dots, \omega_m)$, $\Omega^2 = (\omega_{m+1}, \dots, \omega_{m+n})$, $T^{m+n} = R^{m+n}/Z^{m+n}$, where the coefficients G_{kl} decay exponentially with $|k| + |l|$.

☆ This work is supported by the National Key Basic Research Special Fund (No. G1998020304), Postdoctoral research fund of China and K.C. Wong Research fund.
* Corresponding author.
E-mail address: yqwang31415926@yahoo.com.cn (Y. Wang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.01.003

It is well known that the long-time behavior of a time-dependent nonlinear equation

$$\frac{d^2x}{dt^2} + f(x,t) = 0, \tag{1.3}$$

where f being periodic or quasi-periodic in t, can be very intricate. As a problem proposed by Littlewood [5], people began to study the Lagrangian stability of (1.3) since the early 60's.

The first result was due to Morris [8], who proved that each solution $x(t)$ of the equation

$$\frac{d^2x}{dt^2} + 2x^3 = p(t) \tag{1.4}$$

satisfies $\sup_{R^1}(|x(t)| + |\dot{x}(t)|) < +\infty$, where $p(t)$ is a continuous periodic function.

In 1987, Dieckerhoff and Zehnder [1] generalized the result to

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{i=0}^{l} p_i(t)x^i = 0, \tag{1.5}$$

where $p_i(t+1) = p_i(t)$ are sufficiently smooth.

Subsequently, this result was extended to the more general cases by several authors, we refer to [2,6] and references therein.

The idea of the above-mentioned papers is as follows. By means of the transformation theory, the system is, outside a large disc $D = \{(x,\dot{x}) \in R^2 \mid x^2 + \dot{x}^2 \leqslant A^2\}$ in the $(x,\dot{x})$-plane, transformed into a Hamiltonian equation with the following property. From the Liouville's theorem, it follows that the Poincaré mapping of the equation is area-preserving and is closed to a so-called twist mapping in R^2/D. Then using the KAM theorem [9], one can find large invariant curves diffeomorphic to circles and surrounding the origin in the $(x,\dot{x})$ plane. Every such curve is the base of a time-periodic and under the flow invariant cylinder in the phase space $(x,\dot{x},t) \in R^2 \times R^1$, which confines the solutions in its interior and which therefore leads to a bound of these solutions.

On the other hand, Moser [10] suggested considering the Lagrangian stability of pendulum-type equation

$$\frac{d^2x}{dt^2} + G_x(t,x) = p(t), \tag{1.6}$$

where $G(t+1,x) = G(t,x+1) = G(t,x)$, $p(t+1) = p(t)$ and $\int_0^1 p(t)\,dt = 0$. In 1989 and 1990, Levi [3], Moser [11] and You [14] independently proved that each solution $x(t)$ of (1.5) satisfies $\sup_{R^1}|\dot{x}(t)| < +\infty$. Their proofs are based on the similar idea as above except that the large disc D is replaced by $\{(x,\dot{x}) \in R^2 \mid |\dot{x}| \leqslant A\}$.

Recently, Levi and Zehnder [4], Liu and You [7] independently proved the Lagrangian stability for (1.5) with $p_i(t)$ being quasi-periodic functions with basic frequencies $\omega_1,\ldots,\omega_m$. In their papers, the frequencies $(\omega_1,\ldots,\omega_m)$ satisfy the Diophantine conditions

$$|k_1\omega_1 + \cdots + k_m\omega_m| \geqslant \frac{c}{|k|^\tau}, \quad c > 0,\ \tau > m,\ 0 \neq k \in Z^m.$$

One cannot use the above-mentioned idea to the time quasi-periodic dependent systems because in this case Eq. (1.5) is no longer a time-periodic equation. Instead, they obtain the quasi-periodic solutions and the boundedness of solutions by using the KAM iterations. Roughly speaking, $\forall r \in R^+$, they find a function Φ_r which is quasi-periodic in t and periodic in θ such that the *bounded* set $\{(I,\theta,t) \mid I = \Phi_r(\theta,t),\ t \in R,\ \theta \in S^1\}$ is a slight deformation of the infinitely cylinder $\{(I,\theta,t) \mid I = r^2\}$ in the extend space $R^+ \times S^1 \times R$ and is invariant under the flow of Eq. (1.5). Then the invariant set confines the solutions in its interior and which therefore leads to a bound of these solutions.

In this paper, we will also adopt the method of constructing KAM iterations, but the situation in our case is essentially different from [4,7] because (1.1) is neither a periodic nor a polynomial system on x but a quasi-periodic one. To iterate the KAM step infinitely often, instead using the classical definition of Diophantine condition, we define a modified Diophantine condition. We say $\Omega = (\Omega^1, \Omega^2) = (\omega_1, \dots, \omega_m, \omega_{m+1}, \dots, \omega_{m+n})$ satisfies a modified Diophantine condition if

$$\Omega \in O_\alpha = \left\{\Omega \;\Big|\; \bigl|\langle k, \Omega\rangle\bigr| \geqslant \frac{\alpha}{|k|^\tau},\ \forall 0 \neq k \in \mathbb{Z}^{m+n},\ 1 \gg \alpha,\ \tau > m+n\right\}, \tag{1.7}$$

and

$$\forall l \in \mathbb{N},\ \exists A(l) \geqslant l,\ \text{s.t.}\ \bigl(A(l)\Omega^1, \Omega^2\bigr) \in O_\alpha. \tag{1.8}$$

Under these assumptions, we will prove the following theorem.

Theorem 1. *For the real analytic system* (1.1) *with* (Ω^1, Ω^2) *satisfying* (1.7) *and* (1.8), *we have*:

(1) *All solutions are bounded for all time,* $\sup_{R^1} |\dot{x}(t)| < +\infty$;
(2) *Equation* (1.1) *possesses infinitely many quasi-periodic solutions*

$$x(t) = h\bigl(A(l)\omega_1 t, \dots, A(l)\omega_m t,\ \omega_{m+1} t, \dots, \omega_{m+n} t\bigr), \quad l = 1, 2, \dots,$$

where h *is a* 2π*-periodic in each argument.*

Remark 1.1. By some transformation, Eq. (1.6) can be regarded as a special case of Eq. (1.1).

This paper contains three sections. In Section 2, some lemmas are given which will be useful later. In Section 3, we will construct the KAM iterations and prove Theorem 1.

2. Some lemmas

First we give some definitions.

Let $C \in R^{m+n}$ be any open bounded set. Denote $T_A : R^{m+n} \to R^{m+n}$ the transformation

$$T_A(x_1, \dots, x_{m+n}) = (Ax_1, \dots, Ax_m, x_{m+1}, \dots, x_{m+n}).$$

By Fubini theorem, $\operatorname{mes}(T_A C) = A^m \operatorname{mes}(C)$. From [12], we have

$$\operatorname{mes}(O_\alpha \cap C) > \operatorname{mes}(C)(1 - c_0\alpha) \tag{2.1}$$

and

$$\operatorname{mes}(O_\alpha \cap T_A C) > A^m \operatorname{mes}(C)(1 - c_0\alpha), \tag{2.2}$$

where c_0 is independent on A and α. In the following, we will fix α be a small number such that $1 - 2c_0\alpha > 0$. Hence the right hand of the above two inequalities is larger than 0. We have the same conclusion for the right hand of the inequality in Lemma 2.1.

Denote $O_{\alpha,k} = \{\Omega \in O_\alpha \cap C \mid T_B\Omega \in O_\alpha \text{ for some } B = B(k,\Omega) \geqslant k\}$. Obviously, $O_{\alpha,k+1} \subset O_{\alpha,k}$.

Remark 2.1. To prove the existence of Ω satisfying (1.7) and (1.8), it is sufficient to prove $\bigcap_{n=1}^{\infty} O_{\alpha,k} \neq \emptyset$.

In fact, we will prove

Lemma 2.1. $\operatorname{mes}(\bigcap_{k=1}^{\infty} O_{\alpha,k}) \geqslant \operatorname{mes}(C)(1 - 2c_0\alpha)$.

Proof. Otherwise, there must exist $K \in \mathbb{N}$ and $0 < \delta \ll 1$, such that $\operatorname{mes}(\bigcap_{k=1}^{K} O_{\alpha,k}) = \operatorname{mes}(O_{\alpha,K}) < \operatorname{mes}(C)(1 - 2c_0\alpha) - \delta$. Assume $A > K$. Then, by Fubini theorem, we have

$$\operatorname{mes}\bigl(T_A(O_{\alpha,K})\bigr) < A^m\bigl(\operatorname{mes}(C)(1 - 2c_0\alpha) - \delta\bigr). \tag{2.3}$$

On the other hand, if $x \in T_A(O_\alpha \cap C) \cap O_\alpha$, it means that there exists $y \in O_\alpha \cap C$ such that $x = T_A(y) \in O_\alpha$. Combining the definition of $O_{\alpha,K}$ and the assumption $A > K$, we have $x \in T_A(O_{\alpha,K})$, which yields $T_A(O_{\alpha,K}) \supset T_A(O_\alpha \cap C) \cap O_\alpha$. We shall prove that

$$\operatorname{mes}\bigl(T_A(O_\alpha \cap C) \cap O_\alpha\bigr) > A^m \operatorname{mes}(C)(1 - 2c_0\alpha), \tag{2.4}$$

which contradicts with (2.3).

From Fubini theorem and (2.1), we have

$$\operatorname{mes}\bigl(T_A(O_\alpha \cap C)\bigr) = A^m \operatorname{mes}(O_\alpha \cap C) > A^m \operatorname{mes}(C)(1 - c_0\alpha). \tag{2.5}$$

Obviously $T_A(O_\alpha \cap C) \subset T_A C$ and $O_\alpha \cap T_A C \subset T_A C$. From (2.2), (2.5) and the fact that $\operatorname{mes}(T_A C) = A^m \operatorname{mes}(C)$, we have $\operatorname{mes}(T_A(O_\alpha \cap C) \cap (O_\alpha \cap T_A C)) > A^m \operatorname{mes}(C)(1 - 2c_0\alpha)$, which implies (2.4). □

Remark 2.2. From Lemma 2.1 and the definition of $O_{\alpha,k}$, we know there are infinitely many Ω satisfying (1.7) and (1.8).

In the following, to avoid a flood of constants we will write $u \lessdot v$, $u \cdot\!\!< v$ if there exist positive constants $c > 1$ and $\gamma < 1$ which depend only on α, m, n, τ such that $u \leqslant cv$ and $u \leqslant \gamma v$, respectively.

Equation (1.1) is equivalent to the following analytic system:

$$x' = \frac{\partial H}{\partial y} = y, \qquad y' = -\frac{\partial H}{\partial x} = -G_x(x,t) \tag{2.6}$$

with the Hamiltonian function

$$H(x,y,t) = \frac{1}{2}y^2 + G(x,t) \tag{2.7}$$

with G quasi-periodic in x and t with basic frequencies Ω^1 and Ω^2, respectively. Denote $A_L = \{(I,\theta,t) \mid I \geqslant L,\ (\theta,t) \in R^2\}$.

Lemma 2.2. *Consider the real analytic system* (2.6), (2.7) *with the Diophantine condition* (1.7)*. Then there exists a canonical diffeomorphism ψ depending quasi-periodically on t of the form*

$$\psi: y = I + u(I,\theta,t), \quad x = \theta + v(I,\theta,t),$$

such that $A_{I_+} \subset \psi(A_{I_0}) \subset A_{I_-}$ for some large $I_- < I_0 < I_+$. Moreover the transformed real analytic Hamiltonian vector field $\psi^*(X_H) = X_{\tilde{H}}$ is of the form

$$\tilde{H}(I,\theta,t) = \frac{1}{2}I^2 + \tilde{G}(I,\theta,t), \tag{2.8}$$

where $\tilde{G}$ is quasi-periodic in θ and t with the same basic frequencies as those of G and satisfying $\|\tilde{G}\| \lessdot I^{-1}$.

Proof. We shall look for the required transformation ψ given by means of a generating function $W(I,x,t)$, so that ψ is implicitly defined by

$$\psi: y = I + \frac{\partial W}{\partial x}, \quad \theta = x + \frac{\partial W}{\partial I}.$$

Then $\tilde{H}$ expressed in the variables (I,x,t) instead of (I,θ,t) has the form

$$\begin{aligned}\tilde{H} &= \frac{1}{2}\left(I + \frac{\partial W}{\partial x}\right)^2 + G(x,t) + \frac{\partial W}{\partial t} \\ &= \frac{1}{2}I^2 + I\frac{\partial W}{\partial x} + G(x,t) + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\partial W}{\partial t}.\end{aligned}$$

Now we determine W by the equation

$$I\frac{\partial W}{\partial x} + G(x,t) = [G](t), \tag{2.9}$$

where $[G](t) = \sum_l G_{0l} e^{i\langle l,\Omega^2\rangle t}$.

Write G, W in the Fourier series

$$\begin{aligned}G(x,t) &= \sum_{0\neq k\in\mathbb{Z}^m,\, l\in\mathbb{Z}^n} G_{kl} e^{i\langle k,\Omega^1\rangle x + i\langle l,\Omega^2\rangle t}, \\ W(x,t) &= \sum_{0\neq k\in\mathbb{Z}^m,\, l\in\mathbb{Z}^n} W_{kl} e^{i\langle k,\Omega^1\rangle x + i\langle l,\Omega^2\rangle t}.\end{aligned}$$

Then (2.9) is equivalent to the following equation:

$$\sum_{k\neq 0} \big(i \cdot I\langle k,\Omega^1\rangle W_{kl} + G_{kl}\big) e^{i\langle k,\Omega^1\rangle x + i\langle l,\Omega^2\rangle t} = 0,$$

i.e.,

$$i \cdot I\langle k,\Omega^1\rangle W_{kl} + G_{kl} = 0, \quad \forall 0 \neq k \in \mathbb{Z}^m.$$

By the condition (1.7), we have the estimate

$$|W_{kl}| = \left|\frac{G_{kl}}{iI\langle k, \Omega^1\rangle}\right| \leqslant \frac{G_{kl}|k|^\tau}{|I|\alpha}.$$

With Cauchy inequality, it is easy to prove that $\tilde{H}$ is a real analytic system and $\tilde{G}$ satisfies $\|\tilde{G}\| \lessdot I^{-1}$, where

$$\tilde{G}(I,\theta,t) = \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{\partial W}{\partial t}.$$

From [13, Chapter 3] we know that $\tilde{G}$ is quasi-periodic on θ, t with basic frequencies $\omega_1, \ldots, \omega_m$ and $\omega_{m+1}, \ldots, \omega_{m+n}$, respectively. Because $[G]$ depends only on variable t, we ignore it in Hamiltonian function. □

Suppose $\lambda \gg 1$ and satisfies $(2\lambda\Omega^1, \Omega^2) \in O_\alpha$. From (1.8) and Lemma 2.1, we know such λ exists. For our purpose, it is sufficient to consider the following real analytic system in the domain Σ: $\operatorname{Im} t|, |\operatorname{Im}\theta| < s$, $|I-\lambda| < r$:

$$H(I,\theta,t) = \frac{1}{2}I^2 + R(I,\theta,t) \tag{2.10}$$

with R quasi-periodic on θ and t and satisfying $\|R\| < \epsilon$, where $\epsilon = \epsilon(m,n,\tau,\alpha,s,r)$ is a small parameter.

Set $I = \lambda + \tilde{I}$, where $|\tilde{I}| \leqslant \epsilon^{1/2}$; then (2.10) is equivalent to the analytic parameterized system defined in $D_{r,s} \times O_h$,

$$H(\lambda,\theta,\tilde{I},t) = \lambda\tilde{I} + G(\theta,t,\tilde{I},\lambda)$$

with G quasi-periodic in θ and t and satisfying $\|G\| < \epsilon$, where

$$D_{r,s} = \left\{(\theta,\tilde{I},t) \mid |\tilde{I}| < r,\ |\operatorname{Im}\theta|, |\operatorname{Im} t| < s\right\} \in \mathbb{C}^3,$$
$$O_h = \left\{\lambda \mid \operatorname{dist}\big((\lambda\Omega^1, \Omega^2) - O_\alpha\big) < h\right\} \in \mathbb{C}.$$

Here we ignore the constants term $\lambda^2/2$ in H because the dynamical properties of the Hamiltonian determined by H are independent of this quantity.

Without leading to confusion, we denote variables by λ, x, y, t instead of $\lambda, \theta, \tilde{I}, t$, i.e., we write the Hamiltonian function in the following form:

$$H(x,t,y,\lambda) = \lambda y + G(x,t,y,\lambda) \tag{2.11}$$

with G quasi-periodic in x, t of frequencies Ω^1, Ω^2, respectively.

3. KAM iterations and proof of Theorem 1

3.1. One KAM step

In this subsection, we will propose the necessary assumptions on ϵ, r, s, h and r_+, s_+, h_+ such that the new perturbation terms ϵ_+ is much smaller than ϵ after one KAM step, where the plus sign indicates the corresponding parameter value for the next step.

(a) *Truncation*. We approximate G by a Hamiltonian R, which is linear in y and a trigonometric polynomial in x and t. To this end, let Q be the linearization of G in y at $y = 0$. By Taylor's formula with remainder and Cauchy's estimate, we have

$$|Q|_{r,s} \lessdot \epsilon, \qquad |G - Q|_{\eta r,s} \lessdot \eta^2 \epsilon, \tag{3.1}$$

where $0 < \eta < 1/8$. Then we simply truncate the Fourier series of Q at order K to obtain R. From classical KAM theorem [12], we know

$$|R - Q|_{r,s-\sigma} \lessdot K^{m+n} e^{-K\sigma} \epsilon, \tag{3.2}$$

where $K \geqslant 1$. Since the factor $K^{m+n} e^{-K\sigma}$ will be made small later on, we also have

$$|R|_{r,s-\sigma} \lessdot \epsilon.$$

(b) *Symplectic transformation*. We construct a symplectic transformation with the generating function S,

$$\phi: \begin{cases} \tilde{x} = x + \frac{\partial S}{\partial \tilde{y}}, \\ y = \tilde{y} + \frac{\partial S}{\partial x}. \end{cases}$$

Then we have

$$\begin{aligned} \tilde{H}(x, t, \tilde{y}, \lambda) &= H\left(x, t, \tilde{y} + \frac{\partial S}{\partial x}, \lambda\right) \\ &= \lambda\left(\tilde{y} + \frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} + R\left(x, t, \tilde{y} + \frac{\partial S}{\partial x}, \lambda\right) + G - R. \end{aligned}$$

We determine S by the equation

$$\lambda \frac{\partial S}{\partial x} + \frac{\partial S}{\partial t} + R(x, t, \tilde{y}, \lambda) - [R] = 0, \tag{3.3}$$

where $[R] = G_{00}$.

Write S in Fourier series

$$S(x, t, \tilde{y}, \lambda) = \sum_{k \in \mathbb{Z}^{m+n},\, k \neq 0} S_k(\tilde{y}, \lambda) e^{i(\langle k_1, \Omega^1 \rangle x + \langle k_2, \Omega^2 \rangle t)}, \quad k = (k_1, k_2).$$

From (3.3) we obtain

$$i\big(\langle k_1, \lambda \Omega^1 \rangle + \langle k_2, \Omega^2 \rangle\big) S_k(\tilde{y}, \lambda) + R_k(\tilde{y}, \lambda) = 0,$$

i.e.,

$$S_k(\tilde{y}, \lambda) = \frac{-R_k}{i(\langle k_1, \lambda \Omega^1 \rangle + \langle k_2, \Omega^2 \rangle)}.$$

Suppose

$$h < \frac{\alpha}{2} K^{-\tau-1}, \tag{3.4}$$

then we have

$$\left|\langle k_1, \lambda \Omega^1\rangle + \langle k_2, \Omega^2\rangle\right| \geqslant \left|\langle k_1, \lambda_0 \Omega^1\rangle + \langle k_2, \Omega^2\rangle\right| - \left|\langle k_1, (\lambda - \lambda_0)\Omega^1\rangle\right| \\ \geqslant \frac{\alpha}{|k|^\tau} - \frac{\alpha}{2} K^{-\tau} > \frac{\alpha}{2|k|^\tau},$$

where $\lambda_0 \in \Omega_\alpha$. So $|S_k(\tilde{y}, \lambda)| < (2/\alpha)|R_k(\tilde{y}, \lambda)|k^\tau$. As a consequence,

$$\left\|S(x, t, \tilde{y}, \lambda)\right\|_{r, s-2\sigma} \lessdot \sum_{|k|<K,\, k\neq 0} \|R_k\| |k|^\tau e^{|k|(s-\sigma)} \\ \lessdot \sum_{|k|<K,\, k\neq 0} \|R\|_{r, s-\sigma} |k|^\tau e^{-|k|\sigma} \lessdot \sigma^\tau \epsilon,$$

where $0 < \sigma < s$.

(c) *New error*. We write the new Hamiltonian function into the following form:

$$H_+(x_+, t, y_+, \lambda_+) = \lambda_+ y_+ + G_+(x_+, t, y_+, \lambda_+),$$

where $H_+ = \tilde{H}$, $x_+ = \tilde{x}$, $y_+ = \tilde{y}$, $\lambda_+ = \lambda + [R]_{y_+}$, and

$$G_+ = [R](y) - [R](y_+) + R(y) - R(y_+) + G - R,$$

where we have ignored the variables t and λ.

By the same reason as in the proof of Lemma 2.2, G_+ can be expressed into the form

$$G_+ = \sum_{k\in\mathbb{Z}^m,\, l\in\mathbb{Z}^n} G_{+kl}(y_+, \lambda_+) e^{i\langle k, \Omega^1\rangle x + i\langle l, \Omega^2\rangle t}$$

with the coefficients G_{+kl} decay exponentially with $|k| + |l|$.

If

$$\epsilon \cdot\!< \eta r \sigma^{\tau+1}, \tag{3.5}$$

then we have

$$\left|\frac{\partial S}{\partial x}\right| \leqslant \eta r \leqslant \frac{r}{8}, \quad \left|\frac{\partial S}{\partial y}\right| \leqslant \sigma \quad \text{on } D_{r/2, s-3\sigma}.$$

By direct computation, we have

$$\left|R\left(\tilde{y} + \frac{\partial S}{\partial x}\right) - R(\tilde{y})\right|_{r/2,\, s-3\sigma} \leqslant \left|R_y\left(\tilde{y} + \xi\frac{\partial S}{\partial x}\right)\frac{\partial S}{\partial x}\right|_{r/2,\, s-\sigma} \\ \leqslant \frac{\epsilon^2}{r\sigma^{\tau+1}} := \frac{\epsilon^2}{r\sigma^\nu}. \tag{3.6}$$

Combining (3.1) and (3.2) with the above inequality, we have

$$\|G_+\|_{\eta r, s-3\sigma} \lessdot \eta^2 \epsilon + K^{m+n} e^{-K\sigma}\epsilon + \frac{\epsilon^2}{r\sigma^\nu}.$$

(d) *Transformation of the frequencies*. The new parameter of frequencies is $\lambda_+ = \lambda + [R]_{y_+}/2$. We need the inequality

$$\frac{\epsilon}{r} \lessdot h \tag{3.7}$$

to ensure the existence of a real analytic inverse map $O_{h/4} \to O_{h/2}$, $\lambda_+ \to \lambda$.

Denote

$$\epsilon_+ = \frac{\epsilon^2}{r\sigma^\nu} + (\eta^2 + K^{m+n}e^{-K\sigma})\epsilon.$$

Hence with the assumptions (3.4), (3.5) and (3.7), we have proved that there exist a real analytic transformation $\phi : D_{\eta r, s-5\sigma} \times O_{h/4} \to D_{r,s} \times O_h$ such that $H \circ \phi = N_+ + G_+$ with $\|G_+\| \leqslant \epsilon_+$.

Let

$$E = \frac{\epsilon}{r\sigma^\nu}, \quad r_+ = \eta r, \quad \sigma_+ = \frac{\sigma}{2}.$$

With the assumption that

$$\eta^2 = E, \tag{3.8}$$

$$\sigma^{-(m+n)}K^{m+n}e^{-K\sigma} \leqslant E, \tag{3.9}$$

we have that $E_+ = \epsilon_+/(r_+\sigma_+^\nu)$ satisfies

$$|E_+| < \frac{E\epsilon}{\eta r\sigma^\nu} = \eta^{-1}E\frac{\epsilon}{r\sigma^\nu} = E^{3/2}.$$

In summary, all the necessary assumptions are

(i) $\epsilon \lessdot \eta r\sigma^{\tau+1}$;

(ii) $\dfrac{\epsilon}{r} \lessdot h$;

(iii) $h \lessdot K^{-\tau-1}$;

(iv) $\eta^2 = E$;

(v) $K^{m+n}e^{-K\sigma} \leqslant E$.

3.2. Proof of Theorem 1

We are now ready to set up our parameter sequences. Before doing this, we need to make sure that the above inequalities hold for the initial values $h_0, K_0, \ldots$. But here we may simply define

$$K_0^{-\nu-1} = h_0 = \frac{c_0\epsilon_0}{r_0},$$

and fix

$$\frac{\epsilon_0}{r_0\sigma_0^\nu} = E_0 = \gamma_0$$

to some sufficiently small constant γ_0. This will make $K_0\sigma_0$ large so that the second inequality is satisfied.

Set

$$\sigma_{j+1} = \frac{\sigma_j}{2}, \quad s_{j+1} = s_j - 5\sigma_j, \quad \sigma_0 = \frac{s_0}{20}.$$

Then $s_0 > s_1 > \cdots \to s_0/2$. Set

$$E_{j+1} = c_1^{\kappa-1} E_j^{\kappa}, \quad h_{j+1} = \frac{h_j}{4^{\sigma}}, \quad K_{j+1} = 4K_j,$$

where $\kappa = 3/2$, $\mu = 4/3$. Set

$$r_{j+1} = \eta_j r_j, \quad \eta_j^2 = E_j,$$
$$D_j = \bigl\{|y| < r_j\bigr\} \times \bigl\{|\operatorname{Im} x|, |\operatorname{Im} t| < s_j\bigr\},$$
$$O_j = \bigl\{\lambda \mid \operatorname{dist}\bigl((\lambda\Omega^1, \Omega^2), O_\alpha\bigr) < h_j\bigr\}.$$

It is not difficult to see that the assumptions (i)–(v) hold for the next step only if they hold in the last step.

All the things we have to do further to check the convergence of iterations are similar as classical KAM theorem (see [12]), we omit it here.

Note that all the symplectic transformations ϕ in every step are quasi-periodic on θ and t, we obtain in the neighborhood of the infinitely far point infinitely many invariant sets. Moreover, each of them is a slight deformation of infinite cylinder surrounding the origin and therefore is bounded in y-variable. On these invariant sets the solutions are quasi-periodic on t with basic frequencies expressed as in Theorem 1. By the existence and uniqueness theorem, we obtain the boundedness of all solutions of Eq. (1.1).

Acknowledgments

This work was done when the second author visited the National Center for Theoretical Centers, Taiwan. He acknowledges the hospitality of the institute. The authors thank Prof. Bin Liu for his encouragement and helpful suggestion.

References

[1] R. Dieckerhoff, E. Zehnder, Boundedness of solutions via the twist theorem, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 14 (1987) 79–95.
[2] M. Levi, Quasi-periodic motions in superquadratic time-periodic potentials, Comm. Math. Phys. 143 (1991) 43–83.
[3] M. Levi, KAM theory for particles in periodic potentials, Ergodic Theory Dynam. Systems 10 (1990) 777–785.
[4] M. Levi, E. Zehnder, Boundedness of solutions for quasiperiodic potentials, SIAM J. Math. Anal. 26 (1995) 1233–1256.
[5] J. Littlewood, Some Problems in Real and Complex Analysis, Heath, Lexington, MA, 1968.
[6] B. Liu, Boundedness for solutions of nonlinear periodic differential equations via Moser's twist theorem, Acta Math. Sinica (N.S.) 8 (1992) 91–98.
[7] B. Liu, J. You, Quasiperiodic solutions of Duffing's equations, Nonlinear Anal. 33 (1998) 645–655.
[8] G. Morris, A case of boundedness in Littlewood's problem on oscillatory differential equations, Bull. Austral. Math. Soc. 14 (1976) 71–93.
[9] J. Moser, On invariant curves of area-preserving mappings of an annulus, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1962) 1–20.
[10] J. Moser, Stable and Random Motions in Dynamical Systems, Princeton Univ. Press, 1973.

[11] J. Moser, Quasiperiodic solutions of nonlinear elliptic partial differential equations, Bol. Soc. Brasil. Mat. 20 (1989) 29–45.

[12] J. Poschel, A lectures on the classical KAM theorem, in: Smooth Ergodic Theory and Its Applications (Seattle, WA, 1999), in: Proc. Sympos. Pure Math., vol. 69, American Mathematical Society, Providence, RI, 2001, pp. 707–732.

[13] C. Seigel, J. Moser, Lectures on Celestial Mechanics, Springer-Verlag, 1971.

[14] J. You, Invariant tori and Lagrange stability of pendulum-type equations, J. Differential Equations 85 (1990) 54–65.