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Phase synchronization of coupled chaotic multiple time scales systems

Zheng-Ming Ge *, Chien-Cheng Chen

Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30010, Taiwan, ROC

Abstract

The brushless dc motor (BLDCM) with multi-time scales is an electric machine. By coupled BLDCM, it is discovered that chaotic routes of the uncoupled systems influence synchronous result of coupled identical and nonidentical chaotic systems. Another multi-time scales form, Hindmarsh–Rose (HR) neurons, when the chaotic parameter is selected only in the range of the period-doubling route to chaos, phase synchronization can be predicted via Laypunov exponent. Finally, Laypunov exponent however cannot be used as a criterion for phase synchronization of coupled chaotic systems with either single or multi-time scales in our study.

1. Introduction

Phase synchronization is a major focus of study in chaotic oscillators and was first discussed for coupled Rössler oscillators [1,2]. They announced how the phase synchronization manifests itself in the Lyapunov spectrum. That is, as the coupling is increased, the positive and negative exponents remain, whereas one of the zero exponents becomes negative. On the other hand, the two coupled Hindmarsh–Rose (HR) neurons showed that the Lyapunov exponents cannot be used as a criterion for the occurrence of chaotic phase synchronization [3]. The authors of Ref. [3] announced that it is because that the Rössler oscillators has a single time scale (STS) while HR neurons is a multiple time scales (MTS) mechanism.

In this paper, we simulate three different autonomous coupled systems, brushless dc motor (BLDCM) systems, a HR neurons system and a centrifugal governor system, and found that the chaos route would influence the result no matter what were single time scale systems or multi-time scales ones.

2. Phase synchronization and BLDCM STS system

Here we describe the synchronization of two coupled systems of one and three time scales respectively, and observe the phenomena of their phase synchronizations. The phase of a chaotic trajectory has been defined variously [3,4], we modify the definition of the phase of the system:

$$\phi(t) = \arctan\frac{y(t) - y_0}{x(t) - x_0} \tag{1}$$

where the point (x_0, y_0) is within the rotation center, $x(t)$ and $y(t)$ are the states, and the mean frequency also can be defined as $\Omega = \langle\dot{\phi}\rangle = \lim_{T\to\infty}(\phi_1(T) - \phi_2(0))/T$. The weak locking condition is $|n\phi_1 - m\phi_2| < \text{const}$. Here we restrict

* Corresponding author. Tel.: +886-357-12121; fax: +886-357-20634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

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ourselves to the case $m = n = 1$. In this paper we investigate the relation between Lyapunov exponent and phase synchronization. As the simplest example of the phase synchronization, two coupled Rössler systems were studied [1]. It revealed that one of the vanishing Lyapunov exponents becomes negative, and phase between subsystems is locked at the same time, where the phase difference at the synchronous state fluctuates near the mean value $\frac{\pi}{2}$. The BLDCM is an electromechanical system of which equations of electrical and mechanical dynamics can be described by [5,6]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} i_q &= \frac{1}{L_q}[-Ri_q - n\omega(L_d i_d - k_t) + v_q] \\
\frac{\mathrm{d}}{\mathrm{d}t} i_d &= \frac{1}{L_d}[-Ri_d + nL_q\omega i_q + v_d] \\
\frac{\mathrm{d}}{\mathrm{d}t} \omega &= \frac{1}{J}[T(I,\theta) - T_\ell(t)]
\end{aligned} \tag{2}$$

At the beginning, we consider two coupled nonidentical single time scale BLDCM [5,7].

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} x_{1,2} &= v_q - x_{1,2} - y_{1,2}z_{1,2} + \rho_{1,2}x_{1,2} + k(x_{2,1} - x_{1,2}) \\
\frac{\mathrm{d}}{\mathrm{d}t} y_{1,2} &= v_d - \delta y_{1,2} + x_{1,2}y_{1,2} \\
\frac{\mathrm{d}}{\mathrm{d}t} z_{1,2} &= \sigma(x_{1,2} - z_{1,2}) + \eta x_{1,2}y_{1,2} - T_L
\end{aligned} \tag{3}$$

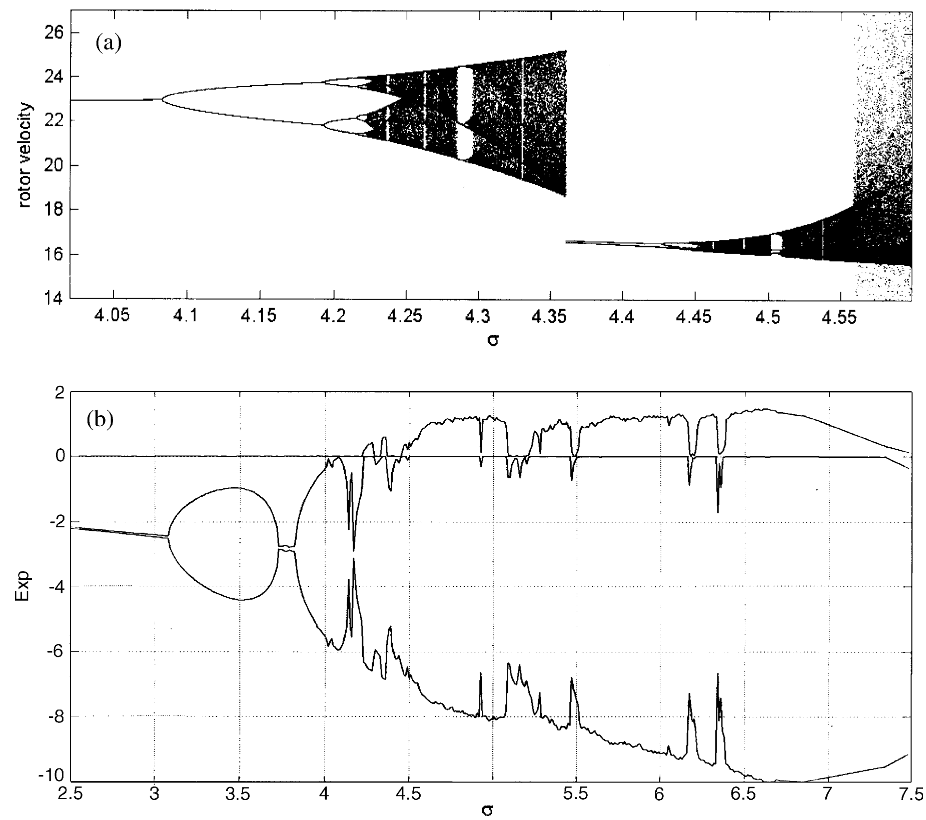

Fig. 1. The bifurcation (a) and Lyapunov spectrum (b) of the uncoupled BLDCM system ($v_q = 0.168$, $v_d = 20.66$, $\delta = 0.875$, $\rho = 60$, $\eta = 0.26$, $T_L = 0.53$) versus σ.

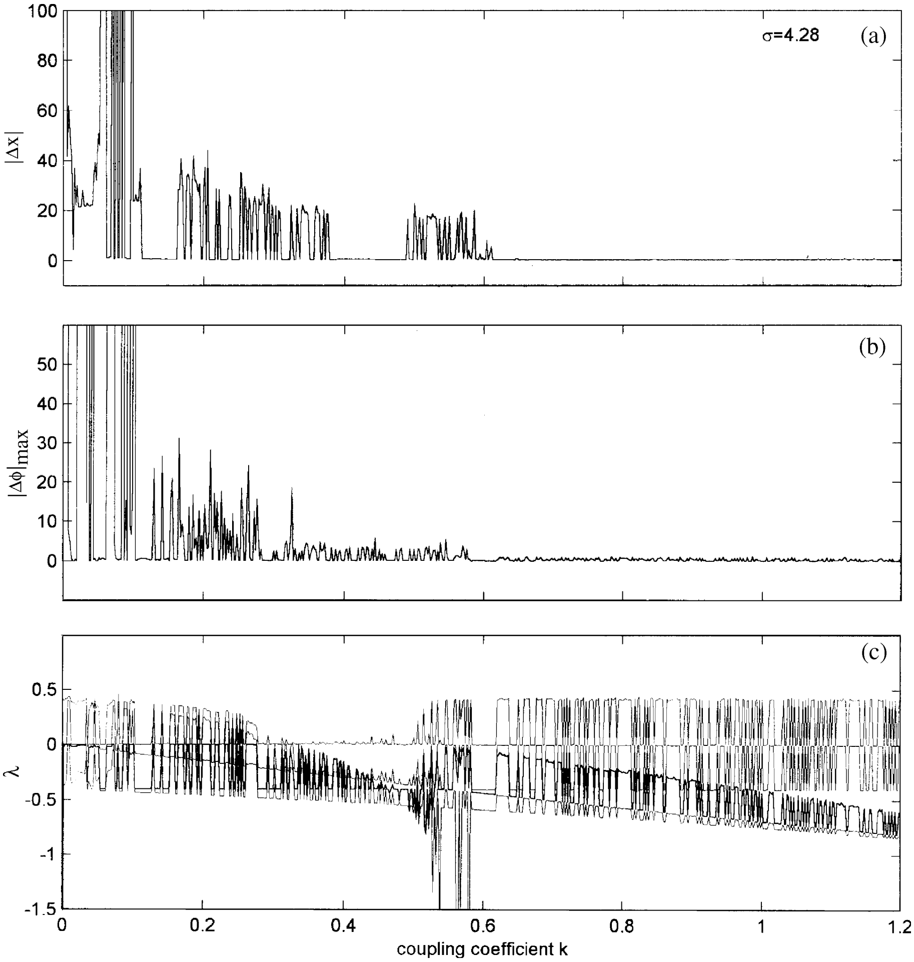

Fig. 2. Maximum state error (a), maximum phase difference (b) and four larger Lyapunov exponents (c) of coupled chaotic BLDCM systems (Eq. (3)) ($\rho_1 = 60$, $\rho_2 = 59.99$) versus coupling strength k ($\sigma = 4.28$). When one of the vanishing Lyapunov exponents falls, phase difference of two coupled systems tends to constant.

The bifurcation and Lyapunov spectrum of the uncoupled original system are shown in Fig. 1. It undergoes period-doubling routes to chaos and has similar behavior with Lorenz system. At the beginning we choose the parameter $\sigma = 4.28$ that is in the first chaotic region and then compute the phase difference, synchronization error and Lyapunov exponents with coupling coefficient k (see Fig. 2). It is shown that phase synchronization occurs when one of two vanishing Lyapunov exponents, whose peak values becomes negative. The condition of occurrence of phase synchronization is similar to that of coupled Rössler oscillators. Second, we choose another parameter, $\sigma = 4.55$, located near the region of chaos with window. The phase difference and Lyapunov exponents vs. coupling strength are shown in Fig. 3. While one of two positive Lyapunov exponents becomes zero, the phases of two subsystems do not approach synchronization yet. When the coupling coefficient k, furthermore, increases, the phase synchronization occurs soon. So we find that Lyapunov spectrum can be used as a criterion for phase synchronization only when the subsystem's parameters located in the chaotic region by period-doubling route.

3. Three time scales BLDCM system

The multiple time scales model of BLDCM is introduced as follows. The three time scales model of BLDCM can be derived by perturbation method. For a general case, we use three time scales $\tau_1 = \frac{L_q}{R}$, $\tau_2 = \frac{L_d}{R}$, $\tau_3 = \frac{JR}{k_t^2}$ and transform Eq. (2) to a compact form. And the mutual coupled nonidentical case is considered here:

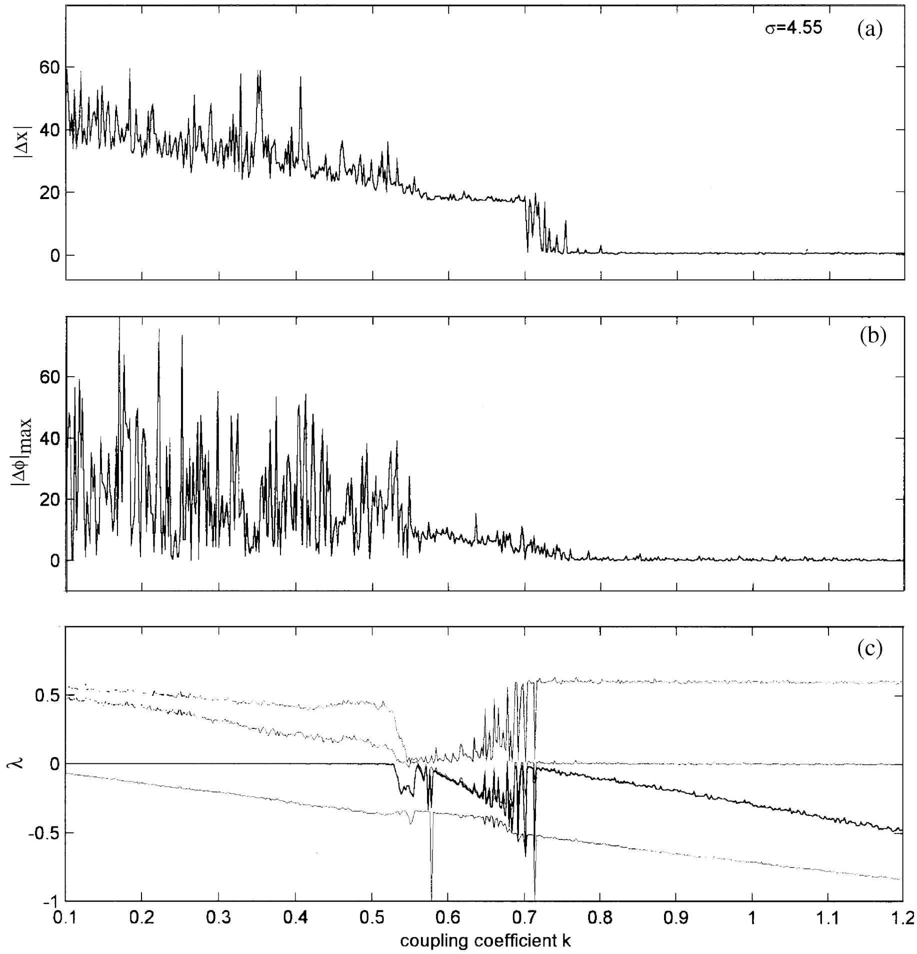

Fig. 3. Maximum state error (a), maximum phase difference (b) and four larger Lyapunov exponents (c) of coupled chaotic BLDCM systems (Eq. (3)) ($\rho_1 = 60$, $\rho_2 = 59.99$) versus coupling strength k ($\sigma = 4.55$). When one of the vanishing Lyapunov exponents falls, phase difference of two coupled systems does not converge to constant.

$$\begin{aligned}
\tau_{1,4}\frac{\mathrm{d}}{\mathrm{d}t}x_{1,2} &= V_q - x_{1,2} - y_{1,2}z_{1,2} - z_{1,2} + k(x_{2,1} - x_{1,2}) \\
\tau_{2,5}\frac{\mathrm{d}}{\mathrm{d}t}y_{1,2} &= V_d + x_{1,2}z_{1,2} - y_{1,2} \\
\tau_{3,6}\frac{\mathrm{d}}{\mathrm{d}t}z_{1,2} &= \sigma x_{1,2} + \rho x_{1,2}y_{1,2} - \eta z_{1,2} - T_L
\end{aligned} \tag{4}$$

The bifurcation and Lyapunov spectrum of the uncoupled original system are shown in Fig. 4. It undergoes period-doubling route to chaos just the same as the single time-scale model does. We choose the parameters $\tau_1 = 6.45$, $\tau_2 = 7.124$, $\tau_4 = 6.451$, $\tau_5 = 7.1238$ and $\tau_3 = \tau_6 = 1$. The results of the simulation of the phase difference, synchronization error and four larger exponents vs. coupling coefficient k are shown in Fig. 5. For this multiple time scales model it is clear that when one of two zero Lyapunov exponents becomes negative, the phase difference does not yet tend to zero at the same coupling coefficient k. We now have the following conclusion: When the original two uncoupled single time-scale subsystems' parameters locate in the chaos region by the period-doubling route, the coupled nonidentical chaotic systems will achieve phase synchronization only if one of two vanishing Lyapunov exponents becomes negative.

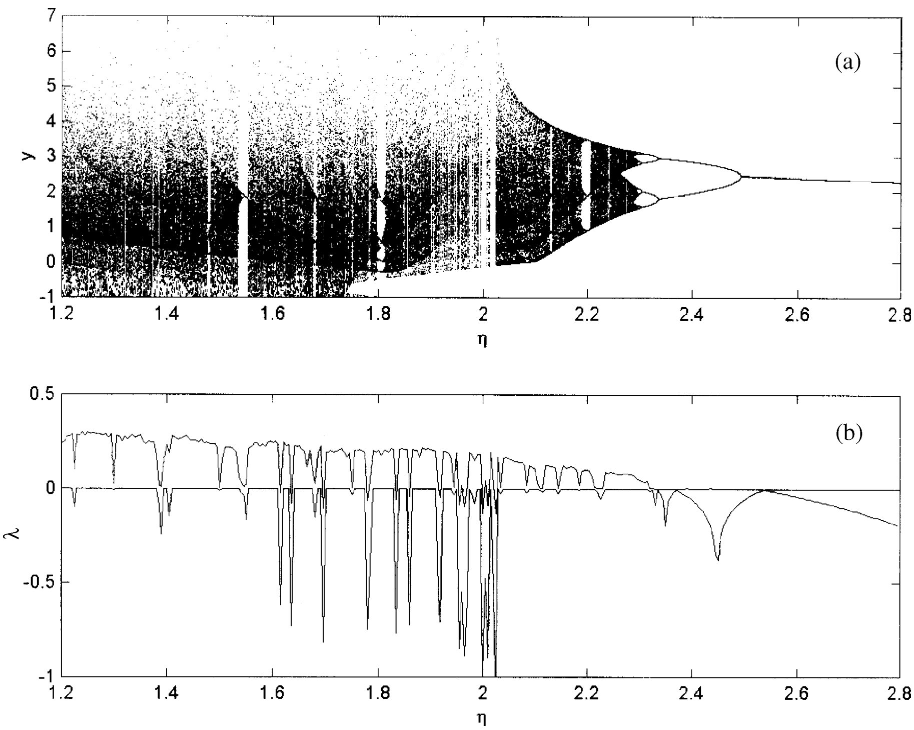

Fig. 4. The bifurcation (a) and Lyapunov spectrum (b) of the uncoupled BLDCM system with multiple time scales (Eq. (4)) versus η, where $v_q = 4.02$, $v_d = -15.31$, $T_L = 2.68$, $\sigma = 16$, $\rho = 1.516$, $\tau_1 = 6.45$, $\tau_2 = 7.124$, $\tau_3 = 1$.

4. Phase synchronization for HR neurons system

But for multiple time scales systems, no distinct relation between the phase synchronization and Lyapunov spectrum is found. We choose two different time scale systems to verify the previous conclusion, i.e. HR neurons [3] and centrifugal governor [8]. HR neuron is a multiple time scales model. Ref. [3] declared that the Lyapunov exponents can not be used as a criterion for a multiple time scales system. In this section we study the bifurcation diagram and Lyapunov spectrum of HR neurons (Fig. 6a and b). With decreasing of γ, when $\gamma < 0.0125$ period-doubling route to chaos in HR neurons can be found (Fig. 6a) with a series of windows and chaos. Now we consider two coupled nonidentical HR neurons with $\gamma = 0.014$. Four larger Lyapunov exponents versus coupling coefficient k are shown in Fig. 7. It is, furthermore, shown that phase synchronization occurs when one of two zero Lyapunov exponents becomes negative. Computing results show that once the parameter γ is chosen in the region of chaos with windows, the phenomena of phase synchronization for coupled HR neurons cannot be revealed by Lyapunov exponents. Therefore, the Lyapunov exponent always could be used as a criterion for phase synchronization, once some parameters are chosen properly, such as for the above uncoupled HR neurons, parameter γ is chosen near the period-doubling routes.

5. Centrifugal governor STS system

The rotational machine with centrifugal governor is a mechanical system [8]. The coupled nonidentical case of this system is as follows:

$$\begin{aligned}
&\frac{d}{dt}x_{1,2} = y_{1,2} + k(x_{2,1} - x_{1,2}) \\
&\frac{d}{dt}y_{1,2} = r_{1,2}z_{1,2}^2 \sin x_{1,2} \cos x_{1,2} - \sin x_{1,2} - cy_{1,2} \qquad (5) \\
&\frac{d}{dt}z_{1,2} = h\cos x_{1,2} - f
\end{aligned}$$

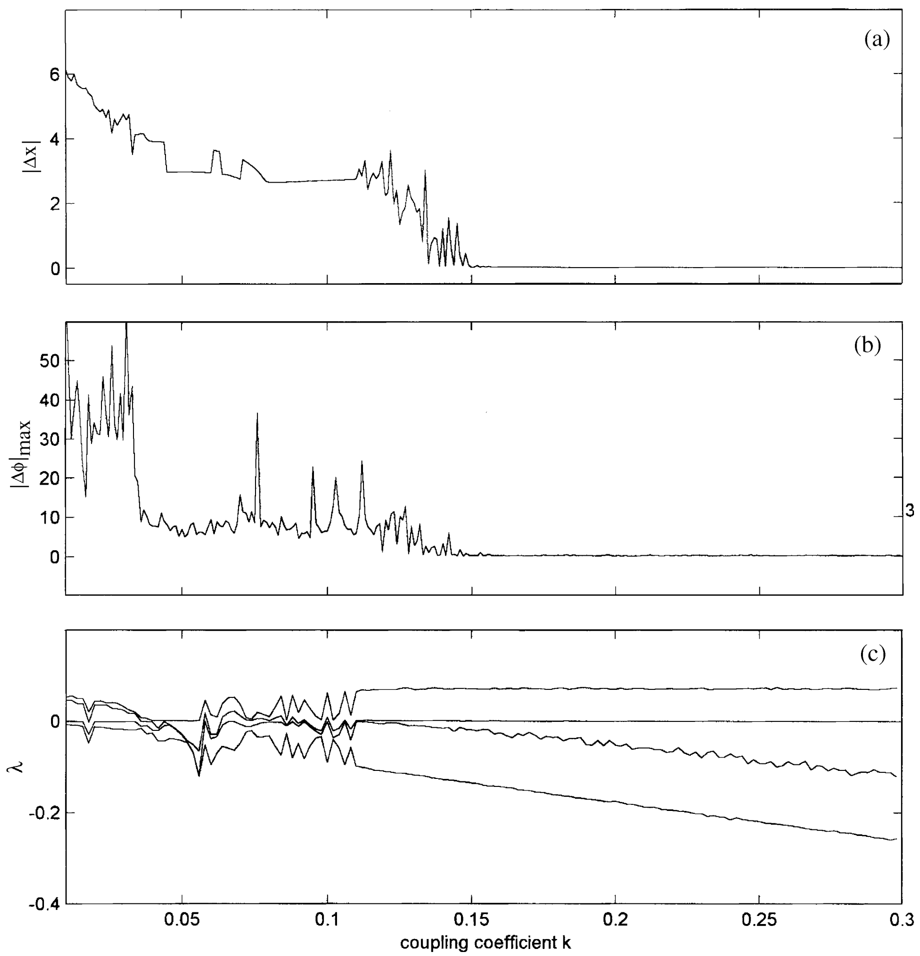

Fig. 5. State error (a), maximum phase difference (b) and four larger Lyapunov exponents (c) of coupled multiple time scales systems (Eq. (4)) versus coupling strength k ($\eta = 16.8$), phase difference of two coupled systems does not tend to constant when one of two zero Lyapunov exponents becomes negative. The synchronous behavior of coupled multiple time scales systems appears behind the fall of one of the vanishing Lyapunov exponents.

where k is a coupling coefficient. The bifurcation diagram of the original uncoupled subsystem is shown in Fig. 8. At the beginning, we choose parameters $h = 5.32$, $r_1 = 0.25$ and $r_2 = 0.248$ to compute the phase function. These parameters are chosen in the region of chaos just after the period-doubling routes. The phase difference and four larger Lyapunov exponents versus coupling k are shown in Fig. 9. The simulation time of these diagrams is set 4000 s. The coupled systems occur respective chaotic motion when $k < 0.12$. The coupled chaotic systems have phase synchronized periodic motion when coupling coefficient $k > 0.632$. The relationship between Lyapunov spectra and synchronous behavior for the coupled centrifugal governors is similar with that of the STS BLDCM system, both of them are single time-scale systems. Then we choose parameters $h = 3.58$, $r_1 = 0.25$ and $r_2 = 0.245$. These parameters are chosen in the region of chaos with windows (Fig. 8). The phase difference and four larger Lyapunov exponents versus coupling coefficient k are shown in Fig. 10. In this case, we cannot detect the weak phase synchronization by Lyapunov exponents, the position at which one of two zero Lyapunov exponents becomes negative is not clear and $|\Delta\phi|$ has not arrive to a nearly constant value. Once we increase the coupled coefficient, the phase difference converges to a nearly constant. So the phase synchronization of the centrifugal governor system with single time scale cannot be observed from Lyapunov spectrum.

6. Conclusions

Finally we mention again, the coupled identical chaotic systems will achieve phase synchronization only if one of the zero Lyapunov exponents becomes negative, no matter the time scale of the chaos system is single or multiple. The

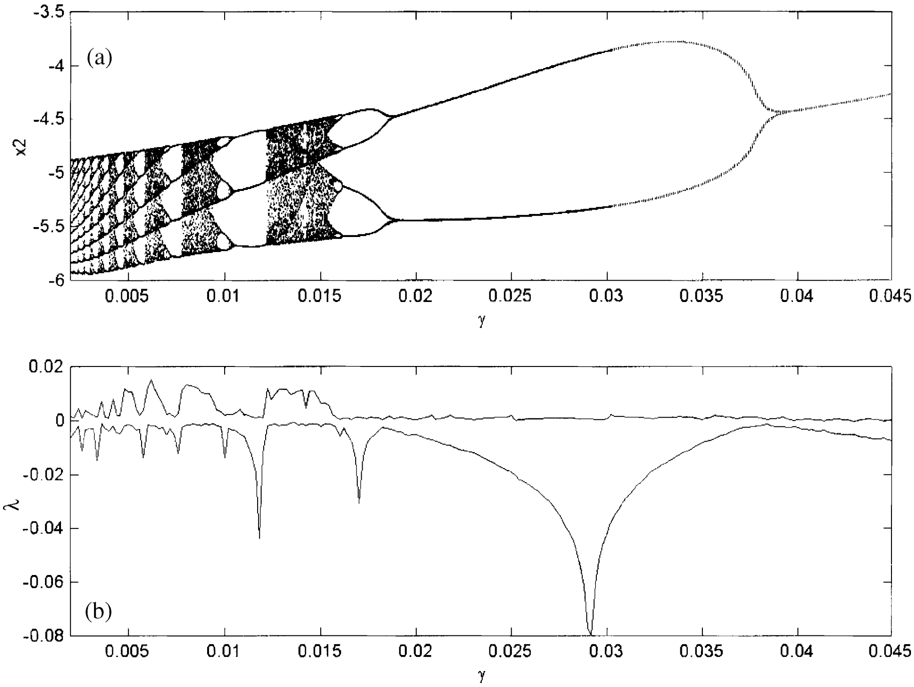

Fig. 6. The bifurcation (a) and Lyapunov spectrum (b) of HR neurons versus γ, where $\chi = 1.56$.

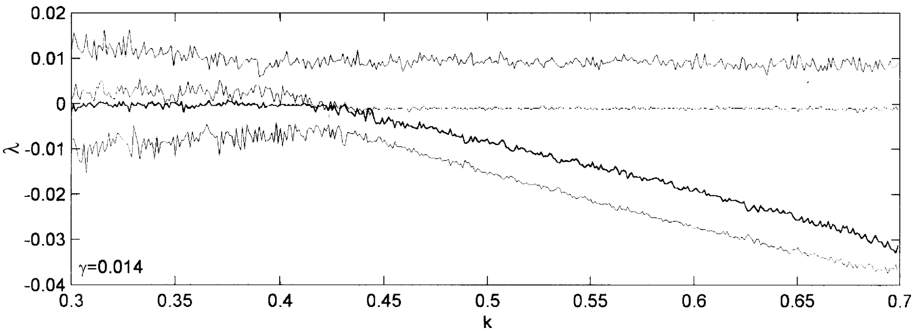

Fig. 7. The four larger Lyapunov exponents versus coupling strength k ($\gamma = 0.014$).

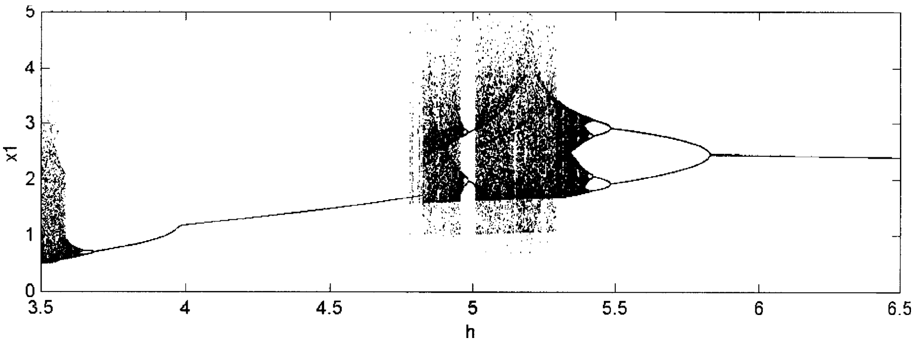

Fig. 8. The bifurcation diagram of the uncoupled subsystem (Eq. (5)) versus h, where $c = 0.7$, $r = 0.25$ and $f = 1.942$.

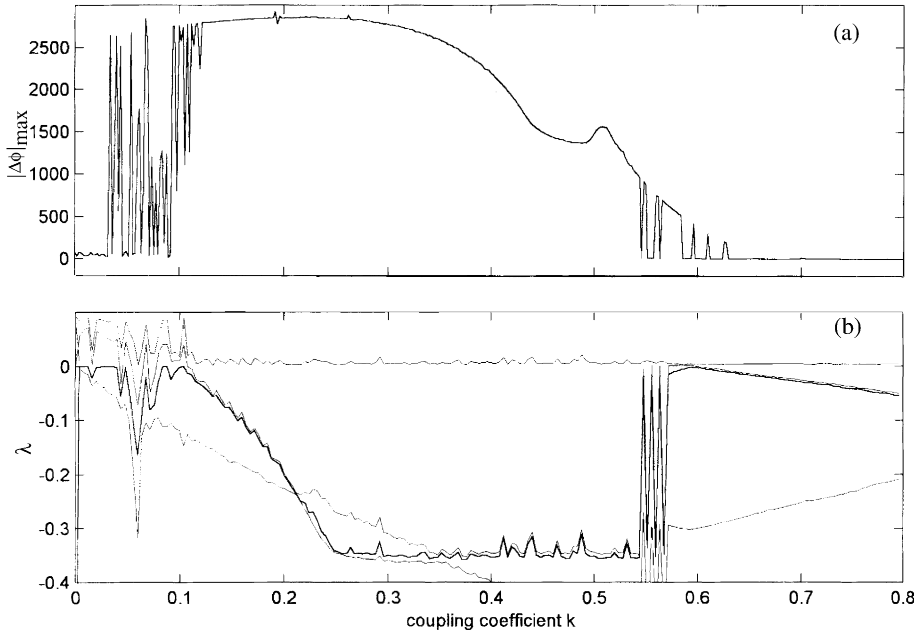

Fig. 9. Phase difference (a) and four larger Lyapunov exponents (b) versus coupling k ($h = 5.32$). The coupled systems (Eq. (5)) ($r_1 = 0.25$, $r_2 = 0.248$) have phase synchronized periodic motions when coupling $k > 0.634$, and Lyapunov exponents cannot predict the occurrence of phase synchronization of periodic motion.

above statement is a necessary, but not sufficient condition when the original two uncoupled subsystems are in the range near the period-doubling routes to chaos. Lyapunov exponent cannot be used as a criterion while the original subsystem is not in the range near the period-doubling routes to chaos, such as in the range of chaos with windows. In our study, the behavior of coupled nonidentical systems shows more complicate. In the case of coupled nonidentical STS BLDCM, the relationship between phase synchronization and Lyapunov exponents can be interpreted by the chaos routes. If the chaotic parameter is chosen in the range of period-doubling routes to chaos, then the Lyapunov exponent can predict the occurrence of the phase synchronization. But if we choose the chaotic parameter near the range of chaos and windows, then the prediction is lost. In the case of the centrifugal governor, we can observe similar situation. That is, when the chaotic parameter is near the range of chaos with windows, the relation between the Lyapunov exponent and the phase synchronization disappears. The possible reason would be explained as: Although the phase used in these two STS systems match the definition of the phase in Eq. (1), we modulate slightly the chaotic parameter, and the phase portraits of two systems tend to form two rotation centers. That is, the main chaotic phase portraits of these systems have two rotational centers. It is possible that the neighborhood of the two centers region would influence intuitively the relationship between the Lyapunov exponents and the phase synchronization.

In our study of coupled nonidentical MTS BLDCM and HR neurons, they show different situations, respectively. In the case of the HR neurons, when the chaotic parameter is only in the range of the period-doubling routes to chaos but not other regions, the Lyapunov exponent can predict the occurrence of the phase synchronization. In the MTS BLDCM, we also cannot observe the relationship between the Lyapunov exponent and phase synchronization. Therefore, the phase synchronization of coupled nonidentical chaos systems is more complex than that of coupled identical ones. Our study shows that the Lyapunov spectrum can hardly used as a criterion for phase synchronization of the coupled nonidentical chaos systems which have single time scale or multiple time scales.

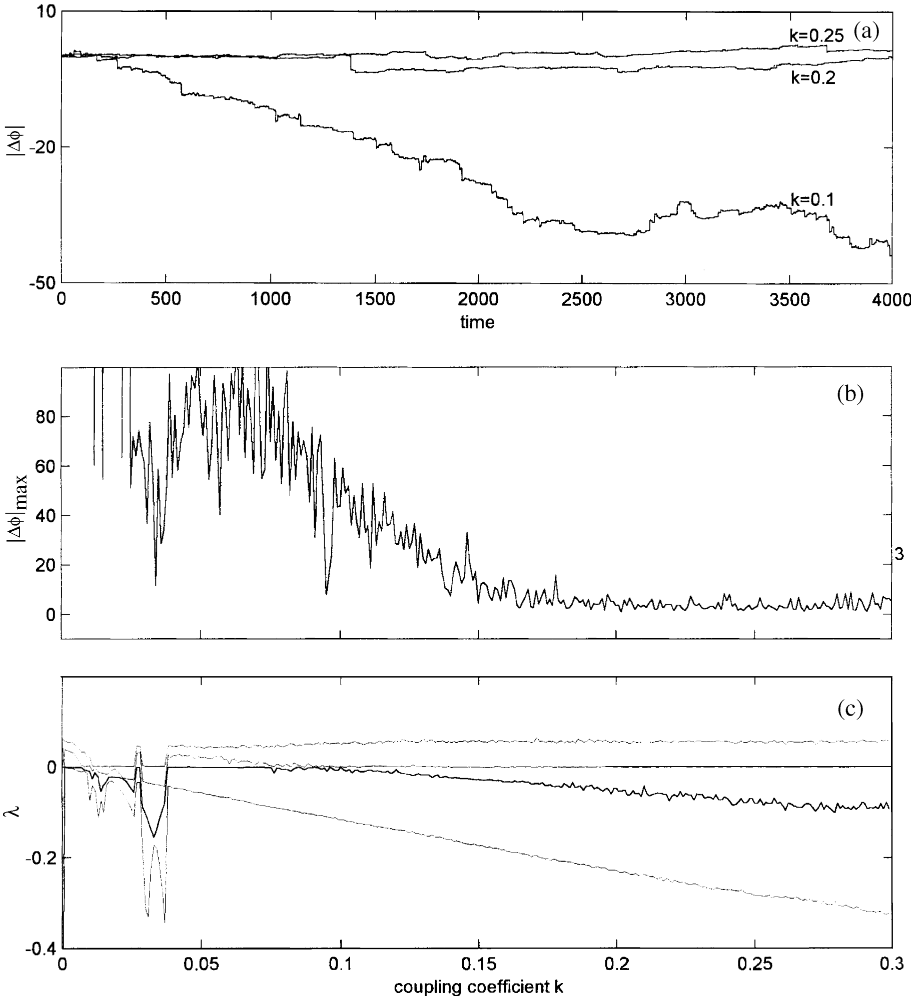

Fig. 10. Phase difference (a) of two coupled chaotic system (Eq. (5)) ($r_1 = 0.25$, $r_2 = 0.245$) versus time for nonsynchronous ($k = 0.1$), nearly synchronous ($k = 0.2$, $k = 0.25$) cases. Maximum phase difference (b) and four larger Lyapunov exponents (c) versus coupling strength k ($h = 3.58$).

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References

[1] Rosenblum M, Pikovsky A, Kurths J. Phys Rev Lett 1996;76:1804.
[2] Rosenblum M, Pikovsky A, Kurths J. Phys Rev Lett 1997;78:4193.
[3] Shuai J-W, Durand D. Phys Lett A 1999;264:289.
[4] Pikovsky A, Rosenblum M, Osipov G, Kurths J. Physica D 1997;104:219.
[5] Hemati N. Ind App Soc Annual Meeting, Conf Rec 1993;1:51.
[6] Hemati N. IEEE Circ Syst 1994;41:40.
[7] Khalil H. Nonlinear systems. Prentice-Hall; 2002.
[8] Ge Z-M, Yang H-S, Chen H-H, Chen H-K. Int J Eng Sci 1999;37:921.