# Path Planning for CNC Contouring around a Corner*

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Corner motion is a common and important case in contouring applications. If the corner error is well-controlled, a machine can own maximal throughput while maintaining the machining accuracy. In this paper, we investigated corner error yielded by different Acc/Dec processing in the command generation phase. At first, we derive some analytical forms of corner error induced by Acc/Dec processing; both linear and FIR Acc/Dec types are studied. Then a simple method of controlling the corner error without changing the Acc/Dec structure is presented. The method is simply to add some delay time to retard the execution of the second linear command. Although some results here do not explicitly present their final exact analytic expressions, their numerical solutions can be easily solved.

***Key Words***: Path Planning, Corner Error, Corner Motion

## 1. Introduction

A sharp corner is formed by two consecutive linear contours with discontinuity in their first derivatives. In general, these two linear contours are fed into the Acc/Dec (feedrate planning) processor one after the other (block by block in motion command) to avoid undesirable vibration and obtain a smoother and faster motion(1). But the Acc/Dec mechanism has a side effect that the corner error will be induced. The corner error occurs because the second linear command starts while the motion of first linear command is still on its decelerating movement; that is to say, the first move blends into the second. In order to meet the requirement of machining accuracy, a method to control the magnitude of corner error becomes inevitable.

If the servo loop is assumed to be well-tuned so that the effect of the servo lag can be ignored(2), the Acc/Dec mechanism becomes the only source of the corner error. This error becomes zero if the machine tool is operating in exact stop mode. However, most machine tools have another operation mode, which is called blended move mode. In this mode, the machine always starts to execute the second command before the first one is completed executed, and as a result, corner error occurs. Despite of this side effect, the blended move mode is still necessary because it shortens machining time and/or meets some operation conditions, e.g., to maintain a minimum tangential velocity. In addition, the blended move mode is especially common for some manufacturing processes, such as sealing, welding and laser cutting. In these operations as mentioned, the cutters are not allowed to dwell at any point to assure the accuracy of a work piece.

In order to reduce or confine the corner error to specific range, researchers proposed several methods from different view points in previous studies. The corner trajectory with clothoide proposed in Ref.(3) is one approach of the path planning view point. This approach allows a continuous increase or decrease in the bend radius during the motion from one straight line to another, so that there is no abrupt change in speed. Tomizuka used a differential equation and preview information for the motion along curves with corners in 1991(4). Nevertheless, these two methods with complex mathematical formulations distress the users because of their computation time. In addition

* Received 5th March, 2003 (No. 03-5026)
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to the path planning view point, the servo control algorithm also provides another approach to overcome the corner error, e.g., Weck and Ye used the IKF control strategy in sharp corner tracking[5]. The cross-coupling control (CCC) is used for contour tracking[6]–[8] and in corner contour case[7]. However, when CCC is encountering a corner, the abrupt change of error could cause a large control force which exceeds the deceleration capability of one servo control loop. An overshoot generally occurs at this corner. This indicates that CCC should be employed in combination with feedrate scheduling to overcome this problem. Besides, both IKF and CCC are complicated control techniques in which their stability problems must be considered carefully.

The main purpose of this paper is to analyze and confine the corner error caused by the conventional linear Acc/Dec scheme and the advanced filter type as the one proposed in Ref.(1). A possible way to reduce the corner error is to change the Acc/Dec structure, such as lengthening the Acc/Dec time. Since the CNC machine tool must perform various operations and contouring, once the Acc/Dec structure is designed it is expected to be fixed. Therefore, we will assume the fixed structure of Acc/Dec in our study.

Sessions are arranged as follows. In session two, the corner error is defined here for later discussion. In session three, the linear Acc/Dec is discussed. In this session, we also show that corner error can be reduced by delay command method. In session four, the FIR Acc/Dec is studied in details, including its compensation for corner error by delay command method. A simulation result is presented to demonstrate the effectiveness of the proposed technique. At last, we make some conclusions in session five.

## 2. Definition of Corner Error

Figure 1 illustrates a common scheme used by machine tools for command generation. The Acc/Dec processor can be regarded as a post-processor for the interpolator. The commands after Acc/Dec processor are capable of producing a smooth motion profile to avoid shock and vibration, so that higher motion speed or improved motion accuracy are achieved. If servo lag problem is ignored, the response will be identical to the command after Acc/Dec processor. In this case, the source of corner error is only coming from the Acc/Dec. Referring to Fig. 1, take a linear command in bi-axis motion for example; the commands generated before and after linear Acc/Dec processors are illustrated in Fig. 2. If the blended move mode is applied in corner motion as shown in Fig. 3, the corner error will exist due to the Acc/Dec. This kind of error is shown in Fig. 4(a). In Fig. 4(b), the trajectory in the $x$-$y$ frame is transformed into a local $a$-$b$ frame. On this frame, the corner error can be defined and analyzed conveniently. In this figure, the $b$-axis divides equally the angle formed by two linear contours. The $a$-axis goes through the corner and is perpendicular to the $b$-axis. Positional point on the trajectory is represented as $(S_a, S_b)$, which is function of time. Define the corner error as

$$\varepsilon=\sqrt{(S_a^*)^2+(S_b^*)^2}=\min\sqrt{(S_a)^2+(S_b)^2} \qquad (1)$$

where $(S_a^*, S_b^*)$ is the point on the trajectory nearest to the origin of local $a$-$b$ frame. The trajectory can be obtained by integrating the velocity command after

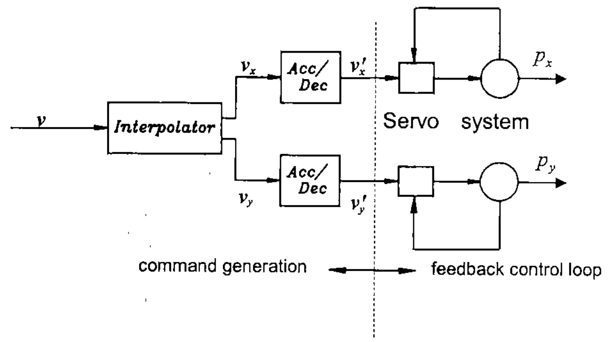


Fig. 1 Layout of the interpolator, Acc/Dec processor and servo system

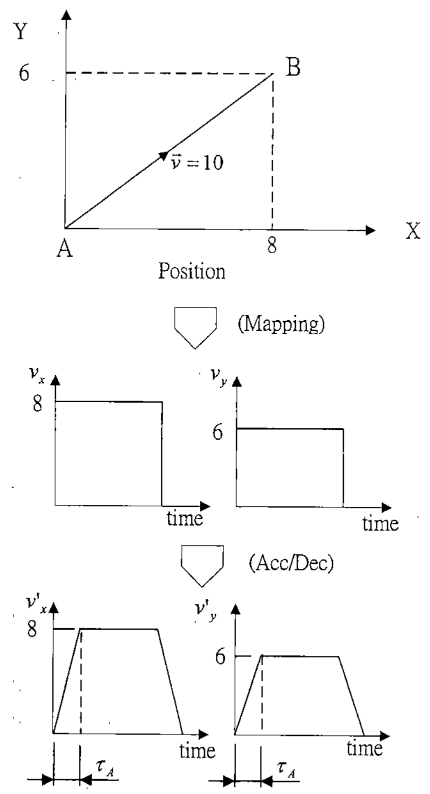


Fig. 2 Command generations before and after the linear Acc/Dec for a linear contour from point A to B

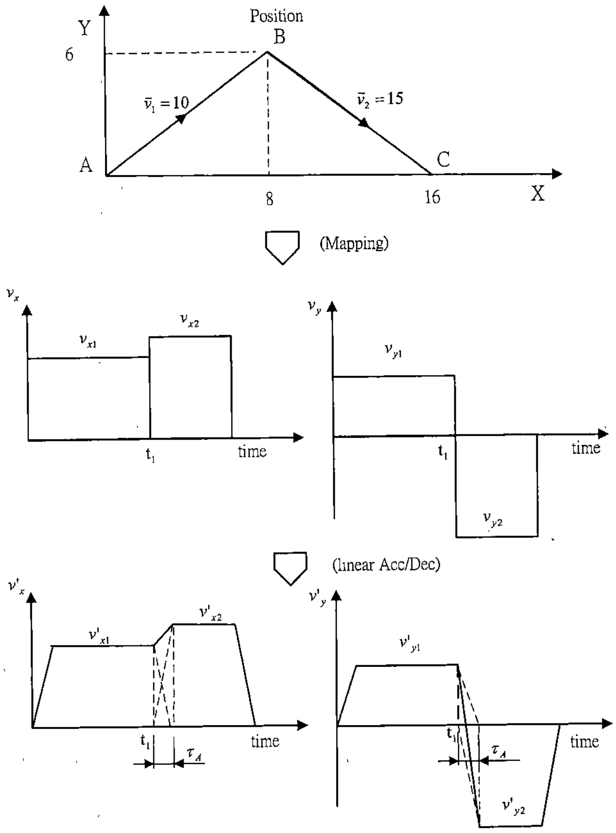


Fig. 3 Example of command generations for the blended mode (for linear Acc/Dec) in corner motion from point A through B to C

Acc/Dec for both axes. By using Eq.(1), we can analyze the corner error due to Acc/Dec mechanism. Let $v_a$ and $v_b$ be the components of instantaneous tangential velocity of command on the $a-b$ frame. At the moment when corner error occurs, the following condition should be satisfied.

$$(v_a, v_b)\cdot(S_a, S_b)=0 \qquad (2)$$

When the $a-b$ frame is chosen as a referenced frame, the velocity commands along the $a$-axis are always positive and commands along $b$-axis will first be negative then become positive, as shown in Fig. 5. When the feedrates of two linear commands are nearly equal, the corner error will happen around the point where $v_b=0$ as will be discussed later.

In the sequel, we will focus on studying corner error due to Acc/Dec processing. The method proposed to reduce the corner error is simply to delay the starting time of the second command. Two different blended velocity profiles are shown in Fig. 6 (a) and (b) for comparison, where one has no delay and the other has delay. As a lower velocity can be obtained due to the delay time (in Fig. 6(b)), the value of corner error will become smaller for delay command. Hereafter all related figures only show the blended portion of velocity profiles which are our

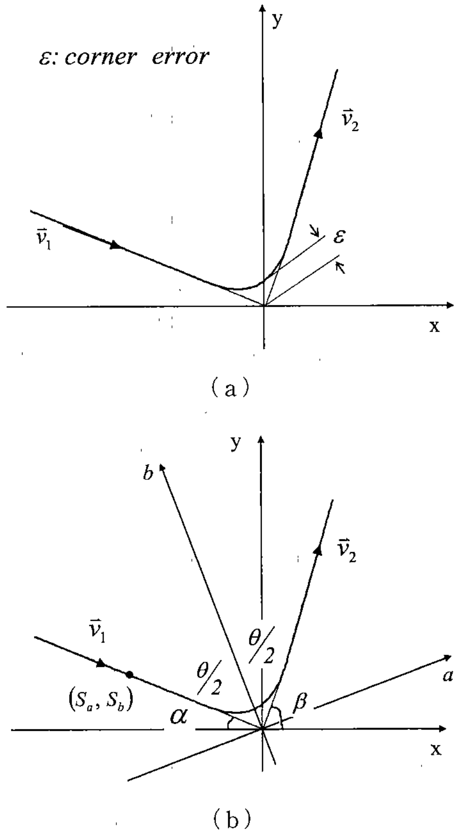


Fig. 4 Corner error and coordinate transformation (a) corner error definition, (b) corner trajectory in different coordinates

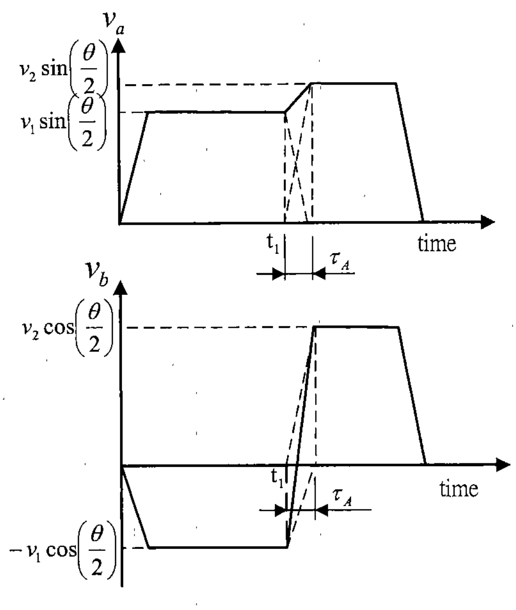


Fig. 5 Blended command analysis on the $a-b$ frame for the case in Fig. 3

concerns.

## 3. Corner Error Analysis for Linear Acc/Dec

Although Acc/Dec operation is digitally im-

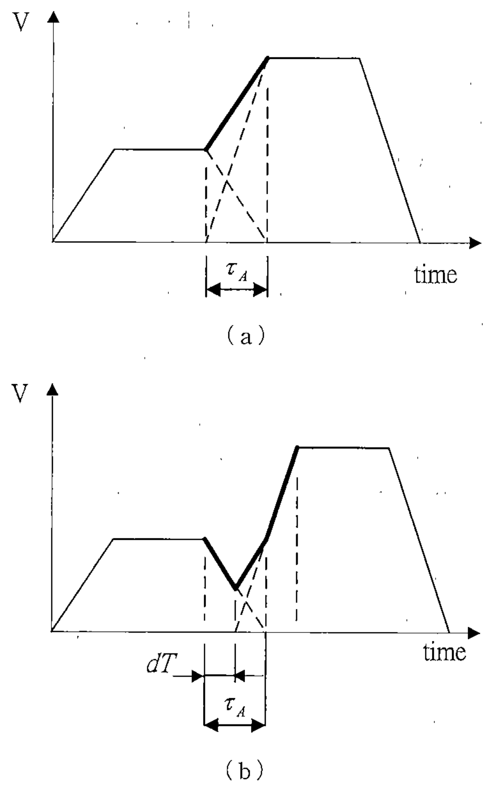


Fig. 6 Velocity profiles after the Acc/Dec for (a) blended commands, (b) blended commands with delay time $dT$

plemented, for the linear Acc/Dec case, we will perform continuous-time analysis for ease of presentation.

### 3.1 Linear Acc/Dec without delay

In most applications of machining, the feedrates of two linear commands are identical, i.e., $v_1=v_2=v_0$. Let the two velocity components of both $v_1$ and $v_2$ mapped onto the $a$-$b$ frame be $(v_{1a}, v_{2a})$ and $(v_{1b}, v_{2b})$, respectively. Figure 7(a) and (b) show the velocity component profiles in both $a$ and $b$ directions, where the initial speeds are

$$v_{a0}=v_0\cdot\sin\left(\frac{\theta}{2}\right) \tag{3.a}$$

$$v_{b0}=v_0\cdot\cos\left(\frac{\theta}{2}\right) \tag{3.b}$$

Since the moment when corner error occurs is during the period of Acc/Dec processing, we can set the starting time of Acc/Dec as $t=0$, and the initial position viewed from $a$-$b$ frame is

$$(S_a, S_b)=\left(-\frac{1}{2}v_{a0}\cdot\tau_A, \frac{1}{2}v_{b0}\cdot\tau_A\right) \tag{4}$$

The resultant velocity and trajectory along the $a$ and $b$ directions during the period of Acc/Dec processing can be described as follows.

$$v_a(t)=v_{1a}+v_{2a}=v_0\cdot\sin\left(\frac{\theta}{2}\right),\quad 0\le t\le\tau_A \tag{5.a}$$

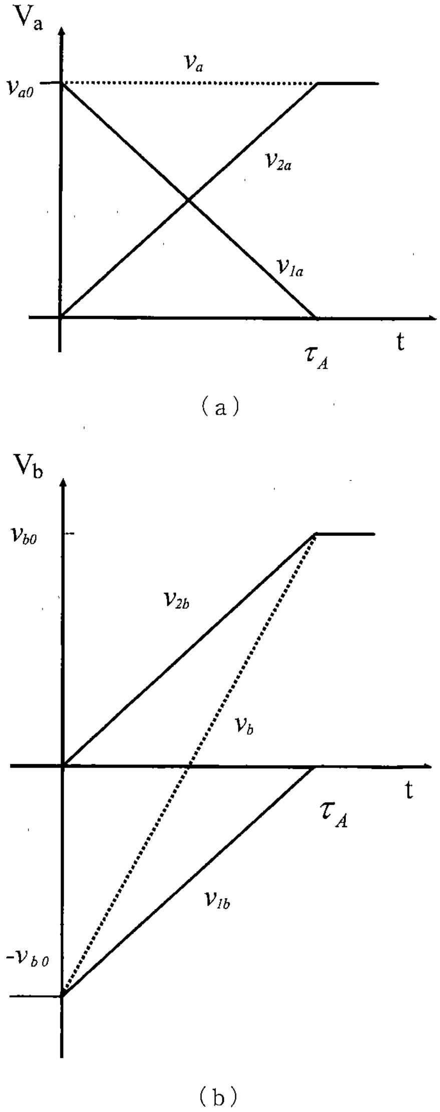


Fig. 7 The velocity components and resultant velocity along the $a$ and $b$ directions in blended portion, (a) along the $a$ direction, (b) along the $b$ direction

$$v_b(t)=v_{1b}+v_{2b}=v_0\cdot\left(\frac{2}{\tau_A}t-1\right)\cdot\cos\left(\frac{\theta}{2}\right),\quad 0\le t\le\tau_A \tag{5.b}$$

$$S_a(t)=\frac{v_0}{2}\cdot(2t-\tau_A)\cdot\sin\left(\frac{\theta}{2}\right),\quad 0\le t\le\tau_A \tag{5.c}$$

$$S_b(t)=\frac{v_0}{2}\cdot\left(\frac{2}{\tau_A}t^2-2t+\tau_A\right)\cdot\cos\left(\frac{\theta}{2}\right),\quad 0\le t\le\tau_A \tag{5.d}$$

In this case the resultant velocity along the $a$-axis is constant as seen from Eq.(5.a), and along the $b$-axis is at a constant acceleration as seen from Eq.(5.b). They are plotted in dotted lines in Fig. 7(a) and (b). Thus, the trajectory is similar to that of a projectile. The velocity profile is symmetric to the $b$-axis, so does the trajectory. The corner error defined by Eq.(1) will occur when the resultant velocity in the $b$ direction is equal to zero. This implies that the corner error occurs at $t=\tau_A/2$. Therefore, corner error can be expressed in a simple analytic form, or

$$\varepsilon=\frac{v_0\tau_A}{4}\cdot\cos\left(\frac{\theta}{2}\right) \quad (6)$$

If the feedrates of two consecutive linear commands are different, i.e., $v_2=c\cdot v_1=c\cdot v_0$, where $c$ is a constant factor, the velocity and position equations of this new trajectory can be derived in a similar way, and they are shown as follows.

$$v_a(t)=v_0\left(\frac{c-1}{\tau_A}t+1\right)\cdot\sin\left(\frac{\theta}{2}\right),\quad 0\leq t\leq\tau_A \quad (7.a)$$

$$v_b(t)=v_0\left(\frac{c+1}{\tau_A}t-1\right)\cdot\cos\left(\frac{\theta}{2}\right),\quad 0\leq t\leq\tau_A, \quad (7.b)$$

$$S_a(t)=\frac{v_0}{2}\cdot\left(\frac{c-1}{\tau_A}t^2+2t-\tau_A\right)\cdot\sin\left(\frac{\theta}{2}\right),\quad 0\leq t\leq\tau_A \quad (8.a)$$

$$S_b(t)=\frac{v_0}{2}\cdot\left(\frac{c+1}{\tau_A}t^2-2t+\tau_A\right)\cdot\cos\left(\frac{\theta}{2}\right),\quad 0\leq t\leq\tau_A, \quad (8.b)$$

In this case, the profile will not be symmetric to the $b$-axis. The solution of corner error is obtained by substituting Eqs. (8) into Eq. (1) to find the minimum value. If $c$ is close to one, the trajectory is almost symmetric with respect to the $b$-axis, and the corner error can be approximated by Eq. (9),

$$\varepsilon=\frac{v_0\cdot\tau_A}{2}\left(\frac{c}{c+1}\right)\cdot\cos\left(\frac{\theta}{2}\right) \quad (9)$$

It is obtained by letting $v_b(t)=0$ to get the value of $t$; then $t$ is brought into $S_b(t)$ to give $\varepsilon$. The result is quite accurate, for example, if $c=2$, the estimation error by using Eq. (9) is only about 2%. Also Eq. (9) is quite useful in investigating the corner error for different factors.

### 3.2 Linear Acc/Dec with delay

Shown in Fig. 8(a) and (b) are the velocity profiles for a linear Acc/Dec with no delay and command delay in the blended portion along the $a$ and $b$ direction, respectively. The notations are the same as the ones used in Fig. 7(a) and (b). Additional notations $v_{dT2a}$ and $v_{dT2b}$ represent $v_{1a}$ and $v_{2b}$ after delay of $dT$, respectively. If $v_1=v_2=v_0$, the velocity and trajectory in time interval $dT\leq t\leq\tau_A$ which is the period of Acc/Dec processing can be described as follows.

$$v_a(t)=v_0\left(1-\frac{dT}{\tau_A}\right)\cdot\sin\left(\frac{\theta}{2}\right) \quad (10.a)$$

$$v_b(t)=v_0\left(\frac{2}{\tau_A}t-\frac{dT}{\tau_A}-1\right)\cdot\cos\left(\frac{\theta}{2}\right) \quad (10.b)$$

$$S_a(t)=\frac{v_0}{2}\cdot\left[2\left(1-\frac{dT}{\tau_A}\right)t+\frac{dT^2}{\tau_A}-\tau_A\right]\cdot\sin\left(\frac{\theta}{2}\right) \quad (11.a)$$

$$S_b(t)=\frac{v_0}{2}\cdot\left[\frac{2}{\tau_A}t^2-2\left(1+\frac{dT}{\tau_A}\right)t+\frac{dT^2}{\tau_A}+\tau_A\right]\times\cos\left(\frac{\theta}{2}\right) \quad (11.b)$$

Due to the symmetry of velocity profile, the corner error will occur when the resultant velocity in the $b$ direction becomes equal to zero, which implies that the corner error will occur at $t=(\tau_A+dT)/2$. Therefore, the corner error can be formulated in a simple analytic form, or,

$$\varepsilon=\frac{v_0\tau_A}{4}\cdot\cos\left(\frac{\theta}{2}\right)\cdot\left(1-\frac{dT}{\tau_A}\right)^2,\quad dT\leq\tau_A \quad (12)$$

From this equation, it is obvious that the corner error is a function of several factors, including Acc/Dec time constant $\tau_A$, the feedrate $v_0$, the corner angle $\theta$, and time delay $dT$. By comparing Eq. (6) with Eq. (12), we found that by adjusting the amount of time delay we can control the magnitude of corner error. Practically, the delay time $dT$ is chosen as a multiple

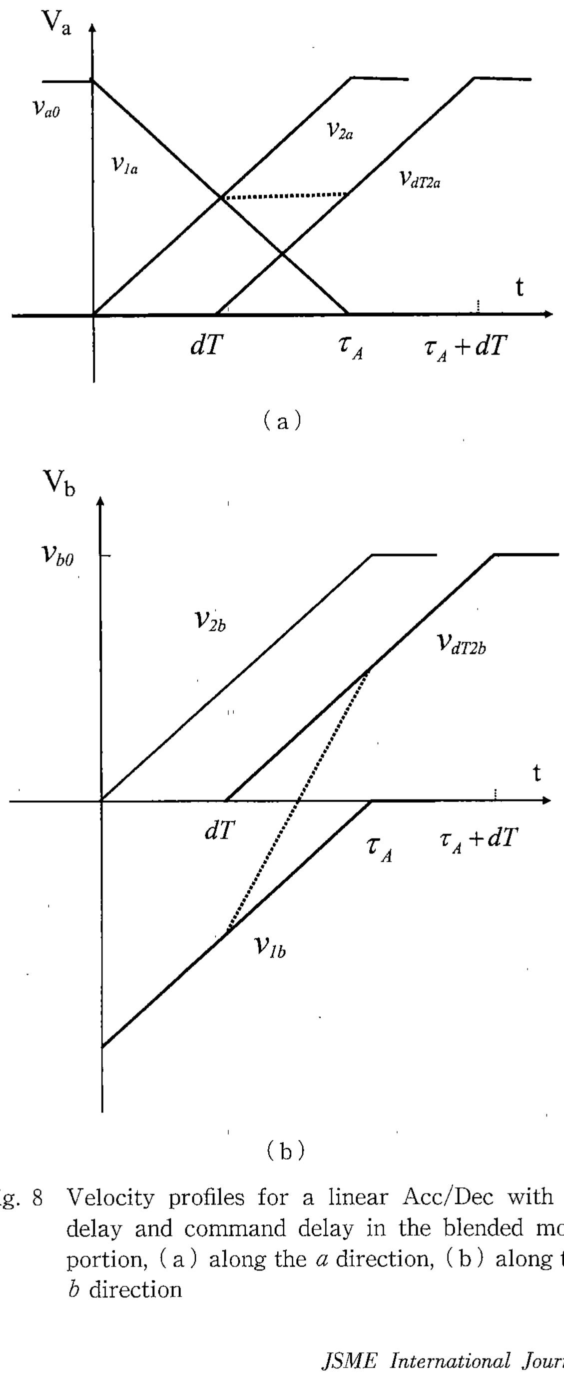


Fig. 8 Velocity profiles for a linear Acc/Dec with no delay and command delay in the blended mode portion, (a) along the $a$ direction, (b) along the $b$ direction

of sample period $T$ since most of the Acc/Dec processors are implemented digitally.

For the more general contouring case, i.e., $v_2=c\cdot v_1=c\cdot v_0$, the velocity and trajectory equations in the $a$- and $b$-axes can be derived as follows.

$$v_a(t)=v_0\cdot\left(\frac{c-1}{\tau_A}t+1-\frac{c\cdot dT}{\tau_A}\right)\cdot\cos\left(\frac{\theta}{2}\right) \quad (13.a)$$

$$v_b(t)=v_0\cdot\left(\frac{c+1}{\tau_A}t-1-\frac{c\cdot dT}{\tau_A}\right)\cdot\cos\left(\frac{\theta}{2}\right) \quad (13.b)$$

$$S_a(t)=\frac{v_0}{2}\cdot\left[\frac{c-1}{\tau_A}t^2+2\left(1-\frac{c\cdot dT}{\tau_A}\right)t+\frac{c\cdot dT^2}{\tau_A}-\tau_A\right]\cdot\sin\left(\frac{\theta}{2}\right) \quad (14.a)$$

$$S_b(t)=\frac{v_0}{2}\cdot\left[\frac{c+1}{\tau_A}t^2-2\left(1+\frac{c\cdot dT}{\tau_A}\right)t+\frac{c\cdot dT^2}{\tau_A}+\tau_A\right]\cdot\cos\left(\frac{\theta}{2}\right) \quad (14.b)$$

where $dT\leq t\leq \tau_A$. The corner error can be obtained by substituting Eq.(14) into Eq.(1) to find the minimum value. If $c$ is close to one, we can let $v_b(t)=0$ to find the occurring time,

$$t=\frac{\tau_A+c\cdot dT}{c+1} \quad (15)$$

It is near the moment when corner error happens. By substituting it into Eq.(14.b) we obtain the approximation of corner error for the most general case of linear Acc/Dec processing.

Table 1 Results of various corner errors for linear Acc/Dec with different $dT$

| Feedrate ($10^5$ BLU/sec) | | corner error (BLU) | | | |
|---|---|---|---|---|---|
| $v_1$ | $v_2$ | $dT$=0 | $dT$=10$T$ | $dT$=20$T$ | $dT$=41$T$ |
| 5 | 15 | 5462 | 3118 | 1430 | 0 |
| 4 | 12 | 4370 | 2494 | 1144 | 0 |
| 3 | 9 | 3277 | 1871 | 858 | 0 |
| 2 | 6 | 2185 | 1247 | 572 | 0 |
| 1 | 3 | 1092 | 624 | 286 | 0 |

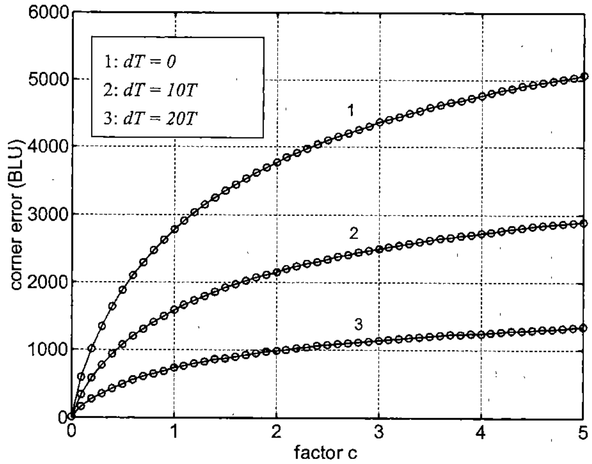


Fig. 9 Change of corner errors for linear Acc/Dec with different factors $c$ and delay time $dT$

Table 1 shows several cases of the corner errors under various velocity combinations and different delay time by linear Acc/Dec operation, where $\tau_A=0.041$ sec, $T=0.001$ sec and the factor $c$ is 3. The results show that the corner error increases proportionally with the increase of velocity. The change of corner errors shown in Fig. 9 are the cases where we set $v_0=4.0\times10^5$ BLU/sec. The BLU represents the basic length unit and is the resolution of a machine. The figure shows that the delay command can truly reduce the corner error and also indicates that corner error is less affected with the increasing of factor $c$, which can also be confirmed from the term $(c/(c+1))$ in Eq.(9).

## 4. Corner Error Analysis for Fir Acc/Dec Scheme

Compared with the conventional Acc/Dec scheme, the FIR Acc/Dec scheme has the advantage that the contour error can be analyzed[1]. The circular contour error caused by the FIR Acc/Dec scheme is also smaller compared with conventional linear Acc/Dec and S-curve Acc/Dec schemes. The systematic design procedure is available and implementation of this Acc/Dec is also as simple as other FIR filters. In the sequel, we will study this FIR Acc/Dec processor and apply it on corner error analysis.

For the case of linear Acc/Dec operation, the derivation for corner error can be manipulated in either continuous or discrete time domain. However, because FIR Acc/Dec processor can not be represented as a continuous time function, we will analyze this case in the discrete time domain. Some definitions and results of the Acc/Dec method proposed in Ref.(1) will be reviewed first for later discussion.

### 4.1 Brief review of FIR Acc/Dec

Let $V[kT]$ represent the velocity command before Acc/Dec, $V'[kT]$ be the current velocity command after the Acc/Dec, and $T$ be the sample period. The $V'[kT]$ is manipulated as a weighted moving average of the recent $N+1$ velocity commands, $V[kT]$, $V[(k-1)T]$, $V[(k-2)T]$, $\cdots$, and $V[(k-N)T]$, and are associated with the appropriate weighted coefficients $b_i$'s ($b_0, b_1, \cdots, b_N$), which are related with the filter we choose. The input-output velocity relationship for the Acc/Dec processor is expressed as

$$V'[kT]=\frac{1}{b_S}\sum_{i=0}^{N}b_i\cdot V[(k-i)T] \quad (16.a)$$

or

$$V'[k]=\frac{1}{b_S}\sum_{i=0}^{N}b_i\cdot V[k-i] \quad (16.b)$$

where $b_S$ is the position convergent factor which is as follows.

$$b_s=\sum_{i=0}^{N} b_i \tag{17}$$

After taking $Z$-transformation of Eq.(16.b), the behavior of the Acc/Dec processor is equivalent to a dynamic system of order $N$ as follows,

$$D(z)\equiv\frac{V'(z)}{V(z)}=\frac{1}{b_s}\sum_{i=0}^{N} b_i\cdot z^{-i} \tag{18}$$

where $z^{-1}$ is a unit delay operator, and $V'(z)$ and $V(z)$ are the $Z$-transformation of $V'[k]$ and $V[k]$, respectively.

In this Acc/Dec design, besides the convergence of travel distance, two more conditions must be considered[1]. First, all the weighted coefficients $b_i$'s should be semi-positive (i.e., $b_i\geqq 0$) ; otherwise, the velocity curve through the Acc/Dec processor will become oscillatory. The second condition is that the final velocity profile must not cause system to violate some system limitations. From Eq. (16.b), the acceleration of each axis can be derived as

$$a'[k]=V[k]\cdot\frac{b_{k-1}}{T\cdot b_s}\quad 1\leq k\leq N+1 \tag{19}$$

The system limitations are following : (a) the velocity $V'[k]$ must be less than the velocity limitation $V_{Max}$ ; (b) the force $F[k]=ma'[k]+cV'[k]$ less than the force limitation $F_{Max}$ ; and (c) the power $P[k]=F[k]\cdot V'[k]$ also less than the power limitation $P_{Max}$, where $m$ is the equivalent mass of load and $c$ is the equivalent viscous damping of the controlled plant.

**4.2 FIR Acc/Dec without delay**

Similar to the linear Acc/Dec case, we first consider the condition that the feedrates of two consecutive linear commands are identical, i.e., $v_1=v_2=v_0$. Let $v_{1a}[k]$, $v_{2a}[k]$ and $v_{1b}[k]$, $v_{2b}[k]$ represent the velocity components for the FIR Acc/Dec along the $a$ and $b$ directions, respectively ; $v_a[k]$ and $v_b[k]$ are their resultant speeds. The index $k$ represents the time variable in digital domain. Let the Acc/Dec processing starts from $k=0$, and its corresponding initial position would be

$$(S_a, S_b)=\left[-\frac{1}{2}v_1\cdot N\cdot T\cdot\sin\left(\frac{\theta}{2}\right),\ \frac{1}{2}v_1\cdot N\cdot T\cdot\cos\left(\frac{\theta}{2}\right)\right] \tag{20}$$

Because the velocity profile is symmetric with respect to the $b$-axis, the trajectory is also symmetric to the $b$-axis. The corner error will occur when the velocity in the $b$ direction is equal to zero. This simply implies that the corner error occurs at $k=N/2$ when $N$ is even, or $k=(N+1)/2$ when $N$ is odd. For ease of discussion, let us assume that $N$ is even from now on. Because the Acc/Dec processor is a digital filter, the speeds can be expressed as follows.

$$v_{1a}[k]=\frac{1}{b_s}\cdot\sum_{i=k}^{N}\left(b_i\cdot v_0\cdot\sin\left(\frac{\theta}{2}\right)\right)$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\sum_{i=k}^{N} b_i \tag{21.a}$$

$$v_{2a}[k]=\frac{1}{b_s}\cdot\sum_{i=0}^{k-1}\left(b_i\cdot v_0\cdot\sin\left(\frac{\theta}{2}\right)\right)$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\sum_{i=0}^{k-1} b_i \tag{21.b}$$

$$v_{1b}[k]=-\frac{1}{b_s}\cdot\sum_{i=k}^{N}\left(b_i\cdot v_0\cdot\cos\left(\frac{\theta}{2}\right)\right)$$

$$=-v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\sum_{i=k}^{N} b_i \tag{21.c}$$

$$v_{2b}[k]=\frac{1}{b_s}\cdot\sum_{i=0}^{k-1}\left(b_i\cdot v_0\cdot\cos\left(\frac{\theta}{2}\right)\right)$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\sum_{i=0}^{k-1} b_i \tag{21.d}$$

Therefore, the resultant velocities can be derived individually as follows.

$$v_a[k]=v_{1a}[k]+v_{2a}[k]$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\left(\sum_{i=0}^{k-1} b_i+\sum_{i=k}^{N} b_i\right)$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right) \tag{22.a}$$

$$v_b[k]=v_{1b}[k]+v_{2b}[k]$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\left(-\sum_{i=k}^{N} b_i+\sum_{i=0}^{k-1} b_i\right)$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\left(-1+\frac{2}{b_s}\sum_{i=0}^{k-1} b_i\right) \tag{22.b}$$

In this case the resultant velocity along the $a$-axis is constant, and along the $b$-axis is at variable acceleration. The trajectory can be obtained by integrating velocity with time as follows.

$$S_a[k]=-v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k} v_a[k]\cdot T$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot T\left(-\frac{N}{2}+k\right) \tag{23.a}$$

$$S_b[k]=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k} v_b[k]\cdot T$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot T\left(\frac{N}{2}-k+\frac{2}{b_s}\sum_{j=1}^{k}\sum_{i=0}^{j-1} b_i\right) \tag{23.b}$$

The corner error will occur when $k=N/2$, and its value is

$$\varepsilon=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot T\cdot\frac{2}{b_s}\sum_{j=1}^{k}\sum_{i=0}^{j-1} b_i \tag{24}$$

For the case that $v_2=c\cdot v_1=c\cdot v_0$, the similar derivations can be performed and shown as follows.

$$v_a[k]=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\left(\sum_{i=0}^{k-1} b_i+c\cdot\sum_{i=k}^{N} b_i\right)$$

$$=v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\left(c+\frac{(1-c)}{b_s}\cdot\sum_{i=0}^{k-1} b_i\right) \tag{25.a}$$

$$v_b[k]=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{b_s}\cdot\left(-\sum_{i=k}^{N} b_i+c\cdot\sum_{i=0}^{k-1} b_i\right)$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\left(-1+\frac{(c+1)}{b_s}\sum_{i=0}^{k-1} b_i\right) \tag{25.b}$$

$$S_a[k]=-v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k} v_a[k]\cdot T \tag{26.a}$$

$$S_b[k]=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k} v_b[k]\cdot T \tag{26.b}$$

Thus the corner error can be obtained by applying Eq. (26) to Eq. (1) to find the minimum value.

### 4.3 FIR Acc/Dec and its delay-command compensation

The delay command can reduce corner error caused by linear Acc/Dec processor. It also works when dealing with the FIR Acc/Dec processor with symmetric filter coefficients. For this Acc/Dec processor, define the velocity command after Acc/Dec with a delay time $dT$ as

$$v_{dT2a}[k]=\begin{cases} v_{2a}[k-d] & \text{when } k>d \\ 0 & \text{when } k\leq d \end{cases} \quad (27.a)$$

$$v_{dT2b}[k]=\begin{cases} v_{2b}[k-d] & \text{when } k>d \\ 0 & \text{when } k\leq d \end{cases} \quad (27.b)$$

It should be noted that $v_{1a}[k]$ and $v_{1b}[k]$ will become zero if index $k$ is greater than $N+1$. The velocity profiles are shown in Fig. 10(a) and (b). The notations used in Fig. 10(a) and (b) are similar to the ones in Fig. 8(a) and (b). Thus, resultant velocities along $a$- and $b$-axes are

$$v_a[k]=v_{1a}[k]+v_{dT2a}[k] \quad (28.a)$$

$$v_2[k]=v_{1b}[k]+v_{dT2b}[k] \quad (28.b)$$

The trajectory will be

$$S_a[k]=-v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k}v_a[k]\cdot T$$

$$=-v_0\cdot\sin\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k}(v_{1a}[k]+v_{dT2a}[k])\cdot T \quad (29.a)$$

$$S_b[k]=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k}v_b[k]\cdot T$$

$$=v_0\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{N}{2}\cdot T+\sum_{j=1}^{k}(v_{1b}[k]+v_{dT2b}[k])\cdot T \quad (29.b)$$

where $1\leq k\leq N+d+1$. The corner error can be obtained by applying Eq. (29) to Eq. (1) to find the minimum value whether $c$ is one or not.

The corner errors for various cases under the Acc/Dec profile with Blackman filter coefficients are presented in Table 2. Similar to the linear Acc/Dec case, the results show that the corner error increases proportionally with the increase of velocity. To visualize how delay time $dT$ and speed factor $c$ influence corner error, the pictorialization is presented in Fig. 11, where the normalized corner error is defined as $E=\varepsilon/v_0$, $dT$ is a multiple of sample period and $N$ is set 40.

It is also possible to calculate corner errors for various types of S-curve Acc/Dec processors when they are transformed into digital filter forms[1]. Following example demonstrates the performance of different Acc/Dec processors in corner contouring.

**4.3.1 Example** For a high-order S-curve, its

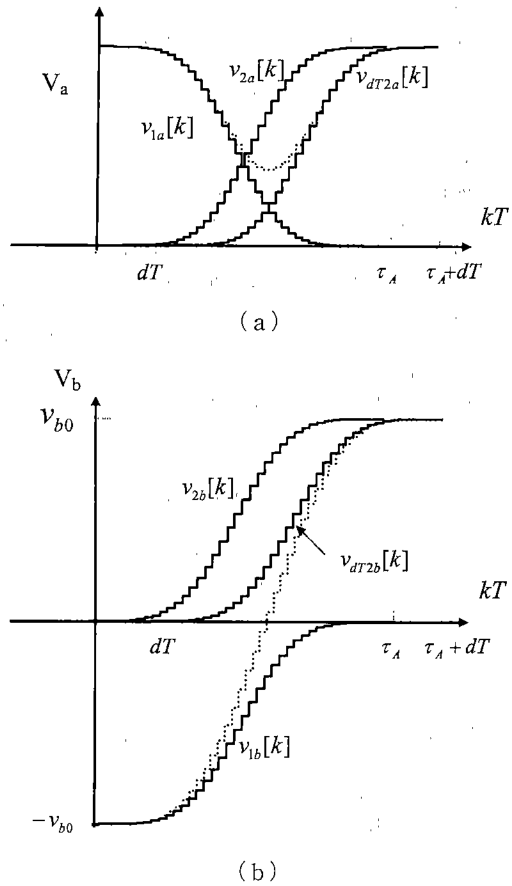


Fig. 10 The velocity profiles for filter-type Acc/Dec with no delay and delay commands, (a) along the $a$ direction, (b) along the $b$ direction

Table 2 Results of various corner errors for FIR Acc/Dec with different $dT$

| Feedrate ($10^5$ BLU/sec) | | corner error (BLU) | | | |
|---|---|---|---|---|---|
| $v_1$ | $v_2$ | $dT$=0 | $dT$=5$T$ | $dT$=15$T$ | $dT$=41$T$ |
| 5 | 15 | 2442 | 1333 | 241 | 0 |
| 4 | 12 | 1953 | 1066 | 193 | 0 |
| 3 | 9 | 1465 | 800 | 145 | 0 |
| 2 | 6 | 977 | 533 | 97 | 0 |
| 1 | 3 | 488 | 267 | 48 | 0 |

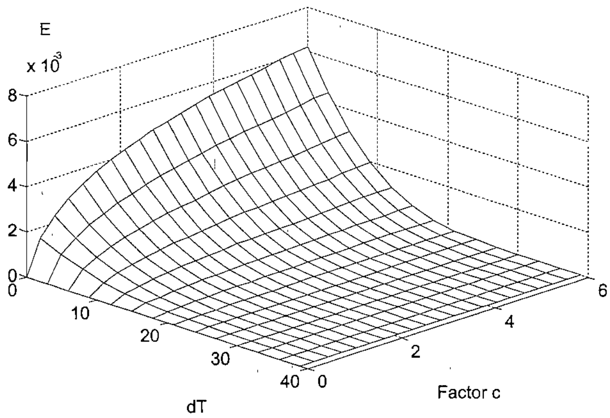


Fig. 11 The normalized corner error $E$ being function of delay time $dT$ and speed factor $c$

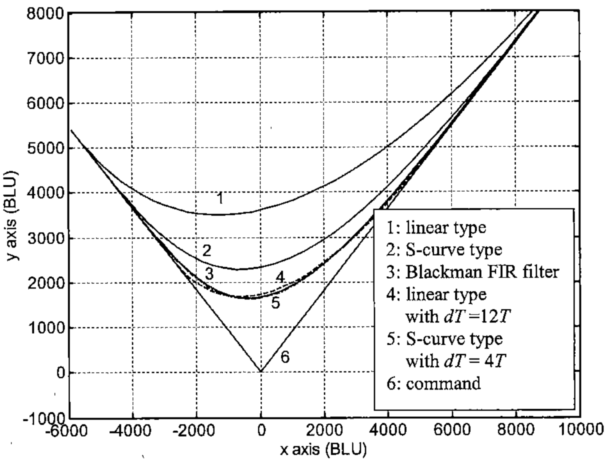


Fig. 12 Comparisons of corner error for different Acc/Dec processors

equation is defined as $s(t)=10t^3-15t^4+6t^5$, where $0\leq t\leq 1$. According to Ref.(1), the normalized velocity and acceleration profile can be respectively expressed in discrete-time as

$$V'[k]=10\left(\frac{k}{N}\right)^3-15\left(\frac{k}{N}\right)^4+6\left(\frac{k}{N}\right)^5,\quad 0\leq k\leq N \tag{30}$$

$$a'[k]=30\left(\frac{k}{N}\right)^2-60\left(\frac{k}{N}\right)^3+30\left(\frac{k}{N}\right)^4,\quad 0\leq k\leq N \tag{31}$$

The normalized velocity is $V[k]=1$, therefore, the coefficients of the FIR filter can be obtained by Eq. (19) or as follows

$$b_{k-1}=a'[k]\cdot b_s\cdot T\quad 1\leq k\leq N+1 \tag{32}$$

Let $N=40$, $c=2$, $dT=0$, $v=6\,000$ mm/sec, and compare the corner errors for three different Acc/Dec processors with same Acc/Dec time constant $\tau_A$, i.e., linear, S-curve and Blackman FIR. In Fig. 12, curve 1, 2 and 3 represent the trajectories after three different Acc/Dec processors, respectively. We can see that the Blackman FIR Acc/Dec has the best performance. To achieve the same corner error as in the FIR Acc/Dec, the other two Acc/Dec processors have to use delay commands. Curve 4 represents trajectory for linear Acc/Dec with a delay command by $12T$, and Curve 5 represents trajectory for S-curve Acc/Dec with delay command by $4T$. It is apparent that the FIR Acc/Dec gives the smallest corner error for the same Acc/Dec time constant, and also provides the shortest machining time for the same corner accuracy.

## 5. Conclusion

In the paper, we discussed the corner error induced in the command generation phase. All Acc/Dec processors induce corner errors, but the FIR Acc/Dec has the smallest one. A strategy is presented by adding a delay time between two successive linear commands to compensate the corner error without altering the Acc/Dec structure. Applying delay command can effectively reduce corner error no matter what kind of Acc/Dec is implemented. The adjustment of delay time depends on what accuracy user desires and what kind of minimum speed the cutting process needs. The simulation confirms the high performance of the FIR Acc/Dec; under the same Acc/Dec time, it has the smallest corner error.

## Acknowledgment

The authors would like to thank the National Science Council of the Republic of China for financial support of this manuscript under Contract No. NSC 88-2212-E-009-033.

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