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Spectral Characterization of Odd Graphs O_k, $k \leq 6$

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Abstract. Let k be an integer with $k \geq 2$. The Odd graph O_k has the $(k-1)$-subsets of $\{1, 2, \ldots, 2k-1\}$ as vertices, and two vertices are adjacent if and only if their corresponding subsets are disjoint. We prove that the odd graphs O_k ($k \leq 6$) are characterized by their spectra among connected regular graphs.

1. Introduction

We shall consider only finite undirected graphs without loops and multiple edges. Now assume Γ is a connected graph with diameter d, let $\Gamma_i(x) = \{y | y \in V(\Gamma)$ and $d(x, y) = i\}$, where $V(\Gamma)$ is the vertex set of Γ and $d(x, y)$ is the distance between vertices x and y. A *distance-regular graph* is one for which the parameters $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ depend not on particular vertices we choose, but only on the distance $i = d(x, y)$ between them. It is clear that $c_0 = a_0 = b_d = 0$, $c_1 = 1$, $a_i = b_0 - b_i - c_i$. The following array

$$\begin{bmatrix} c_0 & c_1 & c_2 & & c_d \\ a_0 & a_1 & a_2 & \ldots & a_d \\ b_0 & b_1 & b_2 & & b_d \end{bmatrix}$$

is called *the intersection array* of Γ.

The adjacency matrix $A(\Gamma)$ of a graph Γ is a square (0, 1) matrix whose rows and columns are indexed by vertices of Γ, and $A(x, y) = 1$ if and only if the vertices x and y are adjacent. The spectrum of A is also called the spectrum of the graph Γ. It is worth mentioning here that the spectrum of a distance-regular graph is determined by its intersection array, refer to [1] and [2, p. 141–143] for details.

Let k be an integer with $k \geq 2$. The Odd graph O_k of characteristic k has the $(k-1)$-subsets of $\{1, 2, \ldots, 2k-1\}$ as vertices, an two vertices are adjacent if and only if their corresponding subsets are disjoint. The small odd graphs are the triangle K_3 ($k = 2$), and the Petersen graph ($k = 3$). In general, the odd graphs O_k are distance-regular graphs of diameter $k-1$ with intersection array.

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & & (k-1)/2 & (k-1)/2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & (k+1)/2 \\ k & k-1 & k-1 & k-2 & k-2 & & (k+1)/2 & 0 \end{bmatrix} \text{ for odd } k, \text{ and}$$

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & & (k/2)-1 & (k/2)-1 & k/2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & k/2 \\ k & k-1 & k-1 & k-2 & k-2 & & (k/2)+1 & (k/2)+1 & 0 \end{bmatrix} \text{ for even } k.$$

The eigenvalues of O_k are the integers $\theta_i = (-1)^i(k-i)$ with multiplicity $m_i = \binom{2k-1}{i} - \binom{2k-1}{i-1}$ for $0 \le i \le k-1$ (refer to [1], or [3]). A. Moon [6, Theorem 3] proved that all distance-regular graphs with the same intersection arrays as those of O_k are isomorphic to O_k. Under a weaker condition, i.e., distance-regularity, this confirmed a conjecture made by N. Biggs [3] that any distance-transitive graph Γ which has the same spectrum as that of O_k is isomorphic to O_k. Based on the above result of A. Moon, we will show in this note that

Main Theorem. *If Γ is a regular connected graphs with the same spectrum as that of O_k, $k \le 6$, then it is isomorphic to O_k.*

The main theorem is proved in Section 3. Its basic idea is to derive all information about $c_i(x, y)$, $a_i(x, y)$ and $b_i(x, y)$, defined in Section 2, for $x, y \in V(\Gamma)$ at distance i, $1 \le i \le k-1$, from $A^j(x, y)$, where $1 \le j \le k-1$, and A is an adjacency matrix of Γ, and then to conclude that Γ must be a distance-regular graph.

Remark.

(1) Cubic lattice graphs [4] and Tetrahedral graphs [5] are characterized by their spectra under extra conditions on the second valency.
(2) We suspect that the cospectral matres of O_k ($k \ge 7$) may exist, though we did not succeed in constructing any one of them yet.

2. Some Notation and Preliminaries

Throughout the rest of this note, we assume that Γ is a connected, regular graph with valency k which has the same spectrum as that of odd graph O_k, i.e., exactly as mentioned in Section 1. Let A be an adjacency matrix of Γ. The following notations are used.

(1) $k_i(x) = |\Gamma_i(x)|$, i.e., the number of all vertices at distance i from $x \in V(\Gamma)$. In particular, $k_1(x) = k$ for all $x \in V(\Gamma)$ by the regularity of Γ.
(2) If x and $y \in V(\Gamma)$ at distance i, define

$$|\Gamma_j(x) \cap \Gamma_1(y)| = \begin{cases} c_1(x, y) & \text{if } j = i-1 \\ a_i(x, y) & \text{if } j = i \\ b_i(x, y) & \text{if } j = i+1 \end{cases}$$

We note that Γ is distance-regular whenever all $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ are functions of $d(x,y) = i$, independent of the choice of x and y, $0 \le i \le k-1$.

In the proof of the main theorem, we need some essential properties of the graph Γ which are reflected on the spectrum of Γ. It is well known that

(1) $A^i(x,y)$ is the number of walks of length i in Γ joining x and y, [2, p. 11], and
(2) $Tr(A^i) = \sum_{j=0}^{k-1} m_j\theta_j^i$ is equal to the number of closed walks in Γ of length i, [7, p. 310]. In particular, $Tr(A^3)$ is equal to six times the number of triangles in Γ. Further,

Lemma 2.1. *If* $d(x,y) = i$, *then*

$$A^{j+1}(x,y) = \sum_{z \in \Gamma_1(y) \cap \Gamma_{i-1}(x)} A^j(x,z) + \sum_{z \in \Gamma_1(y) \cap \Gamma_i(x)} A^j(x,z) + \sum_{z \in \Gamma_1(y) \cap \Gamma_{i+1}(x)} A^j(x,z)$$

Proof. It follows from $A^{j+1}(x,y) = (A^jA)(x,y) = \sum_{z \in \Gamma_1(y)} A^j(x,z)$, and the fact that $\Gamma_j(x) \cap \Gamma_1(y)$ is empty if $j \ne i-1$, i or $i+1$. □

Lemma 2.2.

(1) $A^{2i+1}(x,x) = 0$ *for* $i \le k-2$,
(2) $A^{2i+1-j}(x,y) = 0$ *for* $y \in \Gamma_j(x)$, $1 \le j \le i$, *and*
(3) $a_i(x,y) = 0$ *for all* $y \in \Gamma_i(x)$, $i \le k-2$.

Proof. (1) follows from the fact that the parameters a_i, $i \le k-2$, of O_k are zero, O_k has no cycles of length $2i+1$, $i \le k-2$, so $Tr(A^{2i+1}) = \sum_{j=0}^{k-1} m_j\theta_j^{2i+1} = 0$. (2) and (3) are immediate from (1). □

Lemma 2.3 [2, p. 15]. *Let* $q(x) = \prod_{1 \le i \le k-1} (x - \theta_i)$, *and* $p(x) = |V(\Gamma)|\,q(x)/q(k)$, *then* $p(x)$ *is the unique polynomial of smallest degree such that* $p(A) = J$, *where* J *is the all* 1*'s square matrix of order* $|V(\Gamma)|$.

Lemma 2.4. *For each* $x \in V(\Gamma)$,

(1) $A^2(x,x) = \sum_{y \in \Gamma_1(x)} A(x,y) = k$,
(2) $\sum_{y \in \Gamma_2(x)} A^2(x,y) = k^2 - k \ge k_2(x)$, *and*
(3) $A^4(x,x) \ge 2k^2 - k$

Proof. The first claim is obvious, while the second follows from $\sum_{y \in \Gamma_2(x)} A^2(x,y) = (A^2J)(x,x) - A^2(x,x) - \sum_{y \in \Gamma_1(x)} A^2(x,y)$, $A^2(x,y) = 0$ for x and y at distance 1 by Lemma 2.2(3), and $A^2J = k^2J$. Also, from $A^2(x,y) \ge 1$ for x and y at distance 2, we have $\sum_{y \in \Gamma_2(x)} A^2(x,y) \ge k_2(x)$.

To prove (3), let $B = A^2 - kI$, then $B^2 = A^4 - 2kA^2 + k^2I$, and so, $\sum_{\text{all } y} (B(x,y))^2 = B^2(x,x) = A^4(x,x) - k^2$. Since

$$(B(x,x))^2 + \sum_{y \in \Gamma_1(x)} (B(x,y))^2 + \sum_{y \in \Gamma_2(x)} (B(x,y) - 1)^2 + \sum_{y \in \Gamma_{\geq 3}(x)} (B(x,y))^2$$
$$= \sum_{\text{all } y} (B(x,y))^2 - 2 \sum_{y \in \Gamma_2(x)} B(x,y) + k_2(x) \geq 0$$

replacing $\sum_{\text{all } y} (B(x,y))^2$ by $B^2(x,x) = A^4(x,x) - k^2$, we have

$$A^4(x,x) \geq 2 \sum_{y \in \Gamma_2(x)} A^2(x,y) - k_2(x) + k^2 \geq 2k^2 - k$$

as required. □

3. Proof of the Main Theorem

It is obvious for the case $k = 2$. We give a proof for the case $k = 6$ only. Similar and somewhat simpler arguments work for cases $k = 3, 4, 5$. Let $k = 6$, in this case,

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} \theta_i & 6 & 4 & 2 & -1 & -3 & -5 \\ \hline m_i & 1 & 44 & 165 & 132 & 110 & 10 \end{pmatrix}$$

and $p(x) = (1/12)(x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120)$, i.e.,

$$A^5 + 3A^4 - 23A^3 - 51A^2 + 94A + 120I = 12J \quad (**)$$

The following are immediate consequences of the previous lemmas:

(3.1) For $i \leq 4$, $A^{2i+1-j}(x,y) = 0$ for $x, y \in V(\Gamma)$ at distance j, $0 \leq j \leq i$, by Lemma 2.2(1).
(3.2) $a_i(x,y) = 0$ for $x, y \in V(\Gamma)$ at distance i, $i \leq 4$, by Lemma 2.2(3).
(3.3) $b_0(x,x) = 6$ and $b_1(x,y) = 5$ for $x, y \in V(\Gamma)$ at distance 1, by (3.2).
(3.4) $A^2(x,x) = 6$, by Lemma 2.4(1).
(3.5) $\sum_{y \in \Gamma_2(x)} A^2(x,y) = 30$, by Lemma 2.4(2).
(3.6) $A^4(x,x) \geq 96 - k_2(x) \geq 66$, by Lemma 2.4(3).

Lemma 3.7.

(1) $A^4(x,x) = 66$, *and* $k_2(x) = 30$ *for all* $x \in V(\Gamma)$.
(2) $c_2(x,y) = A^2(x,y) = 1$ *and* $b_2(x,y) = 5$ *for* $x, y \in V(\Gamma)$ *at distance* 2.
(3) $A^3(x,y) = 11$ *for* $x, y \in V(\Gamma)$ *at distance* 1.

Proof. (1) follows from (3.6) and the fact that the average of $A^4(x,x)$ is equal to $\left(\sum_{0 \leq i \leq 5} m_i\theta_i^4\right)\Big/462 = 66$.

(2) Since $\sum_{y \in \Gamma_2(x)} A^2(x,y) = 30$ by (3.5), $A^2(x,y) \geq 1$ if $d(x,y) = 2$, and $k_2(x) = 30$. It follows that $A^2(x,y) = 1$ whenever $d(x,y) = 2$. Hence $b_2(x,y) = 6 - c_2(x,y) = 5$ by (3.2).

(3)

$$\begin{aligned} A^3(x,y) &= \sum_{z \in \Gamma_1(y)} A^2(x,z) \\ &= A^2(x,x) + \sum_{z \in \Gamma_1(y) \cap \Gamma_1(x)} A^2(x,z) + \sum_{z \in \Gamma_1(y) \cap \Gamma_2(x)} A^2(x,z) \quad \text{by Lemma 2.1} \\ &= 6 + |\Gamma_1(y) \cap \Gamma_2(x)| \quad \text{by (3.4) and (3.2)} \\ &= 11, \quad \text{by (3.3)} \end{aligned}$$

as required. □

Lemma 3.8.

(1) $A^5(x,y) = 171$ *for* $x, y \in V(\Gamma)$ *at distance* 1,
(2) $A^4(x,y) = 21$ *for* $x, y \in V(\Gamma)$ *at distance* 2.

Proof. $A^4(x,y) = A^2(x,y) = 0$ by (3.1), and $A^3(x,y) = 11$ by Lemma 3.7(3) for x, $y \in V(\Gamma)$ at distance 1. Also, $A^5(x,y) = A^3(x,y) = 0$ by (3.1) for x, $y \in V(\Gamma)$ at distance 2. Applying these to (**), we obtain both (1) and (2) immediately. □

Lemma 3.9.

(1) $A^4(x,y) = 4$ *for* $x, y \in V(\Gamma)$ *at distance* 4.
(2) $c_3(x,y) = A^3(x,y) = 2$, $b_3(x,y) = 4$, *and* $A^5(x,y) = 58$ *for* x, $y \in V(\Gamma)$ *at distance* 3.

Proof. (1) For $x, y \in V(\Gamma)$ at distance 4, since $3A^4(x,y) + A^5(x,y) = 12$, and $A^5(x,y) = 0$, we have $A^4(x,y) = 4$.

(2) Since $A^5(x,y) - 23A^3(x,y) = 12$, by (**) and (3.1),

$$\begin{aligned} A^5(x,y) &= \sum_{z \in \Gamma_1(y)} A^4(x,z) \\ &= \sum_{z \in \Gamma_1(y) \cap \Gamma_2(x)} A^4(x,z) + \sum_{z \in \Gamma_1(y) \cap \Gamma_4(x)} A^4(x,z) \quad \text{by Lemma 2.1 and (3.2)} \\ &= 21c_3(x,y) + 4b_3(x,y) \quad \text{by Lemma 3.8(2)} \\ &= 4(b_3(x,y) + c_3(x,y)) + 17c_3(x,y) \\ &= 4(6 - a_3(x,y)) + 17c_3(x,y) \\ &= 24 + 17c_3(x,y), \text{ and} \end{aligned}$$

$$\begin{aligned} A^3(x,y) &= \sum_{z \in \Gamma_1(y)} A^2(x,z) \\ &= \sum_{z \in \Gamma_1(y) \cap \Gamma_2(x)} A^2(x,z) \\ &= |\Gamma_1(y) \cap \Gamma_2(x)| \quad \text{by Lemma 3.7(2)} \\ &= c_3(x,y) \end{aligned}$$

we have $24 + 17c_3(x,y) - 23c_3(x,y) = 12$. Hence $c_3(x,y) = 2$, $b_3(x,y) = 6 - c_3(x,y) = 4$, and consequently $A^5(x,y) = 58$. □

Lemma 3.10.

(1) $c_4(x,y) = 2$, $b_4(x,y) = 4$ *for* $x, y \in V(\Gamma)$ *at distance* 4,
(2) $A^5(x,y) = 12$, *and so* $c_5(x,y) = 3$, $a_5(x,y) = 3$ *for* $x, y \in V(\Gamma)$ *at distance* 5.

Proof. (1) Since $A^4(x,y) = \sum_{z \in \Gamma_1(y) \cap \Gamma_3(x)} A^3(x,z) = 2c_4(x,y) = 4$ by Lemma 3.8, it follows that $c_4(x,y) = 2$ and thus $b_4(x,y) = 4$.

(2) For $x, y \in V(\Gamma)$ at distance 5, since $A^5(x,y) = 12$ by (**), and $A^5(x,y) = \sum_{z \in \Gamma_1(y) \cap \Gamma_4(y)} A^4(x,z) = 4c_5(x,y)$ by (1), we have $c_5(x,y) = 3$ and thus $a_5(x,y) = 3$. □

Up to this point, we may conclude that Γ is a distance regular graph of diameter 5 with intersection array

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 6 & 5 & 5 & 4 & 4 & 0 \end{bmatrix},$$

indeed, it is exactly the intersection array of O_6. Hence, Γ is isomorphic to O_6 by Theorem 3 [6]. This completes the proof of the main theorem for the case $k = 6$.

References

1. Bannai, E., Ito, T.: Algebraic Combinatorics I: Association Schemes, Benjamin-Cummings Lecture Note Series (1984)
2. Biggs, N.: Algebraic Graph Theory, Cambridge Univ. Press, Cambridge, 1974
3. Biggs, N.: Some Odd Graph Theory, Proc. Second Internat. Conf. on Comb. Math., Annals of the New York Academy of Science, **319**, 71–81 (1979)
4. Laskar, R.: Eigenvalues of the Adjacency Matrix of Cubic Lattice Graphs, Pacific J. Math. **29** (3), 623–629 (1969)
5. Bose, R.C., Laskar, R.: Eigenvalues of the Adjacency Matrix of Tetrahedral Graphs, Aequationes Math. **4**, 37–43 (1970)
6. Moon, A.: Characterization of the Odd Graphs O_k by Parameters, Discrete Math. **42**, 91–97 (1982)
7. Schwenk, A.J., Wilson, R.J.: On the Eigenvalues of a Graph, pp. 307–336 in: Selected Topics in Graph Theory, L.W. Beineke and R.J. Wilson (eds), Academic P., 1981