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Observer-based indirect adaptive fuzzy-neural tracking control for nonlinear SISO systems using VSS and H^∞ approaches

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Abstract

Fuzzy control is a model free approach, i.e., it does not require a mathematical model of the system under control. An observer-based indirect adaptive fuzzy neural tracking control equipped with VSS and H^∞ control algorithms is developed for nonlinear SISO systems involving plant uncertainties and external disturbances. Three important control methods, i.e., adaptive fuzzy neural control scheme, VSS control design and H^∞ tracking theory, are combined to solve the robust nonlinear output tracking problem. A modified algebraic Riccati-like equation must be solved to compensate the effect of the approximation error via adaptive fuzzy neural system on the H^∞ control. The overall adaptive scheme guarantees the stability of the resulting closed-loop system in the sense that all the states and signals are uniformly bounded and arbitrary small attenuation level of the external disturbance on the tracking error can be achieved. The simulation results confirm the validity and performance of the advocated design methodology.

Keywords: State observer; Indirect adaptive control; FNNVSS,

1. Introduction

The fuzzy controllers provide a systematic and efficient framework to incorporate linguistic fuzzy information from human expert [10,15,21]. Furthermore, fuzzy control is a model free approach,

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i.e., it does not require a mathematical model of the system under control. Hence fuzzy control has found more extensive applications for a wide variety of industrial systems and consumer products. Control engineers are now facing more and more complex systems, and the mathematical models of these systems are more and more difficult to obtain. Thus, in control engineering, model free approaches become more important. There are some model free approaches in conventional control, such as nonlinear adaptive control and PID control. Fuzzy control gives another model free approach [9,1,5,22,23]. In the meantime, variable structure systems (VSS) control [6,7] design technique has been developed as a popular robust strategy to treat uncertain systems with external disturbance, quickly varying parameters and unmodeled dynamics.

Fuzzy systems are structured numerical estimators. They start from highly formalized insights about the structure of categories found in the real world and then articulate fuzzy IF-THEN rules as a kind of expert knowledge. Also, they combine fuzzy sets with fuzzy rules to produce overall complex nonlinear behavior. We have witnessed a rapid growth in the use of fuzzy system in a wide variety of consumer products and industrial systems.

The adaptive control for feedback linearizable nonlinear systems is an approach to nonlinear control design that has attracted a great deal of interest in the nonlinear control community for at least a quarter of a century. The nonlinear adaptive problem is transformed into a linear adaptive control problem by feedback linearization [12,13,19,26,20]. Therefore, the linear adaptive control methodologies can be applied to acquire the desired performance. More recently, an important adaptive fuzzy control system has been proposed to incorporate with the expert information systematically and the stability is guaranteed by universal approximation theorem [2,3,16,17,21,24,25]. An adaptive fuzzy controller is constructed from adaptive fuzzy systems. An adaptive fuzzy system is defined as a fuzzy logic system equipped with a training algorithm, where the fuzzy logic system is constructed from a set of fuzzy IF-THEN rules using fuzzy logic principles, and the training algorithm adjusts the parameters of the fuzzy logic system based on training data. The adaptive fuzzy controllers are classified into direct and indirect adaptive fuzzy controller categories [3,14,25,4]. More specifically, direct adaptive fuzzy controllers use fuzzy logic system as controllers; therefore, linguistic fuzzy control rules can be directly incorporated into the controllers. On the other hand, indirect adaptive fuzzy controllers use fuzzy logic systems to model the plant and construct the controllers assuming that the fuzzy logic systems represent the true plant; therefore, fuzzy IF-THEN rules describing the plant can be directly incorporated into the indirect adaptive fuzzy controller. In this paper, we develop the observer-based indirect adaptive fuzzy-neural tracking control for nonlinear SISO systems by using VSS and H^∞ approaches under the constraint that only the system output is available for measurement. The proposed design method attempts to combine the attenuation technique via H^∞ tracking design scheme, fuzzy logic universal approximation theorem and adaptive control algorithm for the robust tracking design of the nonlinear systems with a large uncertainty or unknown variation in the plant parameters and structures.

This paper is organized as follows. First, the problem formulation is presented in Section 2. A brief description of fuzzy-neural networks is then made in Section 3. VSS indirect adaptive H^∞ tracking control design is given in Section 4. Simulation examples to illustrate the performance of the proposed method is provided in Section 5. Section 6 gives the conclusions of the advocated design methodology.

2. Problem formulation

Consider the nth-order nonlinear dynamical system of the form

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &\doteq x_3\\
&\cdots\\
\dot{x}_n &= f(x_1,x_2,\ldots,x_n)+g(x_1,x_2,\ldots x_n)u+d\\
y &= x_1
\end{aligned} \tag{1}$$

or equivalently the form

$$x^{(n)} = f(x,\dot{x},\ldots,x^{(n-1)})+g(x,\dot{x},\ldots,x^{(n-1)})u+d, \quad y=x, \tag{2}$$

where f and g are unknown but bounded functions, $u\in R$ and $y\in R$ are the control input and output of the system, respectively, and d is the external bounded disturbance. We can rewrite (2) in state space representation

$$\begin{aligned}
\dot{\underline{x}} &= A\underline{x}+B[f(\underline{x})+g(\underline{x})u+d],\\
y &= C^{\mathrm{T}}\underline{x},
\end{aligned} \tag{3}$$

where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix},\quad C=\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix} \tag{4}$$

and $\underline{x}=[x_1,x_2,\ldots,x_n]^{\mathrm{T}}=[x,\dot{x},\ldots,x^{(n-1)}]^{\mathrm{T}}\in R^n$ is a state vector where not all x_i are assumed to be available for measurement. Only the system output y is assumed to be measurable. In order for (2) to be controllable, it is required that $g(\underline{x})\neq 0$ for x in certain controllability region $U_{\mathrm{c}}\subset R^n$. Without loss of generality, we assume $0<g(\underline{x})<\infty$ for $\underline{x}\in U_{\mathrm{c}}$. The control object is to force the system output y to follow a given bounded reference signal y_{r}, under the constraint that all signals involved must be bounded.

To begin with, the reference signal vector $\underline{y}_{\mathrm{r}}$, the tracking error vector $\underline{e}$ will be defined as

$$\begin{aligned}
\underline{y}_{\mathrm{r}} &= [y_{\mathrm{r}},\dot{y}_{\mathrm{r}},\ldots,y_{\mathrm{r}}^{(n-1)}]^{\mathrm{T}}\in R^n,\\
\underline{e} &= \underline{x}-\underline{y}_{\mathrm{r}} = [e,\dot{e},\ldots,e^{(n-1)}]^{\mathrm{T}}\in R^n,\\
\hat{\underline{e}} &= \hat{\underline{x}}-\underline{y}_{\mathrm{r}} = [\hat{e},\dot{\hat{e}},\ldots,\hat{e}^{(n-1)}]^{\mathrm{T}}\in R^n,
\end{aligned}$$

where $\hat{\underline{x}}$ and $\hat{\underline{e}}$ denote the estimate of $\underline{x}$ and $\underline{e}$.

If the functions $f(\underline{x})$ and $g(\underline{x})$ are known and the system is free of external disturbance d, then we can choose the controller u^* to cancel the nonlinearity and design controller. Specially, let $\underline{k}_{\mathrm{c}}=[k_1^{\mathrm{c}},k_2^{\mathrm{c}},\ldots,k_n^{\mathrm{c}}]^{\mathrm{T}}\in R^n$ to be chosen such that all roots of the polynomial $p(s)=s^n+k_n^{\mathrm{c}}s^{n-1}+\cdots+k_1^{\mathrm{c}}$ are in the open left half-plane and control law of the certainty equivalent controller is obtained as [17]

$$u^* = \frac{1}{g(\underline{x})}[-f(\underline{x})+y_{\mathrm{r}}^{(n)}-\underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{e}]. \tag{5}$$

Substituting (5) into (2), we obtain the closed-loop system governed by

$$e^{(n)}+k_n^{\mathrm{c}}e^{(n-1)}+\cdots+k_1^{\mathrm{c}}e=0,$$

where the main objective of the control is $\lim_{t\to\infty} e(t)=0$. However, $f(\underline{x})$ and $g(\underline{x})$ are unknown, the ideal controller (5) cannot be implemented and not all system states $\underline{x}$ can be measured. We have to design an observer to estimate the state vector $\underline{x}$ in the following context.

2.1. State observer scheme

Replacing the functions $f(\underline{x})$, $g(\underline{x})$ and $\underline{e}$ in (5) by the estimation functions $f(\hat{\underline{x}})$, $g(\hat{\underline{x}})$ and $\hat{\underline{e}}$, the control law (5) is rewritten as

$$u = \frac{1}{\hat{g}(\hat{\underline{x}})}[-\hat{f}(\hat{\underline{x}})+y_{\mathrm{r}}^{(n)}-\underline{k}_{\mathrm{c}}^{\mathrm{T}}\hat{\underline{e}}]. \tag{6}$$

Applying (6) to (3) and after some simple manipulations, we can obtain the error equation

$$\begin{aligned}\dot{\underline{e}} &= A\underline{e}-B\underline{k}_{\mathrm{c}}^{\mathrm{T}}\hat{\underline{e}}+B\{f(\underline{x})-\hat{f}(\hat{\underline{x}})+(g(\underline{x})-\hat{g}(\hat{\underline{x}}))u+d\},\\ e_1 &= C^{\mathrm{T}}\underline{e},\end{aligned} \tag{7}$$

where $e_1=y_{\mathrm{r}}-y=y_{\mathrm{r}}-x_1$ denotes the output tracking error. Therefore, the tracking problem can be converted into the regulation problem to design a state observer for estimating the state vector e in (7) in order to regulate e_1 to zero.

From (7), the following is an observer that estimates the state vector $\underline{e}$ in (7)

$$\begin{aligned}\dot{\hat{\underline{e}}} &= A\hat{\underline{e}}-B\underline{k}_{\mathrm{c}}^{\mathrm{T}}\hat{\underline{e}}+\underline{k}_{\mathrm{o}}(e_1-\hat{e}_1),\\ \hat{\underline{e}}_1 &= C^{\mathrm{T}}\hat{\underline{e}},\end{aligned} \tag{8}$$

where $\underline{k}_{\mathrm{o}}^{\mathrm{T}}=[k_n^{\mathrm{o}},k_{n-1}^{\mathrm{o}},\ldots,k_1^{\mathrm{o}}]$ is the state observer gain vector.

The observation errors are defined as $\tilde{\underline{e}}=\underline{e}-\hat{\underline{e}}$ and $\tilde{e}_1=e_1-\hat{e}_1$. Subtracting (8) from (7), we can obtain the output error dynamics

$$\begin{aligned}\dot{\tilde{\underline{e}}} &= (A-\underline{k}_{\mathrm{o}}C^{\mathrm{T}})\tilde{\underline{e}}+B\{f(\underline{x})-\hat{f}(\hat{\underline{x}})+(g(\underline{x})-\hat{g}(\hat{\underline{x}}))u+d\},\\ \tilde{e}_1 &= C^{\mathrm{T}}\tilde{\underline{e}},\end{aligned} \tag{9}$$

where

$$\Lambda = A - \underline{k}_{\mathrm{o}} C^{\mathrm{T}} = \begin{bmatrix} -k_n^{\mathrm{o}} & 1 & 0 & 0 & \cdots & 0 & 0 \\ -k_{n-1}^{\mathrm{o}} & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -k_2^{\mathrm{o}} & 0 & 0 & 0 & \cdots & 0 & 1 \\ -k_1^{\mathrm{o}} & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since (C, Λ) pair is observable, the observer gain vector $\underline{k}_{\mathrm{o}}$ can be chosen such that the characteristic polynomial of Λ is strictly Hurwitz (i.e., the roots of the closed-loop system are in the open left half-plane) and we know that there exists a positive definite symmetric $n \times n$ matrix P which satisfies the Lyapunov equation

$$\Lambda^{\mathrm{T}} P + P\Lambda = -Q, \tag{10}$$

where Q is an arbitrary $n \times n$ positive definite matrix.

Let us rewrite (8) as

$$\dot{\hat{\underline{e}}} = \hat{A}\hat{\underline{e}} + \underline{k}_{\mathrm{o}} C^{\mathrm{T}} \tilde{\underline{e}}, \tag{11}$$

where $\hat{A} = A - B\underline{k}_{\mathrm{c}}^{\mathrm{T}}$ is a strictly Hurwitz matrix. Therefore, there exists a positive definite symmetric $n \times n$ matrix $\hat{P}$ which satisfies the Lyapunov equation

$$\hat{A}^{\mathrm{T}} \hat{P} + \hat{P}\hat{A} = -\hat{Q}, \tag{12}$$

where $\hat{Q}$ is an arbitrary $n \times n$ positive definite matrix. Let $V_{\hat{\underline{e}}} = \frac{1}{2}\hat{\underline{e}}^{\mathrm{T}} \hat{P} \hat{\underline{e}}$, then by using (11) and (12), we have

$$\begin{aligned} \dot{V}_{\hat{\underline{e}}} &= \tfrac{1}{2}\dot{\hat{\underline{e}}}^{\mathrm{T}} \hat{P}\hat{\underline{e}} + \tfrac{1}{2}\hat{\underline{e}}^{\mathrm{T}} \hat{P}\dot{\hat{\underline{e}}} = \tfrac{1}{2}\{\hat{A}\hat{\underline{e}} + \underline{k}_{\mathrm{o}} C^{\mathrm{T}} \tilde{\underline{e}}\}^{\mathrm{T}} \hat{P}\hat{\underline{e}} + \tfrac{1}{2}\hat{\underline{e}}^{\mathrm{T}} \hat{P}\{\hat{A}\hat{\underline{e}} + \underline{k}_{\mathrm{o}} C^{\mathrm{T}} \tilde{\underline{e}}\} \\ &= -\tfrac{1}{2}\hat{\underline{e}}^{\mathrm{T}} \hat{Q}\hat{\underline{e}} + \hat{\underline{e}}^{\mathrm{T}} \hat{P}\underline{k}_{\mathrm{o}} C^{\mathrm{T}} \tilde{\underline{e}}. \end{aligned} \tag{13}$$

Since $\hat{Q}$ and $\underline{k}_{\mathrm{o}}$ are determined by the designer, we can choose $\hat{Q}$ and $\underline{k}_{\mathrm{o}}$, such that $\dot{V}_{\hat{\underline{e}}} \leqslant 0$. Hence, $V_{\hat{\underline{e}}}$ is a bounded function and there exists a constant value $\bar{V}_{\hat{\underline{e}}}$, such that $V_{\hat{\underline{e}}} \leqslant \bar{V}_{\hat{\underline{e}}}$.

3. The Takagi–Sugeno FNN system [23]

Fuzzy logic systems address the imprecision of the input and output variables directly by defining them with fuzzy numbers (and fuzzy sets) that can be expressed in linguistic terms (e.g., *slow*, *medium* and *fast*). The basic configuration of the Takagi–Sugeno (T–S) FNN system includes a fuzzy rule base, which consists of a collection of fuzzy IF-THEN rules in the following form:

$$\begin{aligned} &R^{(l)}: \text{IF } x_1 \text{ is } F_1^l, \text{ and } \cdots, \text{ and } x_n \text{ is } F_n^l, \\ &\qquad \text{THEN } y_l = q_0^l + q_1^l x_1 + \cdots + q_n^l x_n = \underline{\theta}_l^{\mathrm{T}} [1 \underline{x}^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned} \tag{14}$$

where F_i^l are fuzzy sets and $\underline{\theta}_l^{\mathrm{T}} = [q_0^l, q_1^l, \ldots, q_n^l]$ is a vector of the adjustable factors of the consequence part of the fuzzy rule. Also y_l is a linguistic variable, and a fuzzy inference engine to combine the fuzzy IF-THEN rules in the fuzzy rule base into a mapping from an input linguistic vector $\underline{x}^{\mathrm{T}} = [x_1, x_2, \ldots, x_n] \in R^n$ to an output variable $y \in R$. Let M be the number of the fuzzy IF-THEN rules. The output of the fuzzy logic systems with central average defuzzifier, product inference and singleton fuzzifier can be expressed as

$$1\, y(\underline{x}) = \frac{\sum_{l=1}^{M} \upsilon^l \cdot y_l}{\sum_{l=1}^{M} \upsilon^l} = \frac{\sum_{l=1}^{M} \upsilon^l \cdot \underline{\theta}_l^{\mathrm{T}} [1 \underline{x}^{\mathrm{T}}]}{\sum_{l=1}^{M} \upsilon^l}, \tag{15}$$

where $\mu_{F_i^l}(x_i)$ is the membership function value of the fuzzy variable x_i and $\upsilon^l = \prod_{i=1}^{n} \mu_{F_i^l(x_i)}$ is the truth value of the lth implication. The actual membership functions F_i^l in Eq. (14) are normally the bell-shaped functions with parameters to be defined to suit different applications. Eq. (15) can be rewritten as

$$y(\underline{x}) = \underline{\theta}^{\mathrm{T}} \underline{\xi}(\underline{x}), \tag{16}$$

where $\underline{\theta}^{\mathrm{T}} = [\underline{\theta}_1^{\mathrm{T}} \underline{\theta}_2^{\mathrm{T}} \cdots \underline{\theta}_M^{\mathrm{T}}]$ is an adjustable parameter vector and $\underline{\xi}^{\mathrm{T}}(\underline{x}) = [\xi^1(\underline{x}),\ \xi^2(\underline{x}), \ldots, \xi^M(\underline{x})]$ is a fuzzy basis function vector defined as

$$\xi^l(\underline{x}) = \frac{\upsilon^l [1\ \underline{x}^{\mathrm{T}}]}{\sum_{l=1}^{M} \upsilon^l}. \tag{17}$$

When the inputs are fed into the T–S FNN, the truth value υ^l of the lth implication is computed. Applying the common defuzzification strategy, the output of the neural network expressed as (15) is pumped out. The overall configuration of the T–S FNN is shown in Fig. 1.

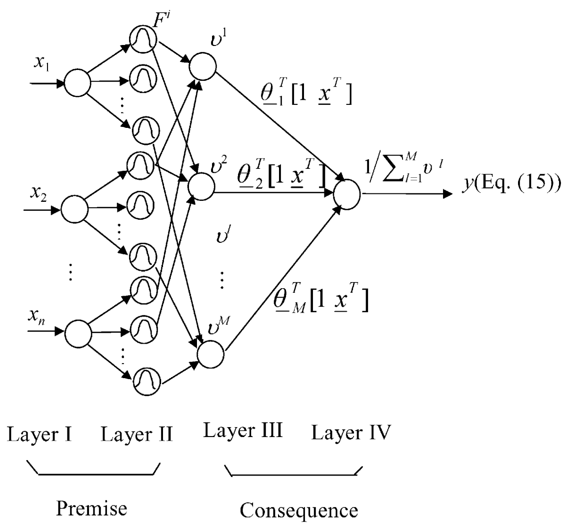

Fig. 1. The configuration of the T–S fuzzy-neural network.

Based on the universal approximation theorem [2,24], the above fuzzy logic system is capable of uniformly approximating any well-defined nonlinear function over a compact set U_c to any degree of accuracy. Also it is straightforward to show that a multi-output system can always be approximated by a group of single-output approximation systems.

4. VSS Adaptive H^∞ tracking control design

To begin with, our task is to use fuzzy neural network to approximate the nonlinear functions $f(\underline{x})$, $g(\underline{x})$ and develop the adaptive laws to adjust the parameters of fuzzy neural networks to attenuate the approximation errors and external disturbance.

First, from (16) the fuzzy systems $\hat{f}(\underline{\hat{x}},\underline{\theta}_f)$ and $\hat{g}(\underline{\hat{x}},\underline{\theta}_g)$ can be described as

$$\hat{f}(\underline{\hat{x}},\underline{\theta}_f) = \underline{\xi}^{\mathrm{T}}(\underline{\hat{x}})\underline{\theta}_f$$

and

$$\hat{g}(\underline{\hat{x}},\underline{\theta}_g) = \underline{\xi}^{\mathrm{T}}(\underline{\hat{x}})\underline{\theta}_g,$$

where $\underline{\theta}_f = [\theta_{f1},\ldots,\theta_{fn}]^{\mathrm{T}}$ and $\underline{\theta}_g = [\theta_{g1},\ldots,\theta_{gn}]^{\mathrm{T}}$ also $\underline{\xi}(\underline{x}) = [\xi_1,\ldots,\xi_n]^{\mathrm{T}}$. The universal fuzzy system $\hat{f}(\underline{\hat{x}},\underline{\theta}_f)$ with input vector $\underline{\hat{x}} \in U_{\hat{x}}$ for some compact set $U_{\hat{x}} \in \Re^n$ is proposed here to approximate the uncertain function $f(\underline{x})$, where $\underline{\theta}_f$ is a parameter vector to be tuned. Similar, the universal fuzzy system $\hat{g}(\underline{\hat{x}},\underline{\theta}_g)$ is defined here to approximate the uncertain functions $g(\underline{x})$, where $\underline{\theta}_g$ is a parameter vector to be tuned.

Next, in order for the linearly parameterized fuzzy model is employed in the approximation procedure of the uncertain dynamics, the membership functions $\mu_{f_i^l}(x_i)$ and $\mu_{g_i^l}(x_i)$ for $1 \leqslant i \leqslant n$, $1 \leqslant l \leqslant M$ should be specified beforehand in this paper. By universal approximate theorem [2,25], there exist optimal approximation parameters $\underline{\theta}_f^*$ and $\underline{\theta}_g^*$ such $f(\underline{x})$ and $g(\underline{x})$ can be approximated as close as possible. By using the adaptive laws, these optimal parameters are artificial quantities required only for analytical purpose as in much previous adaptive fuzzy research.

In order to guarantee the parameters $\underline{\theta}_f$ and $\underline{\theta}_g$ within the given constraint regions $\Omega_{\underline{\theta}_f}$ and $\Omega_{\underline{\theta}_g}$ for all $t \geqslant 0$, respectively, we use the parameter projection algorithm [3,14,16,17]. All constraint regions $\Omega_{\underline{\theta}_f}$ and $\Omega_{\underline{\theta}_g}$ are assumed to be convex. First, let us describe the convex hypercube of $\Omega_{\underline{\theta}_f}$ as

$$\Omega_{0f} = \{\underline{\theta}_f | a_{fi} \leqslant \theta_{fi} \leqslant b_{fi}, 1 \leqslant i \leqslant n\}$$

and

$$\Omega_{\underline{\theta}_f} = \{\underline{\theta}_f | a_{fi} - \delta_f \leqslant \theta_{fi} \leqslant b_{fi} + \delta_f, 1 \leqslant i \leqslant n\},$$

where all values a_{fi}, b_{fi} and $\delta_f > 0$ are specified by the designer, also define $\Phi_f = \underline{\xi} B^{\mathrm{T}} P \underline{e}$ with $P^{\mathrm{T}} = P \geqslant 0$ being the solution of the Riccati-like equation described later. The smooth projection algorithm with respect to $\Omega_{\underline{\theta}_f}$ will be given as [7,27]

$$\dot{\underline{\theta}}_{fi} = \begin{cases} r_1 \bar{\phi}_{fi} & \text{if } \theta_{fi} > b_{fi} \text{ and } \phi_{fi} > 0, \\ r_1 \breve{\phi}_{fi} & \text{if } \theta_{fi} < a_{fi} \text{ and } \phi_{fi} < 0, \\ r_1 \phi_{fi} & \text{otherwise, } 1 \leqslant i \leqslant n, \end{cases} \tag{18}$$

where ϕ_{fi} is the ith component of Φ_f and r_1 denotes the adaptive gain, also

$$\bar{\phi}_{fi} = \left(1 + \frac{b_{fi} - \theta_{fi}}{\delta_f}\right) \phi_{fi}$$

and

$$\breve{\phi}_{fi} = \left(1 + \frac{\theta_{fi} - a_{fi}}{\delta_f}\right) \phi_{fi}.$$

Next, the convex hypercube of $\Omega_{\underline{\theta}_g}$ is given by

$$\Omega_{0g} = \{\underline{\theta}_g | a_{gi} \leqslant \theta_{gi} \leqslant b_{gi}, 1 \leqslant i \leqslant n\}$$

and

$$\Omega_{\underline{\theta}g} = \{\underline{\theta}_g | a_{gi} - \delta_g \leqslant \theta_{gi} \leqslant b_{gi} + \delta_g,\ 1 \leqslant i \leqslant n\},$$

where all values a_{gi}, b_{gi} and $\delta_g > 0$ are specified by the designer too, also define $\Phi_g = \underline{\xi} B^{\mathrm{T}} P \underline{e} u$. The smooth projection algorithm with respect to $\Omega_{\underline{\theta}_g}$ will be given as

$$\dot{\underline{\theta}}_{gi} = \begin{cases} r_2 \bar{\phi}_{gi} & \text{if } \theta_{gi} > b_{gi} \text{ and } \phi_{gi} > 0, \\ r_2 \breve{\phi}_{gi} & \text{if } \theta_{gi} < a_{gi} \text{ and } \phi_{gi} < 0, \\ r_2 \phi_{gi} & \text{otherwise,} 1 \leqslant i \leqslant n, \end{cases} \tag{19}$$

where ϕ_{gi} is the ith component of Φ_g and r_2 denotes the adaptive gain, also

$$\bar{\phi}_{gi} = \left(1 + \frac{b_{gi} - \theta_{gi}}{\delta_g}\right) \phi_{gi}$$

and

$$\breve{\phi}_{gi} = \left(1 + \frac{\theta_{gi} - a_{gi}}{\delta_g}\right) \phi_{gi}.$$

The minimum approximation errors are defined as

$$\Delta f(\hat{\underline{x}}) = f(\underline{x}) - \hat{f}(\hat{\underline{x}}, \underline{\theta}^*_f)$$

and

$$\Delta g(\hat{\underline{x}}) = g(\underline{x}) - \hat{g}(\hat{\underline{x}}, \underline{\theta}^*_g).$$

In order to derive the control laws, we need the following assumptions hold for all $\hat{x} \in U_{\hat{x}}$, $\underline{\theta}_f \in \Omega_{\theta_f}$ and $\underline{\theta}_g \in \Omega_{g_f}$.

A1. There exists an upper bound $\alpha_f > 0$, such that

$$|\Delta f(\underline{\hat{x}})| \leqslant \alpha_f.$$

A2. There exists an upper bound $0 < \alpha_g < 1$, such that

$$|\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}}, \underline{\theta}_g)| \leqslant \alpha_g.$$

A3. There exists a finite constant $B_d > 0$, such that

$$\int_0^\infty \mathrm{d}^2(t)\,\mathrm{d}t \leqslant B_d, \text{ i.e., } \mathrm{d}(t) \in L_2[0,\infty).$$

Theorem 1. *For nonlinear SISO system* (2), *let the assumptions A1–A3 be true. If the VSS adaptive FNN control is chosen as*

$$u = \frac{1}{\hat{g}(\underline{\hat{x}}, \underline{\theta}_g)}\left[-\hat{f}(\underline{\hat{x}}, \underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}} + u_{\mathrm{s}} + u_{\mathrm{h}}\right] \tag{20}$$

where $\dot{\underline{\theta}}_f$ in (18), *$\dot{\underline{\theta}}_g$ in* (19) *with $b_{2i} - \delta_2 > 0$ for all $1 \leqslant i \leqslant m_2$, and the VSS controller u_{s} and the robust H^∞ controller u_{h} are*

$$u_{\mathrm{s}} = -\frac{B_e(\underline{\hat{x}})}{1-\alpha_g}\,\mathrm{sgn}(B^{\mathrm{T}}P\underline{\tilde{e}}), \tag{21}$$

$$u_{\mathrm{h}} = -\frac{1}{2r}B^{\mathrm{T}}P\underline{\tilde{e}}, \tag{22}$$

where $B_e(\underline{\hat{x}}) = \alpha_f + \alpha_g|-\hat{f}(\underline{\hat{x}}, \underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}}|$, r is a positive scalar value and $P = P^{\mathrm{T}} \geqslant 0$ is the solution of the following Riccati-like equation:

$$\Lambda^{\mathrm{T}}P + P\Lambda + Q + PB\left(\frac{1}{\rho^2} - \frac{1-\alpha_g}{r}\right)B^{\mathrm{T}}P = 0 \tag{23}$$

then the H^∞ tracking can be achieved for a prescribed attenuation level ρ.

Proof. Let us reconsider the output error dynamic equation (9) and take into account the minimum approximation errors, the error dynamic equation (9) can be rewritten as

$$\dot{\underline{\tilde{e}}} = \Lambda\underline{\tilde{e}} + B\{-\underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_f - \underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_g u + \Delta f(\underline{\hat{x}}) + \Delta g(\underline{\hat{x}})u + u_{\mathrm{s}} + u_{\mathrm{h}} + d\}, \tag{24}$$

where $\underline{\tilde{\theta}}_f = \underline{\theta}_f - \underline{\theta}_f^*$, $\underline{\tilde{\theta}}_g = \underline{\theta}_g - \underline{\theta}_g^*$, u_{s} is the VSS controller and u_{h} is the robust H^∞ controller.

Consider the Lyapunov-like function candidate

$$V = \frac{1}{2}\underline{\tilde{e}}^{\mathrm{T}}P\underline{\tilde{e}} + \frac{1}{2r_1}\underline{\tilde{\theta}}_f^{\mathrm{T}}\underline{\tilde{\theta}}_f + \frac{1}{2r_2}\underline{\tilde{\theta}}_g^{\mathrm{T}}\underline{\tilde{\theta}}_g. \tag{25}$$

Differentiating (25) with respect to time, and using the control law (20) we obtain

$$\begin{aligned}
\dot{V} = {} & \frac{1}{2}\{\Lambda\underline{\tilde{e}} + B(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g))(u_s+u_h) - B\underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_f - B\underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_g u + B\Delta f(\underline{\hat{x}}) \\
& + B\,\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)(-\hat{f}(\underline{\hat{x}},\underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}}) + Bd\}^{\mathrm{T}}P\underline{\tilde{e}} + \frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}P\{\Lambda\underline{\tilde{e}} \\
& + B(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g))(u_s+u_h) - B\underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_f - B\underline{\xi}^{\mathrm{T}}\underline{\tilde{\theta}}_g u + B\Delta f(\underline{\hat{x}}) \\
& + B\,\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)(-\hat{f}(\underline{\hat{x}},\underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}}) + Bd\} + \frac{1}{r_1}\,\dot{\underline{\tilde{\theta}}}_f^{\mathrm{T}}\underline{\tilde{\theta}}_f + \frac{1}{r_2}\,\dot{\underline{\tilde{\theta}}}_g^{\mathrm{T}}\underline{\tilde{\theta}}_g \\
= {} & \frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}(\Lambda^{\mathrm{T}}P + P\Lambda)\underline{\tilde{e}} + u_h(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g))B^{\mathrm{T}}P\underline{\tilde{e}} + \mathrm{d}B^{\mathrm{T}}P\underline{\tilde{e}} \\
& + u_s(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)B^{\mathrm{T}}P\underline{\tilde{e}} - \underline{\tilde{\theta}}_f^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}} + \frac{1}{r_1}\,\dot{\underline{\tilde{\theta}}}_f^{\mathrm{T}}\underline{\tilde{\theta}}_f - \underline{\tilde{\theta}}_g^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}}u \\
& + \frac{1}{r_2}\,\dot{\underline{\tilde{\theta}}}_g^{\mathrm{T}}\underline{\tilde{\theta}}_g + (\Delta f(\underline{\hat{x}}) + \Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)(-\hat{f}(\underline{\hat{x}},\underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}})B^{\mathrm{T}}P\underline{\tilde{e}}.
\end{aligned} \tag{26}$$

By substituting the robust H^{∞} controller u_h in (22) and completing the square, we get

$$\begin{aligned}
\dot{V} = {} & \frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}\left\{\Lambda^{\mathrm{T}}P + P\Lambda + PB\left(\frac{1}{\rho^2} - \frac{1+\Delta g(\underline{\hat{x}})g^{-1}(\underline{\hat{x}},\underline{\theta}_g)}{r}\right)B^{\mathrm{T}}P\right\}\underline{\tilde{e}} \\
& - \frac{1}{2}\left(\frac{1}{\rho}\,B^{\mathrm{T}}P\underline{\tilde{e}} - \rho d\right)^2 + \frac{1}{2}\,\rho^2 d^2 + u_s(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g))B^{\mathrm{T}}P\underline{\tilde{e}} \\
& - \underline{\tilde{\theta}}_f^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}} + \frac{1}{r_1}\,\dot{\underline{\tilde{\theta}}}_f^{\mathrm{T}}\underline{\tilde{\theta}}_f - \underline{\tilde{\theta}}_g^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}}u + \frac{1}{r_2}\,\dot{\underline{\tilde{\theta}}}_g^{\mathrm{T}}\underline{\tilde{\theta}}_g \\
& + (\Delta f(\underline{\hat{x}}) + \Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)(-\hat{f}(\underline{\hat{x}},\underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}}))B^{\mathrm{T}}P\underline{\tilde{e}}.
\end{aligned} \tag{27}$$

In fact, from adaptive law in (16) we can get $(1/r_1)\dot{\underline{\tilde{\theta}}}_f^{\mathrm{T}}\underline{\tilde{\theta}}_f - \underline{\tilde{\theta}}_f^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}} \leqslant 0$ and $\underline{\theta}_f(t) \in \Omega_{\theta_f}$ for all $t \geqslant 0$ if $\underline{\theta}_f(0) \in \Omega_{0f}$ [7]. Also, from the adaptive law in (19) we obtain $(1/r_2)\dot{\underline{\tilde{\theta}}}_g^{\mathrm{T}}\underline{\tilde{\theta}}_g - \underline{\tilde{\theta}}_g^{\mathrm{T}}\underline{\xi}B^{\mathrm{T}}P\underline{\tilde{e}}u \leqslant 0$ and $\underline{\theta}_g(t) \in \Omega_{\theta_g}$ for all $t \geqslant 0$ if $\underline{\theta}_g(0) \in \Omega_{0g}$. Next, from the assumptions A1–A2 and the VSS controller u_s in (21), we obtain

$$\begin{aligned}
& u_s(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g))B^{\mathrm{T}}P\underline{\tilde{e}} + (\Delta f(\underline{\hat{x}}) + \Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_g)(-\hat{f}(\underline{\hat{x}},\underline{\theta}_f) + y_{\mathrm{r}}^{(n)} - \underline{k}_{\mathrm{c}}^{\mathrm{T}}\underline{\hat{e}}))B^{\mathrm{T}}P\underline{\tilde{e}} \\
& \quad \leqslant -\frac{B_e(1+\Delta g(\underline{\hat{x}})\hat{g}^{-1}(\underline{\hat{x}},\underline{\theta}_f))}{1-\alpha_g}\,|B^{\mathrm{T}}P\underline{\tilde{e}}| + B_e|B^{\mathrm{T}}P\underline{\tilde{e}}| \leqslant 0.
\end{aligned} \tag{28}$$

From above results and Eq. (28), let us consider the Riccati-like equation (23), then the Eq. (27) can be rewritten as

$$\dot{V} \leqslant \frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}\left\{\Lambda^{\mathrm{T}}P + P\Lambda + PB\left(\frac{1}{\rho^2} - \frac{1+\Delta g(\underline{\hat{x}})g^{-1}(\underline{\hat{x}},\underline{\theta}_g)}{r}\right)B^{\mathrm{T}}P\right\}\underline{\tilde{e}} + \frac{1}{2}\,\rho^2 d^2$$
$$\leqslant \frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}\left\{\Lambda^{\mathrm{T}}P + P\Lambda + PB\left(\frac{1}{\rho^2} - \frac{1-\alpha_g}{r}\right)B^{\mathrm{T}}P\right\}\underline{\tilde{e}} + \frac{1}{2}\,\rho^2 d^2$$
$$\leqslant -\frac{1}{2}\,\underline{\tilde{e}}^{\mathrm{T}}Q\underline{\tilde{e}} + \frac{1}{2}\,\rho^2 d^2. \tag{29}$$

Integrating both sides of Eq. (29) from $t=0$ to T and after simple manipulations yields

$$V(T) + \frac{1}{2}\int_0^T \|\underline{\tilde{e}}(t)\|_Q^2\,\mathrm{d}t \leqslant V(0) + \frac{\rho^2}{2}\int_0^T \mathrm{d}^2(t)\,\mathrm{d}t \quad \forall 0 \leqslant T < \infty. \tag{30}$$

Considering the Lyapunov-like function $V(t)$ in (25) and the assumption A3, it yields $\underline{\tilde{e}}^{\mathrm{T}}(t)P\underline{\tilde{e}}(t) \leqslant 2V(0) + \rho^2 B_d$ for all $t \geqslant 0$, therefore the compact set U_x can be constructed as

$$\underline{\hat{x}}(t) \in U_{\underline{\hat{x}}} \triangleq \left\{\underline{\hat{x}}\,|\,\|\underline{\tilde{e}}(t)\| \leqslant \left(\frac{2V(0)+\rho^2 B_d}{\lambda_{\min}(P)}\right)^{1/2},\ \underline{y}_{\mathrm{r}}(t) \in \Omega_{\mathrm{r}}, \forall t \geqslant 0\right\}, \tag{31}$$

where $\underline{y}_{\mathrm{r}} = [y_{\mathrm{r}}, y'_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(n-1)}]^{\mathrm{T}}$.

This demonstrates all states and signals involved of the closed loop system are bounded. Furthermore, the H^∞ performance can be achieved from Eq. (30), i.e.,

$$\int_0^T \|\underline{\tilde{e}}(t)\|_Q^2\,\mathrm{d}t \leqslant 2V(0) + \rho^2\int_0^T \mathrm{d}^2(t)\,\mathrm{d}t, \quad \forall 0 \leqslant T < \infty. \tag{32}$$

From the smooth projection algorithm (19), we know that $a_{gi} - \delta_g \leqslant \theta_{gi} \leqslant b_{gi} + \delta_g$ for $1 \leqslant i \leqslant n$ and all values a_{gi}, b_{gi} and $\delta_g > 0$ can be arbitrary chosen by the designer. If $a_{gl} \equiv \min_{1\leqslant i\leqslant n}(a_{gi} - \delta_g)$ and $b_{gu} \equiv \max_{1\leqslant i\leqslant n}(b_{gi} + \delta_g)$, then it can be obtained that $a_{gl} \leqslant \hat{g}(\underline{\hat{x}},\underline{\theta}_g) \leqslant b_{gu}$ from

$$a_{gl} \leqslant \sum_{i=1}^n a_{gl}\xi_i \leqslant \sum_{i=1}^n (a_{gi} - \delta_g)\xi_i \leqslant \hat{g}(\underline{\hat{x}},\underline{\theta}_g)$$
$$\leqslant \sum_{i=1}^n (b_{gi} + \delta_g)\xi_i \leqslant \sum_{i=1}^n b_{gu}\xi_i = b_{gu}, \tag{33}$$

since $\xi_i(\underline{\hat{x}}) \in (0,1]$ and $\sum_{i=1}^n \xi_i(\underline{\hat{x}}) = 1$. Therefore, $\hat{g}(\underline{\hat{x}},\underline{\theta}_g)$ is invertible if a_{gi} and δ_g can be chosen suitably such that $a_{gl} > 0$. This completes the proof. □

Remark 1. In comparison with previous work [3,8] the VSS adaptive FNN controller developed above can be implemented, i.e., the fuzzy system $\hat{g}(\underline{\hat{x}},\underline{\theta}_g)$ can be guaranteed to be invertible and in turn the indirect control effort in (16) will be well defined.

Remark 2. The upper bound value α_g defined in A2 is supposed to be a state-dependent bound and can be independently chosen as $\alpha_g(\underline{\hat{x}}) \leqslant \varepsilon_g(\underline{\hat{x}})/a_{gl}$, where $\varepsilon_g(\underline{\hat{x}}) = \max|\Delta g(\underline{\hat{x}})|$. It implies that the maximal perturbation of $g(\underline{\hat{x}})$, i.e. $|\Delta g(\underline{\hat{x}})|$, should be less than $\max|g(\underline{\hat{x}}, \underline{\theta}_g)|$.

Remark 3. Our advocated design methodology combines three important control techniques, i.e., adaptive fuzzy neural network control scheme, VSS control design and H^∞ tracking control theory. Two adaptive neural systems, i.e., $\hat{f}(\underline{\hat{x}}, \underline{\theta}_f)$ and $\hat{g}(\underline{\hat{x}}, \underline{\theta}_g)$ equipped with update laws (18) and (19), are constructed to model the unknown systems $f(\underline{x})$ and $g(\underline{x})$, respectively. The VSS controller u_s is required to effectively eliminate the effect of the approximation errors from the universal approximation property. The robust H^∞ controller u_h can be applied such that the effect of the external disturbance on the tracking error can be attenuated to any prescribed level.

To summarize the above analysis, the design algorithm for an observer-based indirect adaptive fuzzy neural tracking control equipped with VSS and H^∞ control is proposed as follows:

Design procedure

Step 1: Specify the feedback and observer gain vector $\underline{k}_c$ and $\underline{k}_o$ such that the characteristic matrices $A - B\underline{k}_c^T$ and $A - \underline{k}_o C^T$ are strictly Hurwitz matrices, respectively.

Step 2: Specify a positive definite $n \times n$ matrix Q and solve the Lyapunov equation (10) to obtain a positive definite symmetric $n \times n$ matrix P.

Step 3: Solve the state equation (8) to obtain estimate state vector $\underline{\hat{x}} = \underline{\hat{e}} + \underline{y}_r$.

Step 4: Specify the design parameters, based on the practical constraints.

Step 5: Define the membership function $\mu_{F_i^l}(\underline{\hat{x}})$ for $i = 1, 2, \ldots, n$ and compute the fuzzy basis functions $\underline{\xi}(\underline{\hat{x}})$. Then the fuzzy logic control systems $\hat{f}(\underline{\hat{x}}, \underline{\theta}_f)$ and $\hat{g}(\underline{\hat{x}}, \underline{\theta}_g)$ can be constructed as

$$\hat{f}(\underline{\hat{x}}, \underline{\theta}_f) = \underline{\xi}^T(\underline{\hat{x}})\underline{\theta}_f$$

and

$$\hat{g}(\underline{\hat{x}}, \underline{\theta}_g) = \underline{\xi}^T(\underline{\hat{x}})\underline{\theta}_g.$$

Step 6: Obtain the control and apply to the plant, then compute the adaptive laws (18) and (19) to adjust the parameter vector $\underline{\theta}_f$ and $\underline{\theta}_g$.

5. The illustrative examples

In this section, we will apply our observer-based indirect adaptive FNN controller using VSS and H^∞ for two cases. The first example is to let the inverted pendulum to track a sine-wave trajectory. The second example is to let the output of mass–spring–damper system to track a sine-wave trajectory as well.

Example 1. Consider the inverted pendulum system as shown in Fig. 2. Let $x_1 = \theta$ be the angle of the pendulum with respect to the vertical line.

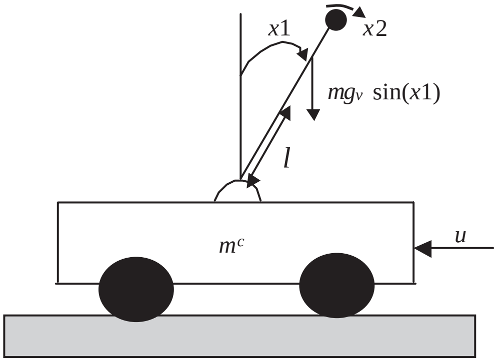

Fig. 2. The inverted pendulum system.

The dynamic equations of the inverted pendulum system [3,25,8,20] and [11] are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (f + gu + d),$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{34}$$

where

$$f = \frac{g_v \sin x_1 - (mlx_2^2 \cos x_1 \sin x_1)/m_c + m}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)}; \qquad g = \frac{\frac{\cos x_1}{m_c+m}}{l\left(\dfrac{4}{3} - \dfrac{m\cos^2 x_1}{m_c + m}\right)}$$

and $g_v = 9.8\ \text{m/s}^2$ is the acceleration due to gravity, m_c is the mass of the cart, l is the half-length of the pole, m is the mass of the pole and u is the control input. In this example, we assume that $m_c = 1$ kg, $m = 0.1$ kg, $l = 0.5$ m. The control object is to control the state x_1 of the system to track the reference trajectory $y_r(t) = 0.5 \sin(t)$. Also the external disturbance d is assumed $0.2 \sin(2t)\exp(-0.1t)$. The choices of r's and h are to improve the convergence rate of the closed-loop system controlled by our proposed controller.

According to the design procedure, the design is given in the following steps:

Step 1: The observer and feedback gain vectors are chosen as $\underline{k}_o^T = [89 \ \ 184]$, and $\underline{k}_c^T = [4 \ \ 4]$, respectively.

Step 2: We select Q in (10) as $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then after solving (10), the positive definite symmetric 2×2 matrix P in (10) is $\begin{bmatrix} 0.9945 & -0.50419 \\ -0.50418 & 0.2956 \end{bmatrix}$.

Step 3: Solve (8) to obtain $\hat{\underline{x}}$.

Step 4: We select $r_1 = 152.0302$, $r_2 = 88.4132$, $\delta_f = 10$, $\delta_g = 0.1$, $a_{gi} = 1.3$, $b_{gi} = 1.5$ and $\rho = [0.50.05 \quad 0.01]$. Also we choose $\alpha_f = 0.1$, $\alpha_g = 0.2$, $B_d = 20$, and $\hat{Q}$ in (12) is chosen as $\begin{bmatrix} 40 & 25 \\ 25 & 30 \end{bmatrix}$ and $\hat{A} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$ in (12). Therefore the positive definite symmetric 2×2 matrix $\hat{P}$ in (12) can be solved as $\begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$. The H^∞ gain $r = 0.7\rho^2$ and the step size is chosen as $h = 0.001895$.

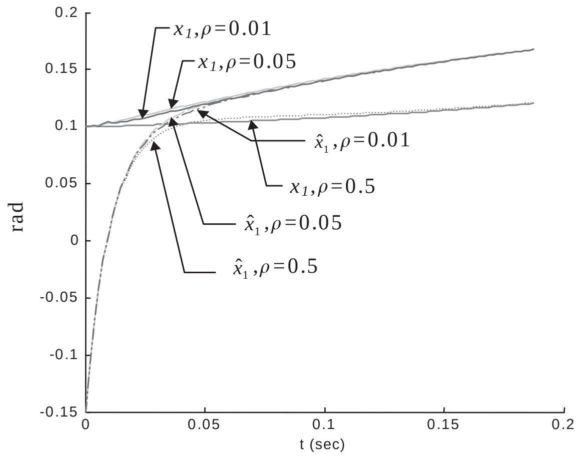

Fig. 3. Trajectories of the state x_1 and the estimated state $\hat{x}_1$.

Step 5: The following membership functions for $\hat{x}_i$, $i=1,2$ are selected as

$$\mu_{F_1^i}(\hat{x}_i)=\{1+\exp[10(\hat{x}_i+0.4)]\}^{-1},\quad \mu_{F_2^i}(\hat{x}_i)=\exp\Big(-2(\hat{x}_i+0.2)^2\Big),$$

$$\mu_{F_4^i}(\hat{x}_i)=\exp(-2\hat{x}_i^2),\quad \mu_{F_4^i}(\hat{x}_i)=\exp\Big(-2(\hat{x}_i-0.2)^2\Big),$$

$$\mu_{F_5^i}(\hat{x}_i)=\{1+\exp[10(\hat{x}_i-0.4)]\}^{-1}.$$

To cover whole cases, we apply (25) fuzzy rules.

Step 6: Obtain control input and compute the adaptive laws (18), (19).

The trajectories of the state x_1 and the estimated state $\hat{x}_1$ for three different levels of attenuation, i.e., $\rho=0.01$, 0.05 and 0.5, are shown in Fig. 3 and it shows that the estimated state $\hat{x}_1$ takes very short time to catch up the system state x_1. For different levels of attenuation, the tracking performances are also very good as shown in Fig. 4, where $\underline{y}_{\mathrm{r}}$ is the reference trajectory.

Also, the generalized velocity $\dot{\theta}(t)$ trajectories for different levels of attenuation and reference are shown in Fig. 5. Under the different prescribed attenuation levels, the integral of the error $\int_0^T \|e(t)\|^2\,\mathrm{d}t$ are indicated in Fig. 6. Therefore, the simulation result shows that the desired H^∞ attenuation requirement can be achieved. Furthermore the effects due to plant uncertainties and external disturbances can be efficiently diminished by proposed observer-based VSS indirect adaptive FNN H^∞ tracking controller.

The trajectories of the control input for different prescribed attenuation levels are shown in Fig. 7(a)–(c), i.e., $\rho=0.01$, 0.05 and 0.5.

Example 2. Consider the mass–spring–damper system as described in Fig. 8, with system parameters as body mass M (kg), spring coefficient K (N/m), friction coefficient B (N/m/s) and applied torque input u (N). The equation of motion for the system can be expressed as [27]

$$M\ddot{y}=u-f_K(x)-f_B(x)-f_C(x)+d, \tag{35}$$

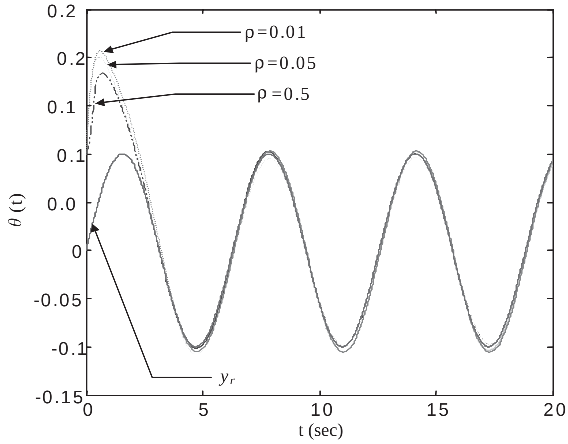

Fig. 4. The output trajectory for different levels of attenuation and reference trajectory.

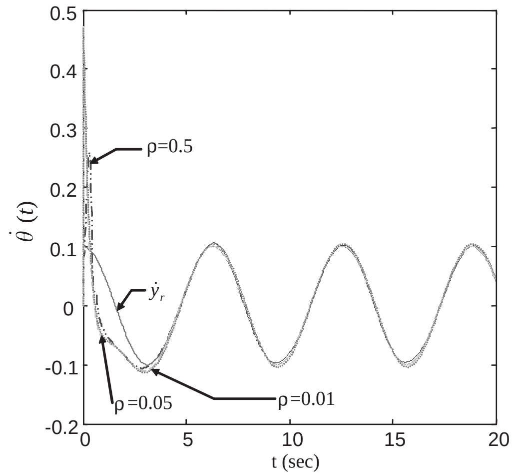

Fig. 5. The velocity trajectory $\dot{\theta}(t)$ for different levels of attenuation and reference trajectory.

where $f_K(x)$ denotes the spring force due to K, $f_B(x)$ is the friction force from and $f_C(x)$ is the coulomb friction force.

Let $x_1 = y$, $x_2 = \dot{x}_1$ and $\underline{x} = [x_1, x_2]^T$, the state space representation of the system can be described as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{M}(-f_K(x) - f_B(x) - f_C(x)) + \frac{1}{M}u + \frac{1}{M}d. \end{aligned} \tag{36}$$

This mass–spring–damper system suffers from plant uncertainties, unmodeled force and external disturbances. The nominal parameters of the system are given by $M_0 = 1$, $K_0 = 2$ and $B_0 = 2$. The perturbations of the system parameters are given as $\Delta M = 0.1\ \sin(y)$, $\bullet\ K = 0.5$ and $\bullet\ B = 0.5$.

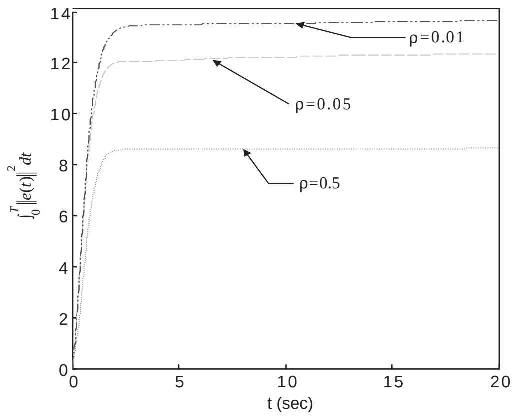

Fig. 6. The integral of the error $\int_0^T \|e(t)\|^2\, \mathrm{d}t$ for different prescribed attenuation levels.

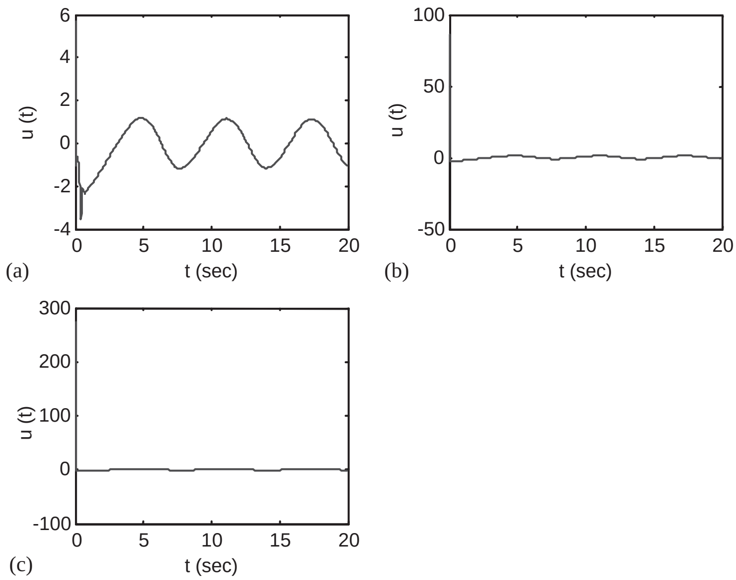

Fig. 7. The trajectories of the control input for different prescribed attenuation levels.

Also the nonlinear spring force and friction force are assumed to be $f_K(x) = K_0 y + \Delta K y^3$ and $f_B(x) = B_0 \dot{y} + \Delta B \dot{y}^2$. In addition, suppose there is a coulomb friction force $f_C(x) = 0.01\mathrm{sgn}(\dot{y})$. Therefore, state space representation can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (f(\underline{x}) + g(\underline{x})u + d_1),$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

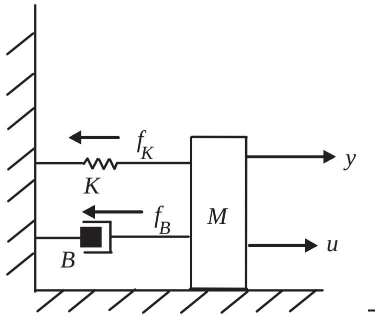

Fig. 8. The mass–spring–damper system.

where

$$f(\underline{x}) = \frac{1}{M_0 + \Delta M}(-f_K(\underline{x}) - f_B(\underline{x}) - f_C(\underline{x})), \tag{37}$$

$$g(\underline{x}) = \frac{1}{M_0 + \Delta M} \tag{38}$$

and

$$d_1 = \frac{1}{M_0 + \Delta M}d. \tag{39}$$

By substituting all parameters into Eq. (37) and (38), we get

$$f(\underline{x}) = \frac{1}{1 + 0.1\sin(x_1)}(-2x_1 - 0.5x_1^3 - 2x_2 - 0.5x_2^2 - 0.01\mathrm{sgn}(x_2)),$$

$$g(\underline{x}) = \frac{1}{1 + 0.1\sin(x_1)}.$$

We also have to determine the bounds f^{U}, g^{U} and g_{L} as follows:

$$\begin{aligned}|f(x_1,x_2)| &\leqslant \left(\frac{1}{|1 + 0.1\ \sin(x_1)|}\right)(0.5|x_1^3| + 2|x_1| + 2 + 0.01|\mathrm{sgn}(x_2)|)\\ &\leqslant 1.5 \times (0.5|x_1^3| + 2|x_1| + 2.01) = f^{\mathrm{U}}(x_1,x_2) \approx f^{\mathrm{U}}(\hat{x}_1,\hat{x}_2),\end{aligned}$$

$$|g(x_1,x_2)| \leqslant 1.5 = g^{\mathrm{U}}(x_1,x_2) \approx g^{\mathrm{U}}(\hat{x}_1,\hat{x}_2), \tag{40}$$

$$|g(x_1,x_2)| \geqslant 0.9 = g_{\mathrm{L}}(x_1,x_2) \approx g_{\mathrm{L}}(\hat{x}_1,\hat{x}_2). \tag{41}$$

The control object is to control the state x_1 of the system to track the reference trajectory $y_r(t) = 0.5\sin(t)$ if only the system output y is measurable. Also the external disturbance d is assumed to

Table 1
Three cases of the initial states

Cases	Initial states
Case1	$x(0)=[0.25 \quad 0]^{\mathrm{T}}$ $\hat{x}(0)=[-0.25 \quad 0]^{\mathrm{T}}$
Case 2	$x(0)=[0.15 \quad 0.15]^{\mathrm{T}}$ $\hat{x}(0)=[-0.15 \quad -0.15]^{\mathrm{T}}$
Case 3	$x(0)=[-0.15 \quad -0.05]^{\mathrm{T}}$ $\hat{x}(0)=[0.15 \quad 0.25]^{\mathrm{T}}$

be 0.2 $\sin(2t)\exp(-0.1t)$. The choices of γ's and h are to improve the convergence rate of the closed-loop system controlled by our proposed controller.

According to the design procedure, the design is given in the following steps:

Step 1: The observer and feedback gain vectors are chosen as $\underline{k}_{\mathrm{o}}^{\mathrm{T}}=[89 \;\; 184]$, and $\underline{k}_{\mathrm{c}}^{\mathrm{T}}=[4 \;\; 4]$, respectively.

Step 2: We select Q in (12) as $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then after solving (10), the positive definite symmetric 2×2 matrix P in (10) is $\begin{bmatrix} 0.9945 & -0.50419 \\ -0.50418 & 0.2956 \end{bmatrix}$.

Step 3: Solve (8) to obtain $\underline{\hat{x}}$.

Step 4: We select $r_1=500$, $r_2=1.0$, $\delta_f=10$, $\delta_g=0.1$, $a_{gi}=1.3$, $b_{gi}=1.5$ and $\rho=0.5$. Also we choose $\alpha_f=0.1$, $\alpha_g=0.2$, $B_d=20$, and $\hat{Q}$ in (12) is chosen as $\begin{bmatrix} 40 & 25 \\ 25 & 30 \end{bmatrix}$ and $\hat{A}=\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$ in (12). Therefore the positive definite symmetric 2×2 matrix $\hat{P}$ in (12) can be solved as $\begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}$. The H^∞ gain $r=0.7\rho^2$ and the step size is chosen as $h=0.001667$.

Step 5: The following membership functions for $\hat{x}_i$, $i=1,2$ are selected as

$$\mu_{F_1^i}(\hat{x}_i) = 1/(1+\exp(10(\hat{x}_i+1))), \quad \mu_{F_2^i}(\hat{x}_i) = \exp(-2(\hat{x}_i+0.5)^2),$$

$$\mu_{F_3^i}(\hat{x}_i) = \exp(-2\hat{x}_i^2), \quad \mu_{F_4^i}(\hat{x}_i) = \exp(-2(\hat{x}_i-0.5)^2),$$

$$\mu_{F_5^i}(\hat{x}_i) = 1/(1+\exp(-10(\hat{x}_i-1))).$$

To cover whole cases, we apply 25 fuzzy rules.

Step 6: Obtain control input and compute the adaptive laws (18) and (19).

According to the initial states, three cases are simulated as shown in Table 1.

The trajectories of the state x_1 and the estimated state $\hat{x}_1$ for three different initial states are shown in Fig. 9 and it shows that the estimated state $\hat{x}_1$ takes very short time to catch up the system state x_1. For different initial states, the tracking performances are also very good as shown in Fig. 10, where $\underline{y}_{\mathrm{r}}$ is the reference trajectory.

Also, the generalized velocity $\dot{y}(t)$ and $y_{\mathrm{r}}(t)$ trajectories for three cases are shown in Fig. 11. Under the different prescribed attenuation levels, the integral of the error $\int_0^T \|e(t)\|^2\,\mathrm{d}t$ are indicated in Fig. 12. Therefore, the simulation result shows that the desired H^∞ attenuation requirement can be achieved. Furthermore the effects due to plant uncertainties and external disturbances can be efficiently diminished by proposed observer-based VSS indirect adaptive FNN H^∞ tracking controller.

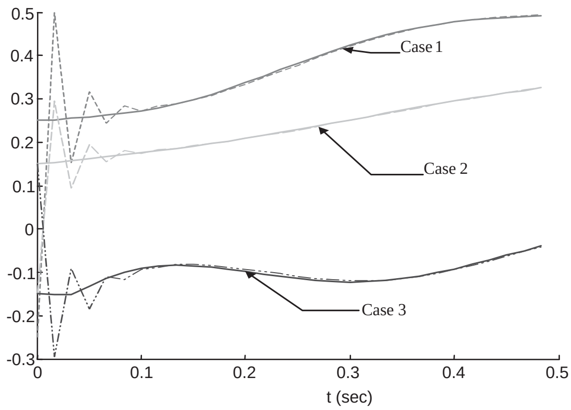

Fig. 9. The trajectories the states x_1 (solid line) and $\hat{x}_1$ (dash line) of 3 cases (time: 0–0.483 s).

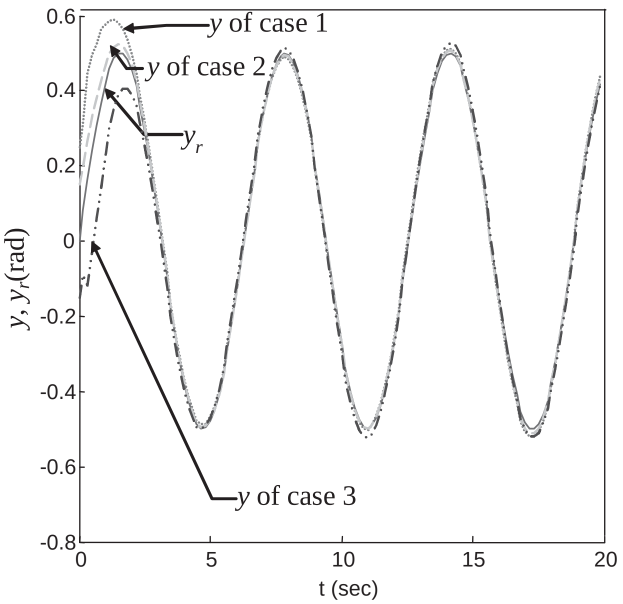

Fig. 10. The output trajectory for three cases and reference trajectory.

The trajectories of the control input of three different initial states are shown in Fig. 13.

6. Conclusions

An indirect adaptive FNN controller with observer design by using VSS and H^∞ control algorithms is developed for nonlinear SISO systems involving plant uncertainties and external disturbances, in which only the system output can be measured. Based on the Lyapunov synthesis approach, the free parameters of the adaptive FNN controller can be tuned on-line by the observer-based output

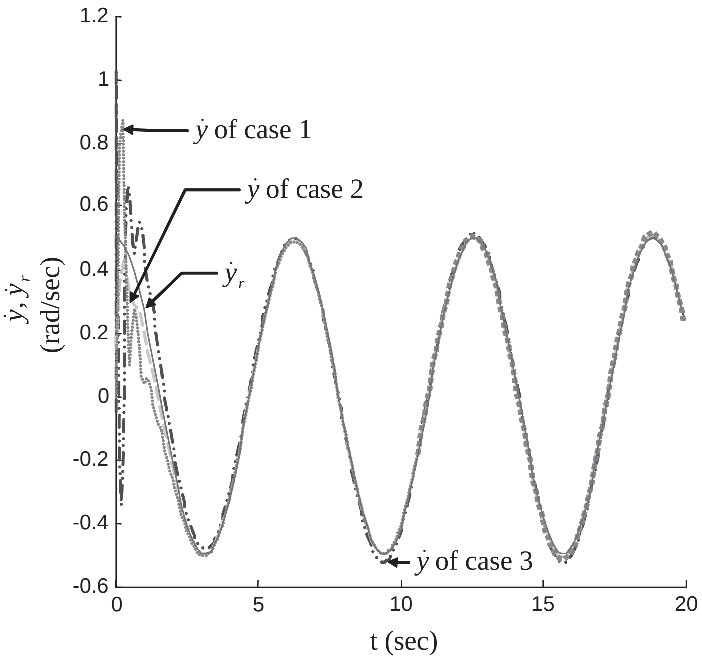

Fig. 11. The generalized velocity $\dot{y}(t)$ and $y_r(t)$ trajectories for three cases.

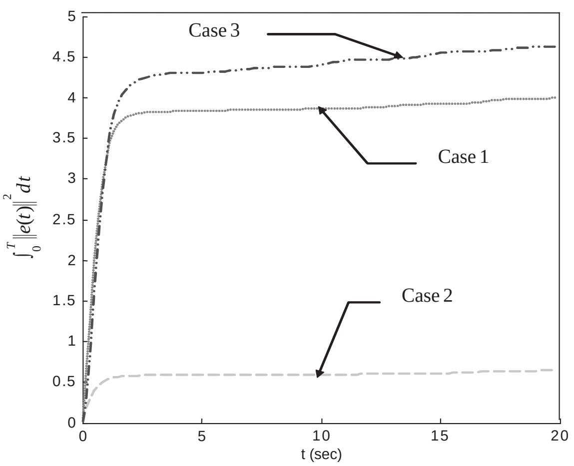

Fig. 12. The integral of the error $\int_0^T \|e(t)\|^2 \,\mathrm{d}t$ for three cases.

feedback control and the adaptive laws. Also the robust nonlinear output tracking requirement can be achieved by three control design techniques, adaptive fuzzy neural control scheme, VSS control design and H^∞ tracking theory. Simulation results show that the overall observer-based adaptive FNN control scheme guarantees stability of the resulting closed-loop system in the sense that all the states and signals are uniformly bounded and H^∞ tracking performance can be achieved.

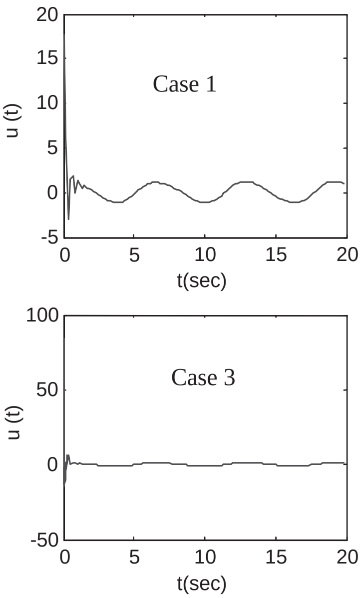

Fig. 13. Trajectories of the control input for three cases.

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