An Optimal Algorithm for Sampled-Data Robust Servomechanism Controller Using Exponential Hold

Yung-Chun Wu and Nie-Zen Yen

***Abstract*— A new structure of sampled-data robust servomechanism controller using exponential hold is developed. An optimal algorithm is also proposed for choosing the controller parameters of two important special classes. The algorithm is derived by minimizing a square-error performance index, and the solution can be solved from a discrete-time algebraic Riccati equation.**

I. Introduction

The problem of robust servomechanism controller design has been widely considered in the literature (see reference). Generally, the purpose for one to construct a robust servomechanism controller is to attain the capability of asymptotic tracking and disturbance rejection with the permission of plant variations. In the literature, a general structure of linear time-invariant robust servomechanism controllers has been characterized [2]–[4], and the well-known "continuous internal model principle" has been given [8], [5], [7]. With this principle, it can be seen that if the steady-state value of the reference input or the disturbance is not constant, then in general, one cannot use sampled data with zero-order hold to construct a ripple-free [7] robust servomechanism controller because ripple errors would occur even if there is no tracking error at the sampling instants.

In this note, a new structure of sampled-data robust servomechanism controllers using exponential hold is developed. Such a structure is convenient for design because it leads to a simple closed-loop form. In particular, controller design for two important special cases classified as the "minimal-order class" and the "one-step prediction class," respectively, are derived. For the former class, the controller has the simplest structure so that it needs less on-line computations. For the later class, on-line control values are calculated by one step ahead of the output measurements so that it allows a leisure time to implement the control scheme. An optimal algorithm for choosing the parameters of the two important special classes is also developed. The algorithm is derived by minimizing a square-error performance index, and the solution can be solved from a discrete-time algebraic Riccati equation. A distinctive feature of the algorithm is that the solution does not depend on the weighting matrix of the performance index, but only on the correlation of the initial values of system state, reference input, and disturbance. Hence, the statistical information of the initial conditions becomes very important to this algorithm.

II. Preliminary

A. System Description

Consider the command tracking and disturbance rejection problem of a linear time-invariant system described as follows:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + F\boldsymbol{d}(t) \qquad (1.a)$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) + G\boldsymbol{d}(t) \qquad (1.b)$$

$$e(t) = \boldsymbol{y}(t) - \boldsymbol{r}(t) \qquad (1.c)$$

where $\boldsymbol{x} \in R^n$ is the state, $\boldsymbol{u} \in R^m$ is the control, $\boldsymbol{y} \in R^m$ is the measurable output, $\boldsymbol{d} \in R^v$ is the disturbance, $\boldsymbol{r} \in R^m$ is the command or reference input, and $\boldsymbol{e} \in R^m$ is the tracking error. The reference input and the disturbance satisfy the following models:

$$\dot{\boldsymbol{x}}_r(t) = A_r\boldsymbol{x}_r(t) \qquad \dot{\boldsymbol{x}}_d(t) = A_d\boldsymbol{x}_d(t)$$
$$\boldsymbol{r}(t) = C_r\boldsymbol{x}_r(t) \qquad \boldsymbol{d}(t) = C_d\boldsymbol{x}_d(t) \qquad (2)$$

where $\boldsymbol{x}_r \in R^{m_r}$ and $\boldsymbol{x}_d \in R^{m_d}$. The system described above is said to have no transmission zero [2] at the eigenvalues of A_r and A_d, if

$$\text{rank}\begin{bmatrix} -sI_n + A & B \\ C & O_m \end{bmatrix} = n + m, \qquad \forall s \in \{\text{eig}(A_r) \cup \text{eig}(A_d)\} \qquad (3)$$

where I_n denotes the $n-$dimensional identity matrix, O_m denotes the m − dimensional zero matrix, eig(#) denotes the set of eigenvalues of a matrix #, and $\text{eig}(A_r) \subset \mathbb{C}^+$, $\text{eig}(A_d) \subset \mathbb{C}^+$, where $\mathbb{C}^+$ is the right-half complex plane including the imaginary axis.

B. Robust Servomechanism Controller

A controller $\boldsymbol{u} = f(\boldsymbol{e}, \boldsymbol{r})$ (with input $\boldsymbol{e}$, $\boldsymbol{r}$ and output $\boldsymbol{u}$) is called a robust servomechanism controller of system (1), if it can satisfy the following three conditions ([4], [6]):

Condition 1: The resultant closed-loop system is asymptotically stable. Thus, if $\boldsymbol{r}(t) \equiv 0$ and $\boldsymbol{d}(t) \equiv 0$, then $\boldsymbol{x}(t) \to 0$ and $\boldsymbol{u}(t) \to 0$ as $t \to \infty$.

Condition 2: Asymptotic tracking action occurs, i.e., $e(t) \to 0$ as $t \to \infty$ for all initial conditions of $\boldsymbol{x}$, $\boldsymbol{x}_r$, $\boldsymbol{x}_d$ and the controller state.

Condition 3: Condition 2 remains true for any parameter variations in A, B, C, F, and G as long as Condition 1 remains true.

C. Deviation Model

It is known [4] that for every linear time-invariant robust servomechanism controller of system (1), there exist matrices $T_{11} \in R^{n\times m_r}$, $T_{12} \in R^{n\times m_d}$, $T_{21} \in R^{m\times m_r}$, and $T_{22} \in R^{m\times m_d}$, such that as $t \to \infty$, then $\boldsymbol{x}(t) \to \boldsymbol{x}_{ss}(t)$ and $\boldsymbol{u}(t) \to \boldsymbol{u}_{ss}(t)$ for all $\boldsymbol{x}(0)$, $\boldsymbol{x}_r(0)$ and $\boldsymbol{x}_d(0)$, where

$$\boldsymbol{x}_{ss}(t) = T_{11}\boldsymbol{x}_r(t) + T_{12}\boldsymbol{x}_d(t) \text{ and } \boldsymbol{u}_{ss}(t) = T_{21}\boldsymbol{x}_r(t) + T_{22}\boldsymbol{x}_d(t) \qquad (4.a)$$

denote the ultimate steady-state trajectories of $\boldsymbol{x}$ and $\boldsymbol{u}$, respectively. Thus, by defining the "deviation variables" as [10]

$$\delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{ss}(t) \qquad \text{and} \qquad \delta\boldsymbol{u}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{ss}(t) \qquad (4.b)$$

and using the fact that $\text{eig}(A_r) \subset \mathbb{C}^+$ and $\text{eig}(A_d) \subset \mathbb{C}^+$, it can be easily checked that the deviation variables satisfy the following model:

$$\dot{\delta\boldsymbol{x}}(t) = A\delta\boldsymbol{x}(t) + B\delta\boldsymbol{u}(t). \qquad (5.a)$$

$$e(t) = C\delta\boldsymbol{x}(t) \qquad (5.c)$$

D. An Augmented Model

Let

$$\lambda(s) = s^p - \sum_{s=0}^{p-1} a_i s^i \qquad (6)$$

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be the lowest order polynomial satisfying

$$\lambda(A_r) = A_r^p - \sum_{i=0}^{p-1} a_i A_r^i = 0$$
$$\lambda(A_d) = A_d^p - \sum_{i=0}^{p-1} a_i A_d^i = 0. \qquad (7)$$

Also, let

$$\Omega = \begin{bmatrix} O_m & I_m & O_m & \cdots & O_m \\ O_m & O_m & I_m & \cdots & O_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_m & O_m & O_m & \cdots & I_m \\ a_0 I_m & a_1 I_m & a_2 I_m & \cdots & a_{p-1} I_m \end{bmatrix} \qquad (8.a)$$
$$\rho = [I_m \mid O_{m\times(p-1)m}] \qquad (8.b)$$

and

$$\xi(t) = W \begin{bmatrix} \boldsymbol{u}_{ss}(t) \\ \frac{d}{dt}\boldsymbol{u}_{ss}(t) \\ \vdots \\ \frac{d^{p-1}}{dt^{p-1}}\boldsymbol{u}_{ss}(t) \end{bmatrix} \qquad (8.c)$$

where $W \in R^{mp\times mp}$ is a selected nonsingular matrix. Then by (2), (4.a), (7), and (8), it is clear that $\boldsymbol{u}_{ss}(t)$ satisfies the following:

$$\dot{\xi}(t) = \phi\xi(t) \qquad (9.a)$$
$$\boldsymbol{u}_{ss}(t) = \Gamma\xi(t) \qquad (9.b)$$

where $\phi = W\Omega W^{-1}$ and $\Gamma = \rho W^{-1}$. Thus, by combining (5) and (9), and using (4.b), one obtains the following augmented model:

$$\begin{bmatrix} \delta\dot{\boldsymbol{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A & -B\Gamma \\ O_{mp\times n} & \phi \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} B \\ O_{mp\times m} \end{bmatrix} \boldsymbol{u}(t) \qquad (10.a)$$
$$\boldsymbol{e}(t) = [C \quad O_{m\times mp}] \begin{bmatrix} \delta\boldsymbol{x}(t) \\ \xi(t) \end{bmatrix}. \qquad (10.b)$$

Notice that the deviation model (5) and the augmented model (10) exist as long as a linear-time invariant robust-servomechanism controller of system (1) can be found.

III. Sampled-Data Robust Servomechanism Controller

A. General Class

Let $T > 0$, and define

$$\begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} = \exp\left(\begin{bmatrix} A & B\Gamma \\ O_{mp\times n} & \phi \end{bmatrix} T \right) \qquad (11)$$

i.e., $\overline{A} = \exp(AT)$, $\overline{\phi} = \exp(\phi T)$ and $\overline{B}_\dagger = \int_0^T \exp(A\theta)B\Gamma\exp[\phi(T-\theta)]\,d\theta$. Also, let $L_2 \in R^{mp\times m}$ and $H_2 \in R^{mp\times m}$ be two constant matrices, $\varphi_1(\theta) \in R^{m\times m}$ and $\varphi_2(\theta) \in R^{m\times m}$ be two piecewise continuous functions to be chosen on $[0, T)$, and define

$$C_\sigma = [C \quad O_{m\times mp}] \begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix}^{-1} \in R^{m\times(n+mp)} \qquad (12.a)$$
$$L_1 = \int_0^T \exp(A\theta)B\varphi_1(T-\theta)\,d\theta \in R^{n\times m} \qquad (12.b)$$
$$H_1 = \int_0^T \exp(A\theta)B\varphi_2(T-\theta)\,d\theta \in R^{n\times m} \qquad (12.c)$$
$$L_3 = C_\sigma \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \in R^{m\times m} \qquad (12.d)$$

and

$$H_3 = C_\sigma \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in R^{m\times m}. \qquad (12.e)$$

Theorem 1: If the matrix

$$A_s = \begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} L_1 & H_1 \\ L_2 & H_2 \end{bmatrix} \begin{bmatrix} C \quad O_{m\times mp} \\ \hline C_\sigma \end{bmatrix} \qquad (13)$$

is stable (i.e., all eigenvalues lie inside the unit complex circle), then the following is a robust servomechanism controller of system (1):

$$\begin{bmatrix} \hat{\xi}((k+1)T) \\ h((k+1)T) \end{bmatrix} = \begin{bmatrix} \overline{\phi} & H_2 \\ O_{m\times mp} & H_3 \end{bmatrix} \begin{bmatrix} \hat{\xi}(kT) \\ h(kT) \end{bmatrix} + \begin{bmatrix} L_2 \\ L_3 + I_m \end{bmatrix} \boldsymbol{e}(kT) \qquad (14.a)$$
$$\boldsymbol{u}(kT+\theta) = [\Gamma\exp(\phi\theta) \quad \varphi_2(\theta)] \begin{bmatrix} \hat{\xi}(kT) \\ h(kT) \end{bmatrix} + \varphi_1(\theta)\boldsymbol{e}(kT) \qquad (14.b)$$

where $k = 0, 1, 2, \cdots$, and $\theta \in [0, T)$.

Proof: By (11) and (12.a)–(12.c), it is clear that a necessary condition for A_s to be stable is that the triple

$$[C \quad O_{m\times mp}], \begin{bmatrix} A & B\Gamma \\ O_{mp\times n} & \phi \end{bmatrix}, \begin{bmatrix} B & O_{n\times mp} \\ O_{mp\times n} & I_{mp} \end{bmatrix} \qquad (15)$$

is stabilizable and detectable. This in turn implies that (C, A, B) is stabilizable and detectable, and the transmission zero assumption (3) holds (a simple rank test easily checks this fact). Thus, a linear time-invariant robust servomechanism controller of system (1) can be found ([2]–[4]), and the deviation model (5) and the augmented model (10) exist. Therefore, by defining

$$\tilde{\xi}(t) = \hat{\xi}(t) - \xi(t) \qquad (16)$$

and subtracting (9) from (14), one obtains

$$\begin{bmatrix} \tilde{\xi}((k+1)T) \\ h((k+1)T) \end{bmatrix} = \begin{bmatrix} \overline{\phi} & H_2 \\ O_{m\times mp} & H_3 \end{bmatrix} \begin{bmatrix} \tilde{\xi}(kT) \\ h(kT) \end{bmatrix} + \begin{bmatrix} L_2 \\ L_3 + I_m \end{bmatrix} \boldsymbol{e}(kT) \qquad (17.a)$$
$$\delta\boldsymbol{u}(kT+\theta) = [\Gamma\exp(\phi\theta) \quad \varphi_2(\theta)] \begin{bmatrix} \tilde{\xi}(kT) \\ h(kT) \end{bmatrix} + \varphi_1(\theta)\boldsymbol{e}(kT). \qquad (17.b)$$

By combining (17) and the deviation model (5), one obtains the following closed-loop system:

$$\begin{bmatrix} \delta\boldsymbol{x}((k+1)T) \\ \tilde{\xi}((k+1)T) \\ h((k+1)T) \end{bmatrix} = \begin{bmatrix} \overline{A} + L_1 C & \overline{B}_\dagger & H_1 \\ L_2 C & \overline{\phi} & H_2 \\ L_3 C + C & O_{m\times m} & H_3 \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \\ h(kT) \end{bmatrix}. \qquad (18)$$

It is easily checked that

$$\begin{bmatrix} I_{n+mp} & O_{(n+mp)\times m} \\ C_\sigma & I_m \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} + L_1 C & \overline{B}_\dagger & H_1 \\ L_2 C & \overline{\phi} & H_2 \\ L_3 C + C & O_{m\times m} & H_3 \end{bmatrix} \begin{bmatrix} I_{n+mp} & O_{(n+mp)\times m} \\ C_\sigma & I_m \end{bmatrix} = \begin{bmatrix} A_s & \begin{matrix} H_1 \\ H_2 \end{matrix} \\ O_{m\times(n+mp)} & O_{m\times m} \end{bmatrix} \qquad (19)$$

so that by giving

$$h(kT) = \overline{h}(kT) + C_\sigma \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \end{bmatrix} \qquad (20)$$

and substituting it into (18), one obtains

$$\begin{bmatrix} \delta\boldsymbol{x}((k+1)T) \\ \tilde{\xi}((k+1)T) \\ \overline{h}((k+1)T) \end{bmatrix} = \begin{bmatrix} A_s & \begin{matrix} H_1 \\ H_2 \end{matrix} \\ O_{m\times(n+mp)} & O_{m\times m} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \\ \overline{h}(kT) \end{bmatrix}. \qquad (21)$$

Since A_s is stable, all the closed-loop poles lie inside the unit complex circle. Thus, it is true that $\delta\boldsymbol{x}(kT) \to 0$, $\tilde{\xi}(kT) \to 0$ and $h(kT) \to 0$ as $k \to \infty$. By (10.b) and (17.b), it is also true that $\boldsymbol{e}(kT) \to 0$ and

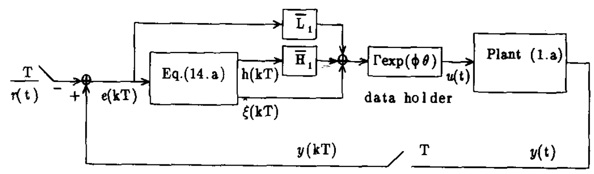

Fig. 1. The sampled-data robust servomechanism controller of (14.a) and (23).

$\delta\boldsymbol{u}(t) \to 0$ as $t \to \infty$. Since the deviation model (5) is an ordinary continuous model, one has $\delta\boldsymbol{x}(t) \to 0$ and $\boldsymbol{e}(t) \to 0$, as $t \to \infty a$. Hence, the asymptotic tracking action occurs as long as A_s is stable. Moreover, from (21), it is clear that the resultant closed-loop system is asymptotically stable if and only if A_s is stable (if $\boldsymbol{r} \equiv 0$ and $\boldsymbol{d} \equiv 0$, then one can treat $\boldsymbol{x}$ as $\delta\boldsymbol{x}$ and $\hat{\xi}$ as ξ), so that for any matrices A, B, and C, the asymptotic tracking action occurs as long as the resultant closed-loop system is asymptotically stable. Thus, the theorem is proved. □

Remark 1: If (A, B) is controllable, then for any matrices L_1 and H_1, there exist infinitely many choices of $\varphi_1(\theta)$ and $\varphi_2(\theta)$ to satisfy (12.b) and (12.c), respectively. In particular, if $n \le mp$ and rank $[B_\dagger] = n$ [no loss of generality by increasing the number of modes of $\lambda(s)$], then a simple possible choice may be the use of an exponential hold as follows:

$$\varphi_1(\theta) = \Gamma \exp(\phi\theta)\overline{L}_1, \qquad \varphi_2(\theta) = \Gamma \exp(\phi\theta)\overline{H}_1 \tag{22.b}$$

where

$$\overline{L}_1 = B_\dagger^\tau (B_\dagger B_\dagger^\tau)^{-1} L_1, \qquad \overline{H}_1 = B_\dagger^\tau (B_\dagger B_\dagger^\tau)^{-1} H_1. \tag{22.c}$$

With this choice, the control (14.b) can be simplified as (see Fig. 1):

$$\boldsymbol{u}(kT+\theta) = \Gamma \exp(\phi\theta)[\hat{\xi}(kT) + \overline{H}_1 h(kT) + \overline{L}_1 \boldsymbol{e}(kT)]. \tag{23}$$

B. Two Special Cases

By letting $\varphi_2(\theta) = 0$ (i.e., $H_1 = 0$) and $H_2 = 0$ or $\varphi_1(\theta) = 0$ (i.e., $L_1 = 0$) and $L_2 = 0$, respectively, then Theorem 1 leads to the following two corollaries.

Corollary 1: (Minimal-order class) If the matrix

$$A_s = \begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C \quad O_{m\times mp}] \tag{24}$$

is stable, then the following is a robust servomechanism controller of system (1):

$$\hat{\xi}((k+1)T) = \overline{\phi}\hat{\xi}(kT) + L_2 \boldsymbol{e}(kT) \tag{25.a}$$

$$\boldsymbol{u}(kT+\theta) = \Gamma \exp(\phi\theta)\hat{\xi}(kT) + \varphi_1(\theta)\boldsymbol{e}(kT). \tag{25.b}$$

Corollary 2: (One-step prediction class) If the matrix

$$A_s = \begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} C_\sigma \tag{26}$$

is stable, then the following is a robust servomechanism controller of system (1):

$$\begin{bmatrix} \hat{\xi}((k+1)T) \\ h((k+1)T) \end{bmatrix} = \begin{bmatrix} \overline{\phi} & H_2 \\ O_{m\times mp} & H_3 \end{bmatrix} \begin{bmatrix} \hat{\xi}(kT) \\ h(kT) \end{bmatrix} + \begin{bmatrix} O_{m\times m} \\ I_m \end{bmatrix} \boldsymbol{e}(kT) \tag{27.a}$$

$$\boldsymbol{u}(kT+\theta) = [\Gamma \exp(\phi\theta) \quad \varphi_2(\theta)] \begin{bmatrix} \hat{\xi}(kT) \\ h(kT) \end{bmatrix}. \tag{27.b}$$

Remark 2: From (21) and (20), one has $\overline{h}(kT) = 0$ and $h(kT) = C_\sigma[\delta_{\boldsymbol{x}}^\tau(kT)\tilde{\xi}^T(kT)]^\tau$ for all $k \ge 1$. Since $h(kT)$ and $\hat{\xi}(kT)$ can be calculated from (27.a) as long as $\boldsymbol{e}((k-1)T)$ is measured, therefore, from (27.b), the values of $\boldsymbol{u}(kT+\theta)$ in-between the sampling instances kT and $(k+1)T$ can be calculated. The class name of (27) reflects this prediction property.

IV. An Optimal Approach

In the rest of this note, one assumes that (C, A, B) is controllable and observable, and both the minimal order class (25) and the one-step prediction class (27) are not empty (i.e., there exist L_1, $L_2 d$ and H_1, H_2 such that both the matrices (24) and (26) are stable). Besides, we select a quadratic performance index as follows:

$$J = E\left(\sum_{k=0}^{\infty} \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \end{bmatrix}^T Q \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \end{bmatrix}\right), \tag{28}$$

where $Q \in R^{(n+mp)\times(n+mp)}$ is positive-definite, and $\tilde{\xi}(kT) = \hat{\xi}(kT) - \xi(kT)$. Notice that the index serves as a measure of the deviation errors from the ultimate steady-state trajectories. Now, it is desired to find the optimal gains L_1, L_2, H_1, and H_2, such that the index J subject to either class of (25) or (27) is minimized.

A. Minimal-Order Class

Since the minimal-order class (25) is a special case of the general class (14) with $\varphi_2(\theta) = 0$ (i.e., $H_1 = 0$), $H_2 = 0$, and $H_3 = 0$, thus the closed-loop system (21) can be simplified as

$$\begin{bmatrix} \delta\boldsymbol{x}((k+1)T) \\ \tilde{\xi}((k+1)T) \end{bmatrix} = \left(\begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C \quad O_{m\times mp}]\right) \cdot \begin{bmatrix} \delta\boldsymbol{x}(kT) \\ \tilde{\xi}(kT) \end{bmatrix}. \tag{29}$$

Since the closed-loop system is asymptotically stable (a necessary condition of the robust servomechanism controller), the index J subject to (29) equals [13]:

$$J = \mathrm{Tr}(V\Psi) \tag{30}$$

where Ψ is a correlation matrix given by

$$\Psi \equiv \mathrm{cor}\left(\begin{bmatrix} \delta\boldsymbol{x}(0) \\ \tilde{\xi}(0) \end{bmatrix}\right) = E\left(\begin{bmatrix} \delta\boldsymbol{x}(0) \\ \tilde{\xi}(0) \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{x}(0) \\ \tilde{\xi}(0) \end{bmatrix}^\tau\right) \tag{31}$$

and $V \in R^{(n+mp)\times(n+mp)}$ is a positive-definite matrix solved from the following Lyapunov equation:

$$\left(\begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C \quad O_{m\times mp}]\right)^\tau V \left(\begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C \quad O_{m\times mp}]\right) - V + Q = 0. \tag{32}$$

By approximating Ψ by $\Psi + \epsilon I_{n+mp}$, where ϵ is a small positive number, and, without loss of generality, by assuming Ψ to be positive-definite, one obtains the following result

Theorem 2: Assume Ψ is positive-definite, then the optimal gains L_1 and L_2 of the sampled-data robust servomechanism controller (25) to minimize the performance index (28) is given by

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = -\begin{bmatrix} \overline{A} & \overline{B}_\dagger \\ O_{mp\times n} & \overline{\phi} \end{bmatrix} P \begin{bmatrix} C^\tau \\ O_{mp\times m} \end{bmatrix} \cdot \left([C \quad O_{m\times mp}] P \begin{bmatrix} C^\tau \\ O_{mp\times m} \end{bmatrix}\right)^{-1} \tag{33.a}$$

where $P \in R^{(n+mp)\times(n+mp)}$ is a positive-definite matrix solved from the following algebraic Riccati equation:

$$\begin{aligned}&\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} P \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}^\tau \\ &\quad - \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} P \begin{bmatrix}C^\tau \\ O_{mp\times m}\end{bmatrix} \\ &\quad \cdot \left([C \quad O_{m\times mp}] P \begin{bmatrix}C^\tau \\ O_{mp\times m}\end{bmatrix}\right)^{-1} \\ &\quad \cdot [C \quad O_{m\times mp}] P \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}^\tau \\ &\quad - P + \Psi = 0. \end{aligned} \tag{33.b}$$

Proof: To minimize (30) subject to (32), we introduce the following augmented cost [12]:

$$\begin{aligned}J_c = \mathrm{Tr}\Bigg(V\Psi + P\Bigg\{&\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right)^\tau \\ &\cdot V\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right) - V + Q\Bigg\}\Bigg)\end{aligned} \tag{34}$$

where P is the associated Lagrange multiplier. Letting $dJ_c/dV = 0$, one obtains

$$\begin{aligned}&\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right) P \left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}\right. \\ &\qquad \left. + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right)^\tau - P + \Psi = 0. \end{aligned} \tag{35.a}$$

On the other hand, by letting $L = [L_1^\tau \quad L_2^\tau]^\tau$ and $dJ_c/dL = 0$, one obtains

$$V\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right) P \begin{bmatrix}C^\tau \\ O_{mp\times m}\end{bmatrix} = 0. \tag{35.b}$$

Since Q is positive-definite, the solution V of the Lyapunov equation (32) is positive-definite, hence (35.b) can be reduced to (33.a). Furthermore, by substituting (33.a) into (35.a), one obtains (33.b). Hence, the necessity of the theorem is proved. Besides, by (30), (32) and (35.a), one has

$$\begin{aligned}J &= \mathrm{Tr}(V\Psi) \\ &= -\mathrm{Tr}\Bigg(V\Bigg\{\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right) \\ &\qquad \times P\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right)^\tau - P\Bigg\}\Bigg) \\ &= -\mathrm{Tr}\Bigg(P\Bigg\{\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right)^\tau \\ &\qquad \times V\left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}L_1 \\ L_2\end{bmatrix}[C \quad O_{m\times mp}]\right) - V\Bigg\}\Bigg) \\ &= \mathrm{Tr}(PQ). \end{aligned} \tag{36}$$

It is known [1], [9] that the algebraic Riccati equation (33.b) and (33.a) has a unique stable solution which minimizes the index (36), so that the theorem is proved. □

B. One-Step Prediction Class

Since the one-step prediction class (27) is a special case of the general class (14) with $\varphi_1(\theta) = 0$ (i.e., $L_1 = 0$), $L_2 = 0$ and $L_3 = 0$, the closed-loop system (21) can be simplified as

$$\begin{bmatrix}\delta_{\boldsymbol{x}}((k+1)T) \\ \bar{\xi}((k+1)T)\end{bmatrix} = \left(\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} + \begin{bmatrix}H_1 \\ H_2\end{bmatrix} C_\sigma\right)\begin{bmatrix}\delta_{\boldsymbol{x}}(kT) \\ \bar{\xi}(kT)\end{bmatrix} \tag{37}$$

for all $k \geq 1$. Assume $h(0) = 0$, then from (18), one has

$$\begin{bmatrix}\delta_{\boldsymbol{x}}(T) \\ \bar{\xi}(T)\end{bmatrix} = \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}\begin{bmatrix}\delta_{\boldsymbol{x}}(0) \\ \bar{\xi}(0)\end{bmatrix} \tag{38}$$

Hence, the correlation of the state $[\delta_{\boldsymbol{x}}^\tau(T)\bar{\xi}^\tau(T)]^\tau$ equals

$$\Psi_\sigma \equiv \mathrm{cor}\left(\begin{bmatrix}\delta_{\boldsymbol{x}}(T) \\ \bar{\xi}(T)\end{bmatrix}\right) = \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}\Psi\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}^\tau. \tag{39}$$

Notice that Ψ_σ is independent of H_1 and H_2, so that the minimization of the index J is equivalent to the minimization of the following index

$$J_1 = E\left(\sum_{k=1}^{\infty}\begin{bmatrix}\delta_{\boldsymbol{x}}(kT) \\ \bar{\xi}(kT)\end{bmatrix}^\tau Q \begin{bmatrix}\delta_{\boldsymbol{x}}(kT) \\ \bar{\xi}(kT)\end{bmatrix}\right). \tag{40}$$

Theorem 3: Assume Ψ_σ is positive-definite, then the optimal gains H_1 and H_2 of the sampled-data robust servomechanism controller (27) to minimize the performance index (28) with initial condition $h(0) = 0$ is given by

$$\begin{bmatrix}H_1 \\ H_2\end{bmatrix} = -\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} P_\sigma C_\sigma^\tau (C_\sigma P_\sigma C_\sigma^\tau)^{-1} \tag{41.a}$$

where $P_\sigma \in R^{(n+mp)\times(n+mp)}$ is a positive-definite matrix solved from the following algebraic Riccati equation:

$$\begin{aligned}&\begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} P_\sigma \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}^\tau - \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix} P_\sigma C_\sigma^\tau \\ &\quad \cdot (C_\sigma P_\sigma C_\sigma^\tau)^{-1} C_\sigma P_\sigma \begin{bmatrix}\bar{A} & \bar{B}_\dagger \\ O_{mp\times n} & \bar{\phi}\end{bmatrix}^\tau - P_\sigma + \Psi_\sigma = 0. \end{aligned} \tag{41.b}$$

Proof: Replacing $[C \quad O_{m\times mp}]$ by C_σ and Ψ by Ψ_σ in Theorem 2, the result follows directly. □

C. Computation of the Correlation Matrix

A convenient method to calculate the correlation matrix Ψ (or Ψ_σ) can be done by way of the augmented model (10). To do so, subtracting (10) from (1) and using (4.b), one obtains

$$\begin{bmatrix}\dot{\boldsymbol{x}}_{ss}(t) \\ \dot{\xi}(t)\end{bmatrix} = \begin{bmatrix}A & B\Gamma \\ O_{mp\times n} & \phi\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{ss}(t) \\ \xi(t)\end{bmatrix} + \begin{bmatrix}O_{n\times m_r} & FC_d \\ O_{mp\times m_r} & O_{mp\times m_d}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t)\end{bmatrix} \tag{42.a}$$

$$0 = [C \quad O_{m\times mp}]\begin{bmatrix}\boldsymbol{x}_{ss}(t) \\ \xi(t)\end{bmatrix} + [-C_r \quad GC_d]\begin{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t)\end{bmatrix}. \tag{42.b}$$

Now, define

$$\boldsymbol{y}_{ss}(t) = [C \quad O_{m\times mp}]\begin{bmatrix}\boldsymbol{x}_{ss}(t) \\ \xi(t)\end{bmatrix} = [C_r \quad -GC_d]\begin{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t)\end{bmatrix} \tag{43}$$

and differentiating (43) continuously, one obtains

$$\begin{bmatrix}\boldsymbol{y}_{ss}(t) \\ \frac{d}{dt}\boldsymbol{y}_{ss}(t) \\ \vdots \\ \frac{d^{g-1}}{dt^{g-1}}\boldsymbol{y}_{ss}(t)\end{bmatrix} = N_1\begin{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t)\end{bmatrix} = N_2\begin{bmatrix}\boldsymbol{x}_{ss}(t) \\ \xi(t)\end{bmatrix} + N_3\begin{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t)\end{bmatrix} \tag{44}$$

where $N_1 = [N_{10}^\tau \cdots N_{1g-1}^\tau]^\tau \in R^{gm\times(m_d+m_r)}$, $N_2 = [N_{20}^\tau \cdots N_{2g-1}^\tau]^\tau \in R^{gm\times(n+pm)}$, $N_3 = [N_{30}^\tau \cdots N_{3g-1}^\tau]^\tau \in$

$R^{gm\times(m_d+m_r)}$, and

$$\begin{aligned} N_{1q} &= [C_r \quad -GC_d]\begin{bmatrix} A_r & O_{m_r x m_d} \\ O_{m_d\times m_r} & A_d \end{bmatrix}^q, \\ N_{2q} &= [C \quad O_{m\times mp}]\begin{bmatrix} A & B\Gamma \\ O_{mp\times n} & \phi \end{bmatrix}^q \\ N_{3q} &= \sum_{f=0}^{q-1}[C \quad O_{m\times mp}]\begin{bmatrix} A & B\Gamma \\ O_{mp\times n} & \phi \end{bmatrix}^f \\ &\quad \cdot \begin{bmatrix} O_{n\times m_r} & FC_dA_d^{q-f-1} \\ O_{mp\times m_r} & O_{mp\times m_d} \end{bmatrix} \end{aligned} \tag{45}$$

for $q = 0, 1, 2, \cdots, g-1$ (except $N_{30} = O_{m\times(m_r+m_d)}$). Since (C, A) is observable, the augmented model (10) is observable, hence rank $N_2 = n + pm$ can be guaranteed by a sufficiently large positive integer g, so that one obtains

$$\begin{bmatrix} \boldsymbol{x}_{ss}(t) \\ \xi(t) \end{bmatrix} = N\begin{bmatrix} \boldsymbol{x}_r(t) \\ \boldsymbol{x}_d(t) \end{bmatrix} \tag{46.a}$$

where

$$N = (N_2^\tau N_2)^{-1}N_2^\tau(N_1 - N_3) \in R^{(n+mp)\times(m_d+m_r)}. \tag{46.b}$$

Now, from (4.b), (16) and (46.a), one has

$$\begin{bmatrix} \delta\boldsymbol{x}(0) \\ \tilde{\xi}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(0) \\ \hat{\xi}(0) \end{bmatrix} - \begin{bmatrix} \boldsymbol{x}_{ss}(0) \\ \xi(0) \end{bmatrix} = \begin{bmatrix} O_{n\times 1} \\ \hat{\xi}(0) \end{bmatrix} + S\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix} \tag{47}$$

where

$$S = \left[\begin{array}{c|c} I_n & \\ & -N \\ O_{mp\times n} & \end{array}\right] \in R^{(n+mp)\times(n+m_d+m_r)}. \tag{48}$$

Hence, the correlation matrix Ψ equals

$$\begin{aligned} \Psi = &\begin{bmatrix} O_{n\times 1} \\ \hat{\xi}(0) \end{bmatrix}\begin{bmatrix} O_{n\times 1} \\ \hat{\xi}(0) \end{bmatrix}^\tau + \begin{bmatrix} O_{n\times 1} \\ \hat{\xi}(0) \end{bmatrix} E\left(\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}^\tau\right) S^\tau \\ &+ SE\left(\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}\right)\begin{bmatrix} O_{n\times 1} \\ \hat{\xi}(0) \end{bmatrix}^\tau + SE\left(\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}^\tau\right) S^\tau. \end{aligned} \tag{49}$$

In particular, if $\hat{\xi}(0) = 0$, then Ψ is simplified to

$$\Psi = SE\left(\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{x}_r(0) \\ \boldsymbol{x}_d(0) \end{bmatrix}^\tau\right) S^\tau. \tag{50}$$

V. Examples

Example 1: Consider a linear time-delay process described as follows (e.g., a tank temperature control [7], or a paper machine [11]):

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= -\boldsymbol{x}(t) + \boldsymbol{u}(t-1) + \boldsymbol{d} \\ \boldsymbol{y}(t) &= \boldsymbol{x}(t) \\ e(t) &= \boldsymbol{y}(t) - \boldsymbol{r}(t) \end{aligned} \tag{51}$$

where $\boldsymbol{u}(\theta) = 0$ for $\theta \in (-1, 0)$, $\boldsymbol{d}$ is a constant disturbance, and

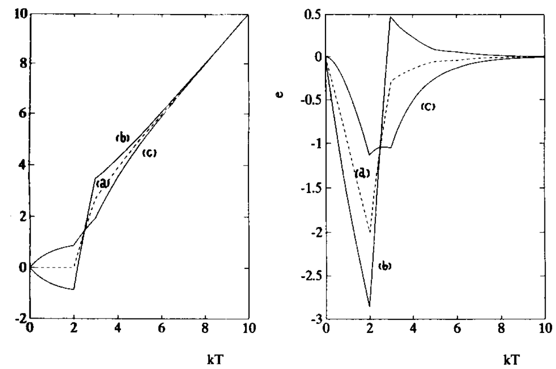

Fig. 2. The ramp tracking response of example 1 using sampled-data controller (55) with initial state $[\boldsymbol{x}^\tau(0) \vdots \boldsymbol{x}_r^\tau(0) \vdots \boldsymbol{x}_d^\tau(0)] =$ (a) $[0 \vdots 0\ 1 \vdots 0]$, (b) $[0 \vdots 0\ 1 \vdots -1]$ and (c) $[0 \vdots 0\ 1 \vdots 1]$.

the reference input satisfies

$$\begin{aligned} \dot{\boldsymbol{x}}_r(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_r(t) \\ \boldsymbol{r}(t) &= [1 \quad 0]\boldsymbol{x}_r(t). \end{aligned} \tag{52}$$

Assume the correlation of the initial states is

$$E\left(\begin{bmatrix} \boldsymbol{x}(0) \\ \hline \boldsymbol{x}_r(0) \\ \hline \boldsymbol{x}_d(0) \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(0) \\ \hline \boldsymbol{x}_r(0) \\ \hline \boldsymbol{x}_d(0) \end{bmatrix}^\tau\right) = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \hline 0 & 0 & 0 & \gamma \end{array}\right] \tag{53}$$

where $\boldsymbol{x}_d(0) = \boldsymbol{d}$ and γ is a positive number. By replacing $\boldsymbol{u}(t-1)$ by $\overline{\boldsymbol{u}}(t)$ to remove the time-delay, selecting $T = 1$, choosing

$$\phi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Gamma = [1 \quad 0] \tag{54}$$

and approximating Ψ_σ [calculated from (50) and (39)] by $\Psi_\sigma + 10^{-5}\gamma I_3$, then from theorem (3), a one-step prediction optimal sampled-data robust servomechanism controller is

$$\begin{aligned} \begin{bmatrix} \hat{\xi}((K+1)T) \\ \hline h((k+1)T) \end{bmatrix} &= \left[\begin{array}{cc|c} 1 & 1 & -3.5209 \\ 0 & 1 & -0.8018 \\ \hline 0 & 1 & -1.8748 \end{array}\right]\begin{bmatrix} \hat{\xi}(kT) \\ \hline h(kT) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ \hline 1 \end{bmatrix} e(kT) \end{aligned} \tag{55.a}$$

$$\begin{aligned} \overline{\boldsymbol{u}}(kT+\theta) &= \boldsymbol{u}((k-1)T+\theta) \\ &= [1\,\theta]\left(\hat{\xi}(kT) + \begin{bmatrix} -3.1948 \\ -1.8593 \end{bmatrix} h(kT)\right) \end{aligned} \tag{55.b}$$

where $h(0) = 0$ and $\hat{\xi}(0) = 0$ are assumed. Notice that (55) is an admissible controller. The responses of the time-delay process with this controller is shown in Fig. 2.

Example 2: Consider the following system (Rosenbrock problem [4]):

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -1/6 & 0 \\ 2/3 & 1 \\ 0 & 1/2 \end{bmatrix} \boldsymbol{u}(t),$$

$$\boldsymbol{x}(0) = 0 \tag{56. a}$$

$$\boldsymbol{y}(t) = \begin{bmatrix} 3 & -3/4 & -1/2 \\ 2 & -1 & 0 \end{bmatrix} \boldsymbol{x}(t). \tag{56. b}$$

This system is to track a sinusoidal signal described as:

$$\dot{\boldsymbol{x}}_r(t) = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix} \boldsymbol{x}_r(t) \tag{57.a}$$

$$\boldsymbol{r}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}_r(t) \tag{57.b}$$

where $\boldsymbol{x}_r(0)$ is a random vector with correlation

$$E(\boldsymbol{x}_r(0)\boldsymbol{x}_r^\tau(0)) = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix} \tag{58}$$

where γ is a positive real number. Selecting $T = 0.5$, choosing

$$\phi = \begin{bmatrix} 0 & \pi & 0 & 0 \\ -\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & -\pi & 0 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{59}$$

and approximating Ψ (calculated from (50)) by $\Psi + 10^{-5}\gamma I_7$, then from Theorem 2, a minimal-order optimal sampled-data robust servomechanism controller is

$$\hat{\xi}((k+1)T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \hat{\xi}((k+1)T) + \begin{bmatrix} 2.9014 & -12.3172 \\ 1.6216 & -4.6239 \\ -1.2756 & 7.5528 \\ -1.5642 & 3.5657 \end{bmatrix} e(kT) \tag{60.a}$$

$$u(kT+\theta) = \begin{bmatrix} \cos(\pi\theta) & \sin(\pi\theta) & 0 & 0 \\ 0 & 0 & \cos(\pi\theta) & \sin(\pi\theta) \end{bmatrix} \cdot \left(\hat{\xi}(kT) + \begin{bmatrix} -6.5989 & 6.6012 \\ 3.5575 & -3.5529 \\ -7.3429 & 7.3395 \\ 8.2651 & -8.2661 \end{bmatrix} e(kT) \right) \tag{60.b}$$

where $\hat{\xi}(0) = 0$ is assumed. The responses of the system with this controller is shown in Fig. 3.

VI. Conclusion

In this note, a new structure of sampled-data robust servomechanism controller using exponential hold is presented. The proposed structure is simple for design and can be easily implemented by digital computers. An optimal algorithm is also derived for choosing the parameters of the two important special classes. The solution of the algorithm can be solved from a discrete-time algebraic Riccati equation.

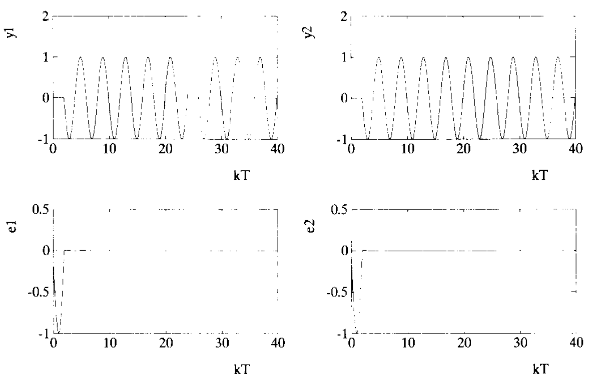

Fig. 3. The sinusoidal tracking response of Example 2 using sampled-data controller (60) with initial state $[\boldsymbol{x}^\tau(0) \vdots \boldsymbol{x}_r^\tau(0)] = [0\,0\,0 \vdots 0\,1]$, where $\boldsymbol{y} = [\boldsymbol{y}1\,\boldsymbol{y}2]^\tau$ and $\boldsymbol{e} = [e1\,e2]^\tau$ are plotted.

It is of interest that the solution of the derived algorithm does not depend on the weighting matrix of the performance index, but only on the correlations of the initial values of the system state, reference input and the disturbance. In a general robust servomechanism controller problem, the uncertain signals to be tracked or the unknown disturbance can be treated as the random vector of the initial values, so that from a statistical viewpoint, the derived algorithm can reflect the capability of treating such uncertainty.

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