Wide-sense nonblocking for symmetric or asymmetric 3-stage Clos networks under various routing strategies

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Abstract

Beneš established the notion of wide-sense nonblocking by constructing an example on the symmetric 3-stage Clos network under packing which requires less hardware compared to strict nonblocking. This has remained the only example of a wide-sense non-blocking 3-stage Clos network which is not strictly nonblocking. In this paper, we study packing as well as several other routing strategies which have been studied in the literature and proved that no other example exists for the symmetric 3-stage Clos network. We then extend the study to asymmetric 3-stage Clos network for the first time. In particular, we extend Beneš example to asymmetric 3-stage Clos network and show that these are the only two possible examples for the strategies under study.

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1. Introduction

A switching network is used to connect a sequence of (input, output) pairs sequentially, while connected pairs can be disconnected by releasing all links on its path. A switching network is said to be strictly nonblocking (SNB) if a pair can always be connected regardless of how the previous pairs are connected; it is said to be
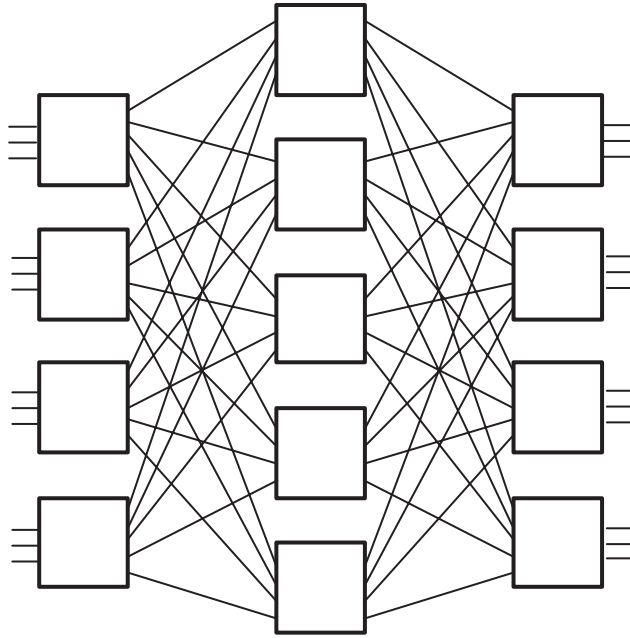
Fig. 1. $C(3, 5, 4)$.

wide-sense nonblocking (WSNB) with respect to a routing strategy A if a pair can always be connected when the routing of every pair follows A. Often, A consists of a few rules and at a given moment, several routes all satisfy A. Then the network is WSNB under A if and only if for each such choice, subsequent pairs can always be connected.

The existence of a WSNB network was first demonstrated by Beneš [1] for the symmetric 3-stage Clos network $C(n, m, r)$ which has r crossbars $F_1, \ldots, F_r$ of size $n \times m$ in the first stage, m crossbars $S_1, \ldots, S_m$ of size $r \times r$ in the second (middle) stage, and r crossbars $T_1, \ldots, T_r$ of size $n \times m$ in the third stage (see Fig. 1). Note that a given pair of (input, output) fixes F_i and T_j; hence the path is determined by the selection of S_k. It is well known [3] that $C(n, m, r)$ is SNB if and only if $m \geqslant 2n - 1$. Beneš [1,2] proved that $C(n, m, 2)$ is WSNB under "packing", which routes a path through a busiest (carrying most paths) middle switch if and only if $m \geqslant \lfloor 3n/2 \rfloor$.

The notion of wide-sense nonblocking in switching networks is a fascinating idea to computer scientists. It suggests that the hardware (network components) can be reduced through intelligent software (routing) without affecting the nonblocking property of the network. However, the only positive result about WSNB $C(n, m, r)$ is the Beneš result for the almost trivial network $r = 2$. Lots of efforts have been spent to expand this result, but without success. This puzzle was eventually answered by results coming from an unexpected direction, the negative side. Namely, the reason that

no WSNB $C(n, m, r)$ have been found for $r > 2$ under packing is that they do not exist.

We review the literature in more detail. Five routing strategies have been proposed in the literature:

(1) Save the unused (STU): Do not route through an empty S_k unless there is no choice.
(2) Packing (P): Route through a busiest, yet available, S_k.
(3) Minimum index (MI): For each pair, route in the order $S_1, S_2, \ldots, S_k$ until the first available one emerges.
(4) Cyclic dynamic (CD): If S_k was used last, try $S_{k+1}, S_{k+2}, \ldots$.
(5) Cyclic static (CS): If S_k was used last, try $S_k, S_{k+1}, \ldots$.

Note that P $\Rightarrow$ STU. So WSNB under STU $\Rightarrow$ WSNB under P since P is a choice of STU. On the other hand, not WSNB under P implies not WSNB under STU.

Smith [6] proved that $C(n, m, r)$ is not WSNB under P or MI if $m < \lfloor 2n - n/r \rfloor$. Du et al. [4] improved to $\lfloor 2n - n/2^r - 1 \rfloor$ which was extended to cover CS in Hwang [5]. For P, Yang and Wang [8] gave a linear programming formulation of the problem and ingeniously found the closed-form solution $m \geqslant \lfloor 2n - n/F_{2r-1} \rfloor$ where F_{2r-1} is the $2r - 1$st Fibonacci number, as a necessary condition for $C(n, m, r)$ to be WSNB. Actually, there was an earlier stronger result of Du et al. reported in the 1998 book of Hwang [5] that $m \geqslant 2n - 1$ is necessary and sufficient for $C(n, m, r)$, $n \geqslant 3$, to be WSNB under P. This result for $r \geqslant 3$ together with Beneš result for $r = 2$ gave a definitive answer to the WSNB property of $C(n, m, r)$ under P. Finally, Tsai et al. [7] proved that for all n, there exists r large enough such that $C(n, m, r)$ is not WSNB under any algorithm.

The proof of the $m \geqslant 2n - 1$ result by Du et al. is quite difficult to check and the proof of Yang and Wang is also complicated. In this paper we give a much simpler proof which not only works for P (hence STU), but also for CD, CS and MI.

We also extend all these results to the asymmetric 3-stage Clos network $C(n_1, n_2, m, r_1, r_2)$ where the first stage has r_1 crossbars with n_1 inputs each, and the third stage has r_2 crossbars with n_2 output each. This is the first time that WSNB is studied for $C(n_1, n_2, m, r_1, r_2)$. Among other things, we are able to extend Beneš positive result on packing to the asymmetric network, thus establishing the second positive result. We also show that no other positive result is possible for the 3-stage Clos network for P, STU, CD, CS and MI.

2. The symmetric case

A state of $C(n, m, r)$ can be represented by an $r \times r$ matrix where cell (i, j) consists of the set of S_k carrying a path from F_i to T_j. Then each row or column can have atmost n entries and the entries must be all distinct. The n-uniform state is the matrix where each diagonal cell contains the set $S_1, \ldots, S_n$ and all other cells are empty. The $\lfloor 2n - n/2^{r-1} \rfloor$ result was actually proved [5] for all algorithms which can reach the n-uniform state, which, as shown in [5], includes P, STU, MI and CS. Hung (private communication)

observed that CD can also reach the n-uniform state. We give a stronger result based on his method.

Lemma 1. *CD can reach any state s from any state s'.*

Proof. Since we can disconnect all paths in s' to reach the empty state, it suffices to prove for s' the empty state. We prove this by adding each S_k in s to its proper cell one by one. Suppose S_k is in cell (i, j). Consider a pair (F_i, T_j). Suppose CD assigns S_h to connect the pair. If $h \neq k$, disconnect the pair and regenerate it immediately. Then CS would assign S_{h+1} to connect the pair. Repeat this until S_k is assigned. Since S_k is arbitrary, s can be reached.　$\square$

Corollary 2. *CD can reach the n-uniform state.*

For CS we prove a weaker property. Let $[i, j]$ denote the set $\{S_i, S_{i+1}, \ldots, S_j\}$ if $i \leqslant j$, and the empty set if $i > j$.

Lemma 3. *Let state s be obtained from s' by adding $[i, j]$, $i < j$, to a cell C. Then s can be reached from s' under CS.*

Proof. Suppose the last assignment is S_k in s'. Since $i < j$, we can add at least two connections in C. Then S_k and S_{k+1} will be assigned. If $k \neq i$, disconnect the connection through S_k and regenerate a connection in C, for which S_{k+2} will be assigned. Continue this until S_i and S_{i+1} are assigned. Then add $j - i - 1$ connections to C for which $S_{i+2}, \ldots, S_j$ will be assigned.　$\square$

Theorem 4. *$C(n, m, r)$ for $r \geqslant 2$ is WSNB under CD and CS if and only if $m \geqslant 2n - 1$.*

Proof. The "if" part is trivial since $C(n, 2n - 1, r)$ is SNB, hence WSNB. To prove the "only if" part, we claim that if $m = 2n - 2$, then there exists a blocking state.

It is well known [5] that it suffices to prove for the minimum r which is 2 here. By Lemmas 1 and 3, the state in which cell $(1, 1)$ contains $[1, n - 1]$ and cell $(2, 2)$ contains $[n, 2n - 2]$ can be reached. But a new pair $(1, 2)$ is blocked.　$\square$

Theorem 5. *For P, hence STU, $C(n, m, r)$, $r \geqslant 3$, is WSNB if and only if $m \geqslant 2n - 1$.*

Proof. The "if" part is trivial. We prove the "only if" part by showing that for $n = 3$ there exists a sequence of calls and disconnections forcing the use of $2n - 1$ middle switches:
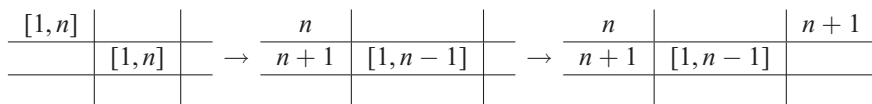
$[1,n]$			n			n		$n+1$
	$[1,n]$	$\rightarrow$	$n+1$	$[1,n-1]$	$\rightarrow$	$n+1$	$[1,n-1]$	

$$\rightarrow \quad \begin{array}{c|c} n & n+1 \\ \hline [1,n-1] & \\ \hline n+1 & \end{array} \qquad \rightarrow \quad \begin{array}{c|c} n & n+1 \\ \hline n+2 \quad [1,n-1] & \\ \hline n+1 & \end{array}$$

$$\rightarrow \quad \begin{array}{c|c} n & [n+1,n+2] \\ \hline n+2 \quad [1,n-1] & \\ \hline n+1 & \end{array}$$

$$\rightarrow \quad \begin{array}{c|c} n & [n+1,n+2] \\ \hline [1,n-1] & \\ \hline [n+1,n+2] & \end{array}$$

$$\rightarrow \cdots \rightarrow \quad \begin{array}{c|c} n & [n+1,2n-2] \\ \hline [1,n-1] & \\ \hline [n+1,2n-2] & \end{array}$$

$$\rightarrow \quad \begin{array}{c|c} n & [n+1,2n-2] \\ \hline 2n-1 \quad [1,n-1] & \\ \hline [n+1,2n-2] & \end{array} \qquad \square$$

Note that this proof is much more elementary than the proof in [4].
For MI, we first prove a lemma.

Lemma 6. *Consider a state s in $C(n,m,2)$ consisting of x (I_1,O_1) calls carried by the set X of middle switches, and y (I_2,O_2) calls carried by the set Y of middle switches such that $X \cap Y = \emptyset$, $X \cup Y = \{1,\ldots,x+y\}$. Then a state s' can be obtained from s, where s' is same as s except that x becomes x', and y becomes $y' = x + y - x'$.*

Proof. Without loss of generality, assume $x' > x$ (otherwise we work with y). Delete $x' - x$ (I_2,O_2) calls carried by $S = \{$the smallest $x' - x$ indices in $Y\}$ from s. Add $x' - x$ new (I_1,O_1) requests. By the MI rule, these new requests must be carried by S. Thus s' is obtained. $\square$

Theorem 7. *$C(n,m,r)$ for $r \geqslant 2$ is WSNB under MI if and only if $m \geqslant 2n - 1$.*

Proof. It suffices to prove that $m = 2n - 1$ is necessary for WSNB for $r = 2$.
   By induction on n, $m = 2n - 3$ is necessary for $C(n-1,m,2)$ to be WSNB. Therefore there exists a state

$$\begin{array}{c|c} X & 2n-3 \\ \hline & Y \end{array}$$

in $C(n,2n-1,2)$, such that $x = y = n - 2$, $X \cup Y = \{1,\ldots,2n-4\}$. Therefore we can obtain a state s' from s by adding $2n - 2$ to the $(1,2)$ cell. Delete the four calls carried by $[2n - 7, 2n - 4]$ in the $(1,1)$ and $(2,2)$ cells, and use Lemma 6 to rebalance the

members of calls carried by them, i.e., each carrying $n-4$ calls. Assign $[2n-7, 2n-4]$ to cell $(2, 1)$.

Next we delete $[2n-11, 2n-8]$ from cells $(1, 1)$ and $(2, 2)$, do the balancing and assign $[2n-11, 2n-8]$ to cell $(1, 2)$. By repeatedly doing so, eventually (the last step may delete only two calls) we reach a state consisting of $2n-2$ distinct indices in cells $(1, 2)$ and $(2, 1)$. Thus a new $(1, 1)$ request must be carried by M_{2n-1}. $\quad\square$

Corollary 8. *For* 3-*stage Clos network* $C(n, m, r)$, *let* s *be the state where* X, Y, *and* Z *are in cells* (i_1, j_2), (i_2, j_1), *and* (i_1, j_1), *respectively*,

X		Z
		Y

where, $\min\{Z\} > k$, $X \cap Y = \emptyset$, $X \cup Y = [1, k]$, $k \leqslant 2(n - |Z|)$.

For each $\alpha \leqslant k$ *and* $\max\{\alpha, k - \alpha\} \leqslant (n - |Z|)$, *let* $f_\alpha(s)$ *be the state which has* $f_\alpha(X)$ *in cell* (i_1, j_2), $|f_\alpha(X)| = \alpha$, *and* $f_\alpha(Y)$ *in cell* (i_2, j_1), *such that* $f_\alpha(X) \cap f_\alpha(Y) = \emptyset$, $f_\alpha(X) \cup f_\alpha(Y) = [1, k]$. *Then* $f_\alpha(s)$ *can be reached from* s *under MI.*

3. The asymmetric case

Without loss of generality, we assume $\lfloor n_1/n_2 \rfloor = k \geqslant 1$ throughout this section. If $n_1 \geqslant r_2 n_2$, then $m = r_2 n_2$ is necessary and sufficient for $C(n_1, n_2, m, r_1, r_2)$ to be either SNB or WSNB. Therefore we assume $r_2 > n_1/n_2$, or $r_2 \geqslant \lceil (n_1 + 1)/n_2 \rceil$.

Theorem 9. $C(n_1, n_2, m, r_1, r_2)$ *for* $r_2 \geqslant 2$ *is WSNB under CS and CD if and only if* $m \geqslant n_1 + n_2 - 1$.

Proof. The "if" part is trivial since $C(n_1, r_1, n_1 + n_2 - 1, n_2, r_2)$ is SNB. To prove the "only if" part, we show that if $m = n_1 + n_2 - 2$, then there exists a blocking state. Clearly, we can reach the state

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\ldots$	$[kn_2 + 1, n_1 - 1]$	
				$[n_1, n_1 + n_2 - 2]$

(if $[kn_2 + 1, n_1 - 1]$ is an empty set, then the corresponding column does not exist). Since row 1 has only $n_1 - 1$ entries and the last column has only $n_2 - 1$ entries, one new connection can be requested in the cell $(1, \lceil n_1/n_2 \rceil + 1)$, but no middle switch is available. $\quad\square$

The MI case is as follows. We first prove a lemma.

Lemma 10. $C(n_1, n_2, m, r_1, r_2)$ *with* $n_1 = n_2 + 1$, $\min\{r_1, r_2\} \geqslant 2$, *is not WSNB under MI if* $m < 2n_2$.

Proof. We prove, by induction on n_2, the existence of a state which must use $2n_2$ middle switches.

(1) $n_2 = 2$,

$$\frac{\Big|}{[1,2]\ \Big|\ 3} \rightarrow \frac{[1,2]\ \Big|}{\Big|\ 3} \rightarrow \frac{[1,2]\ \Big|\ 4}{\Big|\ 3}$$

(2) suppose that for $n_2 = n$ the statement is true.

(3) $n_2 = n + 1$, since for $n_2 = n$ the statement is true, we can reach a state s

$$\frac{X\ \Big|\ 2n}{\Big|\ Y},\ |X| = n, |Y| = n - 1,\ X \cap Y = \emptyset,\ \text{and}\ X \cup Y = [1, 2n - 1].\ \text{Add}\ 2n + 1$$

to cell $(1, 2)$, since $n_2 = n + 1$, and delete the four numbers $[2n - 4, 2n - 1]$ from cell $(1, 1)$ and $(2, 2)$. By noting that Corollary 8 also applies to the asymmetric 3-stage Clos network, we can get a state s_1,

$$\frac{X_1\ \Big|\ [2n, 2n + 1]}{\Big|\ Y_1},\ |X_1| = n - 3, |Y_1| = n - 2,\ X_1 \cap Y_1 = \emptyset \quad \text{and}$$

$$X_1 \cup Y_1 = [1, 2n - 5].$$

Then, we add $[2n - 4, 2n - 1]$ to cell $(2, 1)$, and delete the four numbers $[2n - 8, 2n - 5]$ from cell $(1, 1)$ and $(2, 2)$. By Corollary 8 again, we can reach a state s_2,

$$\frac{X_2\ \Big|\ [2n, 2n + 1]}{[2n - 4, 2n - 1]\ \Big|\ Y_2},\ |X_2| = n - 4, |Y_2| = n - 5,\ X_2 \cap Y_2 = \emptyset \quad \text{and}$$

$$X_2 \cup Y_2 = [1, 2n - 9].$$

Repeat the above steps, without loss of generality, we reach a state s',

$$\frac{\Big|\ X'}{Y'\ \Big|},\ |X'| = n, |Y'| = n + 1,\ X' \cap Y' = \emptyset,\ \text{and}\ X' \cup Y' = [1, 2n + 1].$$

Finally, we add $2n + 2$ to cell $(2, 2)$. □

Corollary 11. $C(n_1, n_2, m, r_1, r_2)$ with $n_2 < n_1 < 2n_2$, $r_1 \geqslant 2$, $r_2 = 2$, is not WSNB under MI if $m < 2n_2$.

Theorem 12. $C(n_1, n_2, m, r_1, r_2)$ with $n_1 > n_2$, $\min\{r_1, r_2\} \geqslant 2$, is WSNB under MI if and only if $m \geqslant \min\{n_1 + n_2 - 1, r_2 n_2\}$.

Proof. The "if" part is trivial. To prove the "only if" part, it suffices to show for $r_1 = 2$.

Case 1: $n_1 \leqslant (r_2 - 1)n_2$, assume $n_1 = pn_2 + q, 0 \leqslant q < n_2$. Clearly, we can reach the state

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\ldots$	$[x, n_1 - n_2]$		

where $x = (p - 2)n_2 + 1$, if $q = 0$; $x = (p - 1)n_2 + 1$, if $q \neq 0$.

We can also move $[1, n_1 - n_2]$ from first row to second row by moving cell by cell in the order from left to right.

Our focus is actually on the last two columns, i.e., the 2×2 submatrix M. The Set $[1, n_1 - n_2]$ in the first $p - 1$ or p columns serves the sole purpose that all entries in M are larger than $n_1 - n_2$. This is achieved by moving the set $[1, n_1 - n_2]$ to the row where entries are to be added in M. The entries are added according to the proof of Lemma 6. Hence, eventually, we reach the state

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\ldots$	$[x, n_1 - n_2]$	$\ldots$	X
					Y

$|X| = |Y| = n_2 - 1$, $X \cap Y = \emptyset$, and $X \cup Y = [n_1 - n_2 + 1, n_1 + n_2 - 2]$.

Finally, add $n_1 + n_2 - 1$ to cell $(1, r_2)$.

Case 2: $(r_2 - 1)n_2 < n_1 < r_2 n_2$, which implies $(n_1 + n_2 - 1) \geqslant r_2 n_2$.

Clearly, we can reach the state

$[1, n_2]$	$\ldots$	$[(r_2 - 3)n_2 + 1, (r_2 - 2)n_2]$		

Similar to Case 1, we can reach the state

$[1, n_2]$	$\ldots$	$[(r_2 - 3)n_2 + 1, (r_2 - 2)n_2]$	X
			Y

$|X| = n_2, |Y| = n_2 - 1$, $X \cap Y = \emptyset$, and $X \cup Y = [(r_2 - 2)n_2 + 1, r_2 n_2 - 1]$.

Finally, add $r_2 n_2$ to cell $(1, r_2)$.

Case 3: $r_2 n_2 \leqslant n_1$. This is a trivial case with $m = r_2 n_2$. $\quad \square$

Finally, we study the packing and STU strategies. Let X_{ij} denote the set of connections from I_i to O_j. We first prove

Lemma 13. *Suppose* $n_1 \geqslant n_2$. *Then* $|X_{11} \cup X_{22}| \leqslant n_2, |X_{12} \cup X_{21}| \leqslant n_2$.

Proof. Suppose not, say, $|X_{11} \cup X_{22}| = n_2 + 1$. Let y denote the $(n_2 + 1)$st middle switch added to cell $(1, 1)$ or cell $(2, 2)$. Without loss of generality, assume y is added to cell $(2, 2)$. Then $X_{11}/X_{22} = \emptyset$ since otherwise, the (I_2, O_2) connection should be routed through a middle crossbar in X_{11}/X_{22} by the packing strategy. Therefore

$$X_{11} \cup X_{22} = X_{22}$$

and

$$|X_{11} \cup X_{22}| \leqslant n_2 - 1$$

since cell $(2,2)$ can have atmost n_2 connections, including y, contradicting the assumption that y is the $(n_2 + 1)$st middle switch in $X_{11} \cup X_{22}$.

Similarly, we can prove $|X_{12} \cup X_{21}| \leqslant n_2$. $\square$

Theorem 14. *Suppose $n_1 \geqslant n_2$. Then $C(n_1, 2, m, n_2, 2)$ is wide-sense non-blocking under the packing or the STU strategy if and only if $m \geqslant \min\{2n_2, n_2 + \lfloor n_1/2 \rfloor\}$.*

Proof. Suppose $n_1 \geqslant 2n_2$. Consider $2n_2$ connections for an input switch. They must be routed through $2n_2$ distinct middle switches. On the other hand, there are atmost $2n_2$ connections, hence $2n_2$ middle switches suffice. Next suppose $n_1 \leqslant 2n_2$.

Necessity.

$$\frac{[1, n_2]}{\qquad \Big| \quad [1, n_2]} \rightarrow \frac{[1, n_2 - \lfloor n_1/2 \rfloor]}{\qquad \Big| \quad [n_2 - \lfloor n_1/2 \rfloor] + 1, n_2]} \rightarrow$$

$$\frac{[1, n_2 - \lfloor n_1/2 \rfloor]}{[n_2 + 1, n_2 + \lfloor n_1/2 \rfloor] \,\Big|\, [n_2 - \lfloor n_1/2 \rfloor] + 1, n_2]}$$

The last state has $n_2 - \lfloor n_1/2 \rfloor + \lfloor n_1/2 \rfloor + \lfloor n_1/2 \rfloor = n_2 + \lfloor n_1/2 \rfloor$ elements.

Sufficiency. Supposed, to the contrary, that there exists a state such that a new request under the packing strategy will force the use of an idle middle crossbar y which will be the $(n_2 + \lfloor n_1/2 \rfloor + 1)$st middle crossbar in use. Without loss of generality, assume y is in cell $(2,2)$. Then by an argument analogous to the one used in proving Lemma 13, $X_{11} \subseteq X_{22}$ in that state. Therefore

$$X_{11} \cup X_{12} \cup X_{21} \cup X_{22} = X_{12} \cup X_{21} \cup X_{22}.$$

Further

$$|X_{12} \cup X_{21}| \leqslant n_2, \qquad \text{(by Lemma 13)}$$

$$|X_{12} \cup X_{22}| \leqslant n_2,$$

$$|X_{21} \cup X_{22}| \leqslant n_1.$$

Hence

$$|X_{12} \cup X_{21} \cup X_{22}| \leqslant (n_2 + n_2 + n_1)/2, \text{ or}$$

$$|X_{12} \cup X_{21} \cup X_{22}| \leqslant n_2 + \lfloor n_1/2 \rfloor. \qquad \square$$

Note that the proof of sufficiency is simpler than Beneš original proof for the symmetric network.

Theorem 15. *Suppose $n_1 \geqslant n_2$ and $\max\{r_1, r_2\} \geqslant 3$. Then $C(n_1, r_1, m, n_2, r_2)$ is wide-sense nonblocking if and only if $m \geqslant \min\{r_2 n_2, n_1 + n_2 - 1\}$.*

Proof. The "if" part is trivial since the condition already guarantees strict nonblockingness by an extension of Clos result [3] to the asymmetric case. We now prove the "only if" part. If $n_1 \geqslant r_2 n_2$, then trivially, $m \geqslant r_2 n_2$ is necessary. Therefore we assume $n_1 < r_2 n_2$.

   Case (i): $r_2 = 2, \ r_1 \geqslant 3$.

$$
\begin{array}{|c|c|}
\hline
[1, n_2] & \\
\hline
& n_2 \\
\hline
& \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
[1, n_2 - 1] & \\
\hline
n_2 + 1 & n_2 \\
\hline
& \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
[1, n_2 - 1] & \\
\hline
n_2 + 1 & n_2 \\
\hline
& n_2 + 1 \\
\hline
\end{array}
\rightarrow
$$

$$
\begin{array}{|c|c|}
\hline
[1, n_2 - 1], n_2 + 1 & \\
\hline
& n_2 \\
\hline
& n_2 + 1 \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
[1, n_2 - 1], n_2 + 1 & \\
\hline
& [n_2, n_2 + 1] \\
\hline
& \\
\hline
\end{array}
\rightarrow
$$

$$
\begin{array}{|c|c|}
\hline
[1, n_2 - 1] & \\
\hline
n_2 + 2 & [n_2, n_2 + 1] \\
\hline
& \\
\hline
\end{array}
$$

Repeat such an operation, eventually we obtain

$$
\begin{array}{|c|c|}
\hline
[1, n_2 - 1] & \\
\hline
& [n_2, n_2 + \lfloor n_1/2 \rfloor - 1] \\
\hline
& \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
[1, n_2 - 1] & \\
\hline
n_2 + \lfloor n_1/2 \rfloor & [n_2, n_2 + \lfloor n_1/2 \rfloor - 1] \\
\hline
& \\
\hline
\end{array}
$$

   Case (ii): $r_2 \geqslant 3, r_1 = 2$.
First,

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
[1, n_2] & [n_2 + 1, 2n_2] & \cdots & [(r_2 - 3)n_2 + 1, (r_2 - 2)n_2 - 1] & [(r_2 - 2)n_2, (r_2 - 1)n_2 - 2] & \\
\hline
& & & & & \\
\hline
\end{array}
$$

now, consider the last three columns,
define $A = [(r_2 - 3)n_2 + 1, (r_2 - 2)n_2 - 1], B = [(r_2 - 2)n_2, (r_2 - 1)n_2 - 1]$

$$
\begin{array}{|c|c|}
\hline
A & B, (r_2 - 1)n_2 - 1 \\
\hline
& \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
A & B, (r_2 - 1)n_2 - 1 \\
\hline
& (r_2 - 1)n_2 - 1 \\
\hline
\end{array}
$$

$$
\rightarrow
\begin{array}{|c|c|}
\hline
A & B \quad (r_2 - 1)n_2 \\
\hline
& (r_2 - 1)n_2 - 1 \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
A & B, (r_2 - 1)n_2 \\
\hline
(r_2 - 1)n_2 & (r_2 - 1)n_2 - 1 \\
\hline
\end{array}
$$

$$
\rightarrow
\begin{array}{|c|c|}
\hline
A & B \\
\hline
(r_2 - 1)n_2 & (r_2 - 1)n_2 - 1 \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|}
\hline
A & B, (r_2 - 1)n_2 \\
\hline
& (r_2 - 1)n_2 - 1 \\
\hline
\end{array}
$$

$$
\rightarrow
\begin{array}{|c|c|}
\hline
A & B \\
\hline
& (r_2 - 1)n_2 - 1, (r_2 - 1)n_2 \\
\hline
\end{array}
\rightarrow \cdots \rightarrow
\begin{array}{|c|c|}
\hline
A & B \\
\hline
& [(r_2 - 1)n_2 - 1, r_2 n_2 - 3] \\
\hline
\end{array}
$$

define $C = [(r_2-1)n_2-1, r_2n_2-3] \rightarrow$

A	B	$r_2n_2 - 2$
		C

$\rightarrow$

A	B	$r_2n_2 - 2$
	$r_2n_2 - 2$	C

$\rightarrow$

$A, r_2n_2 - 2$	B	
	$r_2n_2 - 2$	C

$\rightarrow$

$A, r_2n_2 - 2$	B	$r_2n_2 - 1$
		C

$\rightarrow$

$A, r_2n_2 - 2$	B	
	$r_2n_2 - 1$	C

$\rightarrow$

A	B	
	$r_2n_2 - 1$	$C, r_2n_2 - 2$

$\rightarrow$

$A, r_2n_2 - 1$	B	
		$C, r_2n_2 - 2$

$\rightarrow$

$A, r_2n_2 - 1$	$B, r_2n_2 - 2$	
		C

$\rightarrow$

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\cdots$	$A, r_2n_2 - 1$	$B, r_2n_2 - 2$	r_2n_2
					C

Case (2): $n_1 \leqslant (r_2 - 1)n_2$.
First,

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\cdots$	$[n_1 - 2n_2 + 1, n_1 - n_2 - 1]$	$[n_1 - n_2, n_1 - 2]$	

also, consider the last three columns,
  define, $A = [n_1 - 2n_2 + 1, n_1 - n_2 - 1], B = [n_1 - n_2, n_1 - 2]$
  similarly, we can get the following state

A	B	
		$[(n_1 - 1, n_1 + n_2 - 3]$

define $C = [(n_1 - 1, n_1 + n_2 - 3]$

$\rightarrow$

A	B	$n_1 + n_2 - 2$
		C

$\rightarrow$

A	B	$n_1 + n_2 - 2$
	$n_1 + n_2 - 2$	C

$\rightarrow$

A	$B, n_1 + n_2 - 2$	
		C

$\rightarrow$

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\cdots$	A	$B, n_1 + n_2 - 2$	$n_1 + n_2 - 1$
					C

Case (iii): $r_1 \geqslant 3, r_2 \geqslant 3$.
Let $p = \lfloor n_1/n_2 \rfloor$. Then $n_2 \leqslant n_1 < r_2n_2$ implies $1 \leqslant p < r_2$. It suffices to prove the state

$\mathscr{S} = $

$[1, n_2]$	$[n_2 + 1, 2n_2]$	$\cdots$	$[(p - 2)n_2 + 1, (p - 1)n_2]$	

can be reached since we can use Case (i) for the remaining $r_1 \times (r_2 - p + 1)$ array with $n_2' \leqslant n_1 < 2n_2'$. Hence the total number of middle crossbars used is $(p - 1)n_2 + (n_1' + n_2 - 1) = n_1 + n_2 - 1$.

$$\underline{[1, n_2]} \;\Big| \quad \longrightarrow \quad \underline{[1, n_2]} \;\Big|\; \underline{[n_2 + 1, 2n_2]} \;\Big| \quad \longrightarrow$$

$$\underline{[1, n_2]} \;\Big|\; \underline{[n_2 + 1, 2n_2]} \;\Big|\; \underline{[2n_2 + 1, 3n_2]} \;\Big| \quad \longrightarrow \cdots \longrightarrow \mathscr{S} \qquad \square$$

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