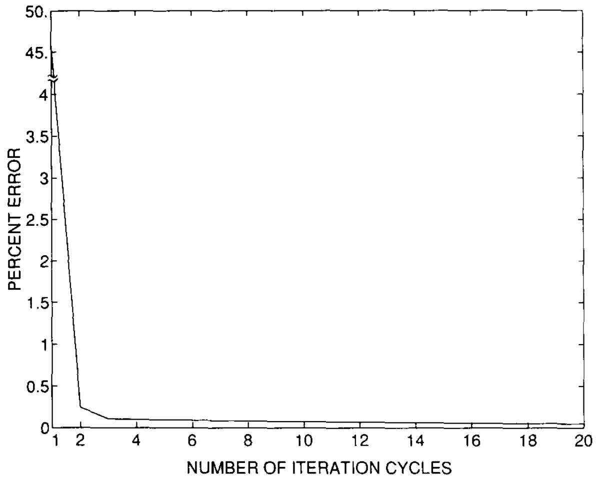

Fig. 1. Percent error versus the number of iterations cycles.

good representation of the original signal. We verified this observation in this paper. If there are multiple solutions of the signal recovery problem then our algorithm converges to one of the solutions because every solution is a member of the intersection set, C_o, as described in the beginning of this section.

The signal recovery technique can also be extended to multidimensional wavelets.

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Inverting Periodic Filters

Ching-An Lin and Chwan-Wen King

***Abstract*—We consider linear periodic filters. We give simple necessary and sufficient conditions for the filter to be invertible and a simple formula to compute its inverse. If the filter is not invertible, we propose a method to compute its optimal approximate inverse. An illustrative example is given.**

I. Introduction

Periodic filters have been found useful in speech scrambling [3], in filtering of cyclostationary signals [1], and in decimator-interpolator filter design to reduce the required computations [7]. The inverse or an "approximate inverse" of a periodic filter is required to recover the scrambled signal [3]. Inverting a class of periodic filters is discussed in [10].

We study the problem of finding the inverse or an approximate inverse of a linear periodic filter. We use the state equation description. We give necessary and sufficient conditions for the existence of the inverse, and we give a simple formula to compute it as a periodic filter with the same period. In case the inverse does not exist, i.e., in not implementable as a causal filter, we propose a method to find an approximate inverse which has a property that, when it cascades with the original periodic filter, the overall cascade connection is a pure delay of minimal possible length.

In our analysis, a single-input single-output (SISO) N-periodic digital filter is represented as an $N \times N$ proper rational matrix in z with strictly proper upper off-diagonal entries, as discussed in [5] and [8]. This model corresponds to the block signal processing structure [9, ch. 10]. This representation yields considerable simplification in analysis.

In Section II, we state precisely the problem under consideration and the transfer matrix representation for periodic filters. In Section III, we derive the necessary and sufficient condition for the existence of the inverse and a simple formula for computing it. A method of finding the optimal approximate inverse is proposed in Section IV. An illustrative example is given in Section V. Section VI is a brief conclusion.

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C.-A. Lin is with the Department of Control Engineering, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.

C.-W. King is with the Institute of Electronics, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.

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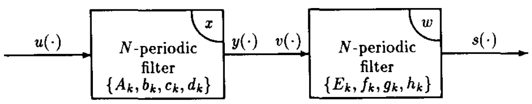

Fig. 1. Cascade connection of two periodic filters.

II. Problem Statement and Transfer Matrix Representation

A. Problem Statement

Consider the N-periodic linear causal filter with input u and output y, described by

$$\begin{aligned} x_{k+1} &= A_k x_k + b_k u_k \\ y_k &= c_k x_k + d_k u_k, \qquad k \geq 0 \end{aligned} \tag{2.1}$$

where $A_k \in \mathbb{R}^{n\times n}, b_k \in \mathbb{R}^{n\times 1}, c_k \in \mathbb{R}^{1\times n}$, and $d_k \in \mathbb{R}$ are N-periodic, i.e., $A_{k+N} = A_k, b_{k+N} = b_k, c_{k+N} = c_k$, and $d_{k+N} = d_k$ for all $k \geq 0$.

The inverse of filter (2.1) is an N-periodic linear causal filter of the form

$$\begin{aligned} w_{k+1} &= E_k w_k + f_k v_k \\ s_k &= g_k w_k + h_k v_k, k \geq 0 \end{aligned} \tag{2.2}$$

such that the cascade connection (shown in Fig. 1) satisfies that, with $x_0 = 0$ and $w_0 = 0$, the output s is identical to the input u, that is,

$$s_k = u_k, \qquad \text{for all } k \geq 0, \qquad \text{for all } u^1.$$

If the inverse exists, then the filter is said to be *invertible*. The questions addressed in this correspondence are 1) under what conditions is the filter (2.1) invertible and if so, how do we compute its inverse; and 2) if the inverse does not exist, how do we construct a filter of the form (2.2) that is an optimal approximate inverse?

We say the filter (2.2) is an approximate inverse of the filter (2.1) if the corresponding cascade connection shown in Fig. 1 is a pure delay. More precisely, with $x_0 = 0$ and $w_0 = 0$, for all input u,

$$s_k = \begin{cases} u_{k-L} & \text{for some fixed } L > 0, \quad \text{for all } k \geq L \\ 0 & \text{for } 0 \leq k < L. \end{cases} \tag{2.3}$$

The optimal approximate inverse is the filter of form (2.2), so that (2.3) is satisfied with the smallest possible L. The construction of the optimal approximate inverse is described in Section IV.

B. Transfer Matrix Representation

Consider again the filter (2.1). It is shown in [8] that the filter (2.1) is equivalent to an N-input N-output linear time-invariant system

$$\begin{aligned} \overline{x}_{k+1} &= \overline{A}\overline{x}_k + \overline{B}\overline{u}_k \\ \overline{y}_k &= \overline{C}\overline{x}_k + \overline{D}\overline{u}_k \end{aligned} \tag{2.4}$$

[1] By this definition, the filter (2.1) may have infinitely many *inverses*. The inverse refers to the *equivalent class* of filters yielding the same input–output relation.

where

$$\begin{aligned} \overline{x}_k &= x_{kN}, \overline{u}_k = [u_{kN}\ u_{kN+1} \cdots u_{kN+N-1}]^T, \\ \overline{y}_k &= [y_{kN}\ y_{kN+1} \cdots y_{kN+N-1}]^T \\ \overline{A} &= A_{N-1}A_{N-2}\cdots A_1 A_0 \\ \overline{B} &= [\overline{b}_0\, \overline{b}_1 \cdots \overline{b}_{N-1}] \\ \text{with } \overline{b}_i &= \begin{cases} A_{n-1}\cdots A_{i+1}b_i, & i = 0, \cdots, N-2 \\ b_{N-1}, & i = N-1 \end{cases} \\ \overline{C}^T &= \left[\overline{c}_0^T\, \overline{c}_1^T \cdots \overline{c}_{N-1}^T\right] \\ \text{with } \overline{c}_i &= \begin{cases} c_0, & i = 0 \\ c_i A_{i-1}\cdots A_0, & i = 1, \cdots, N-1 \end{cases} \\ \overline{D} &= [\overline{d}_{i,j}] \\ \text{with } \overline{d}_{i,j} &= \begin{cases} 0, & i < j \\ d_i & i = j \\ c_i Ai - 1 \cdots A_{j+1}b_j, & i > j. \end{cases} \end{aligned} \tag{2.5}$$

We note that with $x_0 = \overline{x}_0 = 0$, the systems (2.1) and (2.4) exhibit identical input–output relation, except that in (2.4) the input and output are sequences of vectors of dimension N.

The transfer matrix of the system (2.4), defined as

$$G(z) = \overline{C}(zI - \overline{A})^{-1}\overline{B} + \overline{D} \tag{2.6}$$

is an $N \times N$ proper rational matrix with $G(\infty) = \overline{D}$. From the definition (2.5), we note that $\overline{D}$ is lower triangular.

Thus, each linear N-periodic SISO filter is represented by an $N \times N$ proper rational matrix $G(z)$ with $G(\infty)$ lower triangular. The converse is also true.

Proposition 2.1 [5], [6]: Let $G(z)$ be an $N \times N$ proper rational matrix. Then $G(z)$ can be realized as an N-periodic SISO system of the form (2.2) if and only if $G(\infty)$ is lower triangular. □

The following results will be used in establishing the main results of this correspondence.

Proposition 2.2 [5]: The system (2.1) is BIBO stable if and only if $G(z)$, defined in (2.6), has all its poles inside the unit disk.

Proposition 2.3 [5]: Let $G(z)$ and $F(z)$ be the $N \times N$ proper rational matrices representing the N-periodic filters (2.1) and (2.2), respectively. Let $H(z)$ be the transfer matrix associated with the cascade system shown in Fig. 1. We have

$$H(z) = F(z)G(z).$$

□

III. Necessary and Sufficient Condition for the Existence of the Inverse

Consider the filter (2.1) and the associate transfer matrix $G(z)$ defined by (2.4), (2.5), and (2.6). Clearly, the filter (2.2) is the inverse of filter (2.1) if and only if the transfer matrix associated with the cascade connection in Fig. 1 is $H(z) = I$, the $N \times N$ identity matrix. From Proposition 2.3, this implies that

$$F(z) = G(z)^{-1}. \tag{3.1}$$

By Proposition 2.1, for the transfer matrix $F(z)$ to be realizable as an N-periodic system of the form (2.2), we must have that 1) $F(z)$ is proper and 2) $F(\infty)$ is lower triangular.

It is well known that the transfer matrix $G(z)^{-1}$ is proper iff $G(\infty)$ is nonsingular. And, if $G(z) = \overline{C}(zI - \overline{A})^{-1}\overline{B} + \overline{D}$ with $\overline{D}$ nonsingular, then [4, p. 656]

$$G(z)^{-1} = -\overline{D}^{-1}\overline{C}(zI - \overline{A} + \overline{B}\overline{D}^{-1}\overline{C})^{-1}\overline{B}\,\overline{D}^{-1} + \overline{D}^{-1}. \tag{3.2}$$

We note that if $\overline{D}$ is nonsingular and lower triangular, then so is $\overline{D}^{-1}$. The following result follows directly from the above discussions.

Proposition 3.1: The periodic filter (2.1) has an inverse of the form (2.2) if and only if the transfer matrix $G(z)$ associated with (2.1) has a proper inverse. □

Comment: It follows from Proposition 2.2 that the inverse of the filter (2.1) is BIBO stable if and only if $G(z)^{-1}$ has all its poles inside the unit disk. From (3.2), the inverse is BIBO stable if the matrix $\overline{A} - \overline{B}\,\overline{D}^{-1}\overline{C}$ has all its eigenvalues inside the unit disk.

Clearly, since $G(\infty) = \overline{D}$ is lower triangular, it is nonsingular if and only if $d_k \neq 0$ for $k = 0, \cdots, N-1$. The following theorem gives a necessary and sufficient condition for the existence of the inverse.

Theorem 3.2: The periodic filter (2.1) is invertible if and only if $d_k \neq 0$ for all k. □

If the system (2.1) is invertible, then $G(z)^{-1}$ is the transfer matrix associated with its inverse. A minimal realization of $G(z)^{-1}$ in the form (2.2) can be obtained by the method proposed in [6]. A straightforward realization is given as follows.

Theorem 3.3: Suppose the filter (2.1) is invertible, then its inverse of the form (2.2) can be computed as

$$E_k = A_k - b_k d_k^{-1} c_k, \qquad f_k = b_k d_k^{-1}, \qquad g_k = -d_k^{-1} c_k,$$

and

$$h_k = d_k^{-1}. \tag{3.3}$$

Proof: We have to show that the cascade connection in Fig. 1 with $x_0 = 0, w_0 = 0$ satisfies that, for all u,

$$s_k = u_k, \qquad \text{and } k \geq 0.$$

We prove it by induction. For $l = 0$, we have $x_0 = w_0 = 0$ and

$$s_0 = h_0 y_0 = d_0^{-1}(d_0 u_0) = u_0$$

where we have used that $v_0 = y_0$. Assume that for $l = i$, we have $s_i = u_i$ and $w_i = x_i$. Then

$$\begin{aligned} w_{i+1} &= E_i w_i + f_i y_i \\ &= (A_i - b_i d_i^{-1} c_i) x_i + b_i d_i^{-1}(c_i x_i + d_i u_i) \\ &= A_i x_i + b_i u_i = x_{i+1}. \end{aligned}$$

and

$$\begin{aligned} s_{i+1} &= g_{i+1} w_{i+1} + h_{i+1} y_{i+1} \\ &= -d_{i+1}^{-1} c_{i+1} x_{i+1} + d_{i+1}^{-1}(c_{i+1} x_{i+1} + d_{i+1} u_{i+1}) \\ &= u_{i+1}. \end{aligned}$$

Thus it follows that $s_k = u_k$, for $k \geq 0$. □

IV. Computing the Optimal Approximate Inverse

If the filter (2.1) is not invertible, the associated transfer matrix $G(z)$ does not have a proper inverse. Then we wish to construct a system of the form (2.2) so that the input–output relation of the cascade connection shown in Fig. 1 is a pure delay. A delay of m steps is a linear shift-invariant system and thus can be regarded as an N-periodic system. Let $m = p + qN$, where p, q are nonnegative integers with $0 \leq p < N$. The transfer matrix associated with the m-step delay, viewed as an N-periodic system, is

$$H_m(z) = \begin{bmatrix} O & z^{-1} I p \\ I_{N-p} & O \end{bmatrix} z^{-q} \tag{4.1}$$

where I_L denotes the $L \times L$ identity matrix. Since a cascade connection of an m-step delay and an l-step delay is an $(m+l)$-step delay, the following lemma follows from Proposition 2.3.

Lemma 4.1: Let m and l be positive integers, then

$$H_m(z) H_l(z) = H_{m+l}(z). \tag{4.2}$$

□

Finding the optimal approximate inverse of (2.1) is the same as finding the minimal L and a proper transfer matrix $F(z)$ with $F(\infty)$ lower triangular such that

$$F(z)G(z) = H_L(z) \tag{4.3}$$

where $H_L(z)$ is the transfer matrix associated with the L -step delay. This is equivalent to finding the minimal L so that 1) $H_L(z)G(z)^{-1}$ is proper, and 2) $H_L(z)G(z)^{-1}|_{z=\infty}$ is lower triangular. We shall first find the smallest m_1 such that $H_{m_1}(z)G(z)^{-1}$ is proper.

Write $G(z)^{-1} = [\beta_{ij}(z)/\alpha_{ij}(z)]_{1 \leq i,j \leq N}$, where $\beta_{ij}(z)$ and $\alpha_{ij}(z)$ are polynomials in z with real coefficients. Let $\delta(p)$ denote the degree of the polynomial p. Let $\overline{m} = \max_{ij}[\delta(\beta_{ij}) - \delta(\alpha_{ij})]$. The $\overline{m}$ so defined is the largest relative degree of entries of $G(z)^{-1}$. Since $G(z)^{-1}$ is not proper, $\overline{m} \geq 1$. Suppose i_0 is such that

$$\begin{aligned} &\max_j[\delta(\beta_{i_0 j}) - \delta(\alpha_{i_0 j})] \\ &\quad = \overline{m} \text{ and } \max_{i<i_0, j}[\delta(\beta_{ij}) - \delta(\alpha_{ij})] < \overline{m}. \end{aligned}$$

In other words, i_0 is the smallest row number in which $\overline{m}$ is attained. Let

$$m_1 = (N - i_0 + 1) + (\overline{m} - 1)N = \overline{m}N - i_0 + 1. \tag{4.4}$$

Then

$$H_{m_1}(z) = \begin{bmatrix} O & z^{-1} I_N - i_0 + 1 \\ I_{i_0 - 1} & O \end{bmatrix} z^{-(\overline{m}-1)}$$

and $H_{m_1}(z)G(z)^{-1}$ is proper. To see that $H_{m_1}(z)G(z)^{-1}$ is proper, let us partition

$$G^{-1}(z) = \begin{bmatrix} Q_1(z) \\ Q_2(z) \end{bmatrix}$$

where $Q_1(z) \in \mathbb{R}_p(z)^{(i_0-1) \times N}$ and the maximal relative degree of entries of $Q_1(z)$ is $\overline{m} - 1$. Thus,

$$\begin{aligned} H_{m_1}(z)G(z)^{-1} &= \begin{bmatrix} O & z^{-1} I_{N-i_0+1} \\ I_{i_0-1} & O \end{bmatrix} z^{-(\overline{m}-1)} \\ &\quad \cdot \begin{bmatrix} Q_1(z) \\ Q_2(z) \end{bmatrix} = \begin{bmatrix} z^{-\overline{m}} Q_2(z) \\ z^{-\overline{m}+1} Q_1(z) \end{bmatrix} \end{aligned}$$

which is clearly proper. It can be checked that for all $l < m_1$, $H_l(z)G(z)^{-1}$ is not proper. Thus, $m = m_1$, defined in (4.4), is the smallest possible integer such that $H_m(z)G(z)^{-1}$ is proper.

Let $\hat{F}(z) = H_{m_1}(z)G(z)^{-1}$. If $\hat{F}(\infty)$ is lower triangular, then $F(z) := \hat{F}(z)$ is the transfer matrix which yields the optimal approximate inverse. This would result in m_1 steps delay in the cascade connection of Fig. 1. If $\hat{F}(\infty)$ is not lower triangular, then more delay is required. The resulting $F(z)$ will have the form

$$F(z) = H_{m_2}(z)\hat{F}(z)$$

with $F(\infty)$ lower triangular. Write $\hat{F}(\infty) = [\hat{f}_{ij}]$. Let $1 \leq r \leq N - 1$ be the smallest integer such that $\hat{f}_{ij} = 0$ for all $j > i + r$. The number r is called the *upper bandwidth* of $\hat{F}(\infty)$ [2, p. 6] and is a measure of how far the matrix is from being lower triangular. If $\hat{f}_{ij} \neq 0$ for all $j > i$, then $r = N-1$. The following result shows that the smallest number m_2 that yields $H_{m_2}(\infty)\hat{F}(\infty)$ lower triangular is the upper bandwidth of $\hat{F}(\infty)$.

Proposition 4.2: Let r be the upper bandwidth of $\hat{F}(\infty)$. Then $m_2 = r$ is the smallest integer such that $H_{m_2}(\infty)\hat{F}(\infty)$ is lower triangular.

Comment:

1) Thus, the best approximate inverse of (2.1) is given by the transfer matrix

$$\begin{aligned} F(z) &= H_{m_2}(z)\hat{F}(z) = H_{m_2}(z)H_{m_1}(z)G(z)^{-1} \\ &= H_{m_1+m_2}(z)G(z)^{-1} = H_L(z)G(z)^{-1} \end{aligned} \tag{4.5}$$

where we have used Lemma 4.1, $L = m_1 + m_2$ is the smallest amount of delay, m_1 defined in (4.4) relates to the relative degree of $G(z)^{-1}$, and m_2 is the upper bandwidth of $\hat{F}(\infty)$.

2) The method proposed in [6] can then be used to realize $F(z)$ as an N-periodic filter of the form (2.2), which then is the optimal approximate inverse of (2.1).

Proof: Suppose $l \leq N - 1$; we have from (4.1)

$$H_l(\infty) = \begin{bmatrix} O & O \\ I_{N-l} & O \end{bmatrix}. \tag{4.6}$$

Let us partition $\hat{F}(\infty)$ as

$$\hat{F}(\infty) = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ \hat{F}_{21} & \hat{F}_{22} \end{bmatrix}$$

where $\hat{F}_{12}$ is $(N-l)\times(N-l)$, $\hat{F}_{11}$, $\hat{F}_{21}$, and $\hat{F}_{22}$ are with compatible dimensions. Thus,

$$\begin{aligned} H_l(\infty)\hat{F}(\infty) &= \begin{bmatrix} O & O \\ I_{N-l} & O \end{bmatrix}\begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ \hat{F}_{21} & \hat{F}_{22} \end{bmatrix} \\ &= \begin{bmatrix} O & O \\ \hat{F}_{11} & \hat{F}_{12} \end{bmatrix}. \end{aligned} \tag{4.7}$$

The above product is lower triangular if and only if $\hat{F}_{12}$ is lower triangular. But $\hat{F}_{12}$ is lower triangular if and only if $l \geq r$. Thus, the result follows. □

V. An Illustrative Example

In this section, we give an example to demonstrate the computation of optimal approximation inverse. Consider the three-periodic filter given by

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & 2 \\ 0 & 1 & 2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$b_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

$$c_0 = [0 \quad 1 \quad 0], \qquad c_1 = [1 \quad 1 \quad 4],$$

$$c_2 = [0 \quad 0 \quad 1],$$

$$d_0 = 0, \qquad d_1 = 4, \qquad d_2 = 0.$$

The transfer matrix associated with this periodic filter is

$$G(z) = \frac{1}{z - \frac{1}{2}} \begin{bmatrix} 1 & 3 & -1 \\ 4z+1 & 4z+1 & 0 \\ 2z+1 & 0 & 1 \end{bmatrix}.$$

By computation

$$\begin{aligned} G(z)^{-1} &= \frac{1}{2(4z+1)} \\ &\cdot \begin{bmatrix} 4z+1 & -3 & 4z+1 \\ -4z-1 & 2z+2 & -4z-1 \\ -(4z+1)(2z+1) & 3(2z+1) & -2(4z+1) \end{bmatrix}. \end{aligned} \tag{5.1}$$

Since $G(z)^{-1}$ is improper, the inverse does not exist. This also can be easily seen from that $d_0 = d_2 = 0$. The optimal approximate inverse is computed as follows. From (5.1), the largest relative degree of $G(z)^{-1}$ is 1 and occurs at the third row. Thus, $\overline{m} = 1$ and $i_0 = 3$. Then $m_1 = 1 \cdot 3 - 3 + 1 = 1$ and

$$\begin{aligned} \hat{F}(z) &= H_1(z)G(z)^{-1} \\ &= \frac{1}{2(4z+1)} \\ &\cdot \begin{bmatrix} -z^{-1}(4z+1)(2z+1) & 3z^{-1}(2z+1) & -2z^{-1}(4z+1) \\ 4z+1 & -3 & 4z+1 \\ -4z-1 & 2z+2 & -4z-1 \end{bmatrix} \end{aligned}$$

$$\hat{F}(\infty) = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

The upper bandwidth of $\hat{F}(\infty)$ is 1, thus $m_2 = 1$. The smallest delay then is 1 + 1 = 2, and the rational matrix of the optimal approximate inverse is

$$\begin{aligned} F(z) &= H_2(z)G(z)^{-1} \\ &= \frac{1}{2z(4z+1)} \\ &\cdot \begin{bmatrix} -4z-1 & 2z+2 & -4z-1 \\ -(4z+1)(2z+1) & 3(2z+1) & -2(4z+1) \\ z(4z+1) & -3z & z(4z+1) \end{bmatrix}. \end{aligned}$$

By using the algorithm proposed in [6], a realization of $F(z)$ as a three-periodic filter of the form (2.2) is described by

$$E_0 = E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{4} & 0 & -\frac{1}{2} \end{bmatrix},$$

$$f_0 = f_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{bmatrix},$$

$$g_0 = [1 \quad 0 \quad \tfrac{3}{2}], \qquad g_1 = [1 \quad \tfrac{3}{2} \quad -1],$$

$$g_2 = [\tfrac{3}{4} \quad \tfrac{1}{2} \quad 0],$$

$$h_0 = 0, \qquad h_1 = 0, \qquad h_2 = \tfrac{1}{2}.$$

□

VI. Conclusion

Periodic filters and their inverses are important in speech scrambling applications. In this correspondence, we give simple necessary and sufficient conditions for the filter to be invertible, and a simple formula to compute its inverse. If the filter is not invertible, we propose a method to compute its optimal approximate inverse. An example is given to demonstrate the proposed method for constructing the optimal approximate inverse.

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Reduction of the MSE in R-times Oversampled A/D Conversion from $\mathcal{O}(1/R)$ to $\mathcal{O}(1/R^2)$

Nguyen T. Thao and Martin Vetterli

***Abstract*—In oversampled analog-to-digital conversion, the usual reconstruction method using lowpass filtering leads to a mean squared error (MSE) inversely proportional to the oversampling ratio R. In this correspondence, we prove, under certain assumptions and with periodic analog input signals, that optimal reconstruction achieves an MSE with an oversampling ratio dependence order of at least $\mathcal{O}(1/R^2)$. That is, an MSE slope of -6 dB per octave of oversampling is obtained, rather than the conventional -3 dB/octave slope of classical schemes.**

I. INTRODUCTION

Analog-to-digital conversion consists of discretizing an analog signal in time and in amplitude. Shannon's well-known sampling theorem [1] guarantees that when a bandlimited signal is sampled only in time at the Nyquist rate or above, no information is lost. It also gives an analytical expression for the reconstruction of the bandlimited signal from its samples. Results on reconstruction were also obtained by Logan [2] when the analog signal is discretized only in amplitude. Under certain assumptions, he showed that an octave band signal is uniquely defined by its zero crossings, up to a multiplicative constant. This corresponds to amplitude quantization to regions having positive and negative values. However, in practical A/D conversion, analog signals are discretized both in time and amplitude (see Fig. 1). Few analytical results have been derived about the reconstruction of an analog signal from its discrete-time discrete-amplitude (DTDA) version.

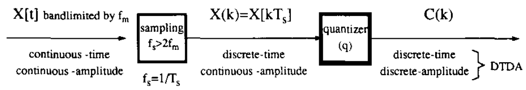

Fig. 1. Discretization scheme of a bandlimited analog signal with maximum frequency f_m.

Of course, if the signal is sampled in time at the Nyquist rate and uniformly quantized with a step size q, then the quantization error is given by $q^2/12$. A more interesting scenario results when the samples are taken above the Nyquist rate, i.e., when oversampling occurs. The classical reconstruction method consists in lowpass filtering the quantized signal, thus preserving the original bandlimited signal but reducing the power of the quantization error signal in proportion to the oversampling ratio R (under certain assumptions [3], [4]).

However, the following insight indicates that the classical reconstruction method may not be optimal in the mean squared error (MSE) sense. Halving the amplitude quantization step size will reduce the quantization error by a factor of 4 in the MSE sense, but halving the sampling period will only reduce the quantization error by 2. This inhomogeneity in the time and amplitude dimensions is counterintuitive. We will show that optimal reconstruction leads to homogeneity, that is, halving either amplitude or time quantization leads to a reduction of quantization error by a factor of 4.

The suboptimality in classical reconstruction stems from the fact that a *requantization* of the lowpass filtered reconstruction does not in general lead to the same quantized signal [5], [6]. That is, the DTDA version of the original signal is different from the DTDA version of the lowpass filtered reconstruction. It was shown in [5] and [6] that an estimate which does not reproduce the DTDA version of the original signal can be automatically improved in terms of MSE. Therefore, by necessity, any *optimal* reconstruction scheme should at least provide an estimate which reproduces the DTDA version of the original signal.

In this correspondence, we analyze the MSE of an estimate given by an *optimal* reconstruction scheme, with the assumption that the analog signals are periodic in the time interval in which they are coded. Assuming that the original signal has a minimum number of quantization threshold crossings (QTC's), we show that the MSE is at least inversely proportional to R^2 instead of R, for R high enough.

This result is the consequence of an analysis of the information present in the DTDA signal. After defining the mathematical context of our derivations in Section II, we show in Section III that when the oversampling ratio is high enough, the DTDA signal gives the location of the analog signal's QTC's with a time uncertainty equal to the sampling period. As shown in Section IV, this implies the $\mathcal{O}(1/R^2)$ behavior of the MSE.

II. MATHEMATICAL CONTEXT AND NOTATIONS

As mentioned in the Introduction, we consider that the bandlimited signals are sampled and quantized on the time interval $[0, T]$ and are T-periodic. We designate such signals using boldface italic capital letters, like $\boldsymbol{X}$. We denote the value of $\boldsymbol{X}$ at time t by $X[t]$. Bandlimited and T-periodic real signals necessarily have a finite

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The authors are with the Department of Electrical Engineering, and Center for Telecommunications Research, Columbia University, New York, NY 10027-6699.

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