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Contour of the attenuated length of an evanescent wave at constant frequency within a band gap of photonic crystal

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Abstract

The plane wave method is normally applied to determine the eigenfrequency of a two-dimensional (2D) photonic crystal. A slight change to this eigenvalue equation makes the wave number its eigenvalue providing a direct means to determine the attenuated length of the evanescent modes at the frequency within the photonic band gap. The contour of the length of attenuation of the evanescent modes in a square lattice can be determined using the proposed wave number eigenvalue equation. The wave number eigenvalue equation for the two-dimensional (3D) photonic crystal can also be obtained using a derivation similar to that for the 2D photonic crystal. Possible applications of the proposed calculation-method are presented.

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1. Introduction

Photonic crystals are generally artificial three-dimensional (3D) periodic dielectric structures in which electromagnetic waves with frequencies in a certain range cannot propagate [1]. They have unique characteristics and several useful applications, including for example, suppressed spontaneous emission [2] and thermal emission [3] and strong confinement of light in defects [4]. Several waveguides can be constructed from the defects created in the photonic crystal. The coupling between defects is dependent on the attenuation length of the electromagnetic (EM) field. The attenuation length of the EM field has not been determined in detail for any structure. However, all the penetration depth in all $\vec{k}$ directions can be obtained for a given frequency if the contours of the map of the attenuation length at a constant frequency can be obtained. This study proposed a method for so doing.

Constant-frequency contours have recently been applied to tunable photonic band gaps for tuning static electric fields or static magnetic fields [5]. Also, Luo et al. [6] applied the constant frequency contour in $\vec{k}$ space to find the condition under which all-angle negative refraction occurs in a square lattice that includes cylindrical dielectric rods in a two-dimensional (2D) photonic crystal. Most researchers begin with the frequency eigenvalue equation obtained by the plane wave method, to determine $\vec{k}$ values indirectly at a given frequency, and thus obtain the constant frequency contour. This study turns the frequency eigenvalue equation into the wave number k eigenvalue equation in any arbitrary $\vec{k}$ direction. Section 2 will derive these expressions. Section 3 presents the resulting field attenuated length in the square lattice that includes the cylindrical rods to illustrate the method. Finally, the results are discussed and a conclusion is drawn.

2. Method

The plane wave method has been developed and extensively applied to calculate the photonic band structure of the photonic crystals. Consider some key equations. The

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frequency eigenvalue equation for the magnetic field in a periodic dielectric and magnetic medium is

$$-\sum_{\vec{G}'}(\vec{k}+\vec{G})\times\epsilon^{-1}_{\vec{G}\vec{G}'}(\vec{k}+\vec{G}')\times\vec{H}_{\vec{k},\vec{G}'}=\frac{\omega^2}{c^2}\sum_{\vec{G}'}\mu_{\vec{G},\vec{G}'}\vec{H}_{\vec{k},\vec{G}'}(1)$$

where $\vec{H}(\vec{r})=\sum_{\vec{G}}e^{i(\vec{k}+\vec{G})\cdot\vec{r}}\vec{H}_{\vec{k},\vec{G}}$, the dielectric permittivity $\epsilon(\vec{r})=\sum_{\vec{G}}e^{i\vec{G}\cdot\vec{r}}\epsilon(\vec{G})$, the magnetic permeability $\mu(\vec{r})=\sum_{\vec{G}}e^{i\vec{G}\cdot\vec{r}}\mu(\vec{G})$, $\epsilon^{-1}_{\vec{G}\vec{G}'}\equiv\epsilon^{-1}(\vec{G}-\vec{G}')$ and $\mu_{\vec{G}\vec{G}'}\equiv\mu(\vec{G}-\vec{G}')$.

For a 2D photonic crystal, the propagated light exhibits E-polarization (TE) modes and H-polarization (TM) modes. E or H-polarization is E or H along the axis of the rods and in the z-direction. Also, set $k_z=0$. In Eq. (1), $(\mu,\vec{H})$ and $(\epsilon,\vec{E})$ can be exchanged for each other. The TM modes are considered here because the TE modes can be derived similarly. Eq. (1) can be written as

$$\sum_{\vec{G}'}\epsilon^{-1}_{\vec{G}\vec{G}'}(\vec{k}+\vec{G})\cdot(\vec{k}+\vec{G}')H_{z,\vec{k},\vec{G}'}=\frac{\omega^2}{c^2}\sum_{\vec{G}'}\mu_{\vec{G}\vec{G}'}H_{z,\vec{k},\vec{G}'}\qquad(2)$$

This equation is the eigen frequency equation for a given wave vector $\vec{k}$. Now, multiply both sides of Eq. (2) by $\epsilon_{\vec{G}''\vec{G}}$, and then sum over $\vec{G}$ on both sides:

$$\sum_{\vec{G},\vec{G}'}\epsilon_{\vec{G}''\vec{G}}\epsilon^{-1}_{\vec{G}\vec{G}'}[k^2+k\hat{k}\cdot(\vec{G}+\vec{G}')+\vec{G}\cdot\vec{G}']H_{z,\vec{k},\vec{G}'}$$

$$=\frac{\omega^2}{c^2}\sum_{\vec{G}'',\vec{G}}\epsilon_{\vec{G}''\vec{G}}\mu_{\vec{G}\vec{G}'}H_{z,\vec{k},\vec{G}'}\qquad(3)$$

From Eq. (3),

$$k\cdot kH_{z,\vec{k},\vec{G}''}=\sum_{\vec{G}'\vec{G}}\left[\epsilon_{\vec{G}''\vec{G}}\left(\frac{\omega^2}{c^2}\mu_{\vec{G}\vec{G}'}-\epsilon^{-1}_{\vec{G}\vec{G}'}\vec{G}\cdot\vec{G}'\right)\right]H_{z,\vec{k},\vec{G}'}$$

$$-\sum_{\vec{G}'\vec{G}}\epsilon_{\vec{G}''\vec{G}}\epsilon^{-1}_{\vec{G}\vec{G}'}\hat{k}\cdot(\vec{G}+\vec{G}')kH_{z,\vec{k},\vec{G}'}\qquad(4)$$

Combining Eq. (4) with an identity $kH_{z,\vec{k},\vec{G}''}=k\sum_{\vec{G}'}\times\delta_{\vec{G}''\vec{G}'}H_{z,k,\vec{G}'}$, yields a matrix equation,

$$\begin{pmatrix}\varnothing & \mathbf{I}\\ \mathbf{A} & \mathbf{B}\end{pmatrix}\begin{pmatrix}H_{z,\vec{k},\vec{G}'}\\ kH_{z,\vec{k},\vec{G}'}\end{pmatrix}=k\begin{pmatrix}H_{z,\vec{k},\vec{G}''}\\ kH_{z,\vec{k},\vec{G}''}\end{pmatrix}\qquad(5)$$

$$\mathbf{A}=\epsilon_{\vec{G}''\vec{G}}\left[\left(\frac{\omega^2}{c^2}\right)\mu_{\vec{G}\vec{G}'}-\epsilon^{-1}_{\vec{G}\vec{G}'}\vec{G}\cdot\vec{G}'\right],$$

$$\mathbf{B}=-\epsilon_{\vec{G}''\vec{G}}\epsilon^{-1}_{\vec{G}\vec{G}'}\hat{k}\cdot(\vec{G}+\vec{G}')$$

where (respectively) $\varnothing$ and $\mathbf{I}=(\delta_{\vec{G}''\vec{G}'})$ are the $N\times N$ zero matrix and the $N\times N$ identity matrix, $\mathbf{A}$ and $\mathbf{B}$ are $N\times N$ matrices. In Eq. (5), that two different quantities have the same subindex implies that they are summed. More specifically, in Eq. (5), the column vector has $2N$ elements and the matrix has $2N\times 2N$ elements. Here, N is the number of the reciprocal lattice vectors truncated in the plane wave expansion of $\vec{H}$. Eq. (5) is clearly an eigenvalue equation for the wave number k at a given angular frequency ω in the wave vector direction, $\hat{k}$. Hence, a constant frequency contour in k-space can be obtained from Eq. (5). Eq. (4) may be rewritten in another form and recast into a matrix equation as follows.

$$\begin{pmatrix}\varnothing & \mathbf{I}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}H_{z,\vec{k},\vec{G}'}\\ k_xH_{z,\vec{k},\vec{G}'}\end{pmatrix}=k_x\begin{pmatrix}H_{z,\vec{k},\vec{G}''}\\ k_xH_{z,\vec{k},\vec{G}''}\end{pmatrix}\qquad(6)$$

for a given k_y and angular frequency ω, where

$$\mathbf{C}=\epsilon_{\vec{G}''\vec{G}}\left[\frac{\omega^2}{C^2}\mu_{\vec{G}\vec{G}'}-\epsilon^{-1}_{\vec{G}\vec{G}'}(\vec{G}+k_y\hat{y})\cdot(\vec{G}'+k_y\hat{y})\right]$$

$$\mathbf{D}=\epsilon_{\vec{G}''\vec{G}'}\epsilon^{-1}_{\vec{G}\vec{G}'}(G_x+G'_x)$$

are $N\times N$ matrices. The wave number eigenvalue equation for a 3D photonic crystal can also be derived in a manner similar to the derivation for a 2D photonic crystal. The wave number eigenvalue equations involve $6N\times 6N$ matrix. The computation time is much greater than that for a 2D photonic crystal, which is one reason why a 2D photonic crystal was chosen to illustrate the proposed method. Eq. (6) can also be used to determine a constant frequency contour in $\vec{k}$ space. However, the eigenvalue k and k_x in Eqs. (5) and (6), respectively, may be complex [8]. Then we may get positive value of imaginary part of k. The reciprocal of the length of attenuation of the evanescent wave hence may be obtained when its frequency is within the band gap of E-polarization or H-polarization or both.

3. Results and discussion

A well-known result concerning the photonic band structure of a square lattice with a cylindrical rods is considered to illustrate the method. The chosen dielectric material is GaAs with dielectric constant $\epsilon_a=11.43$. The background is air (with a dielectric constant $\epsilon_b=1$). Here, the number of plane waves is $N=19\times 19$ and the radius of each cylindrical rod $r=0.15a$, where a is the side of the unit cell of the square lattice. Fig. 1 depicts the resulting band structure. The lowest band gap is in bands 1 and 2 in E-polarization mode. Between bands 1 and 2 is one H-polarized band. Hence, at a frequency ω of $0.4(2\pi c/a)$, the frequency is in the first band gap of E-polarized band and a wave propagates in the H-polarized mode. Therefore, the constant-frequency map of H-polarization for pure real k values exists, as plotted in Fig. 2. For E-polarization at that frequency, no pure real k value is allowed for any waves. In such a case, the allowed k values are complex. Thus, the length of the attenuation λ_p must be defined to enable an evanescent wave to be $\exp[-I_{\min}(k)\lambda_p]=\exp(-2\pi)$, where $I_{\min}(k)$ is the minimum positive value of the imaginary part of complex k. Fig. 3 shows the results of the contour map of $a/\lambda_p(\theta)$ in the main figure. The inset in the figure shows the numerical values (dots and thick curve) of the real part of k. The thick curve represents the real part of k of which the imaginary part is the smallest at each direction. The

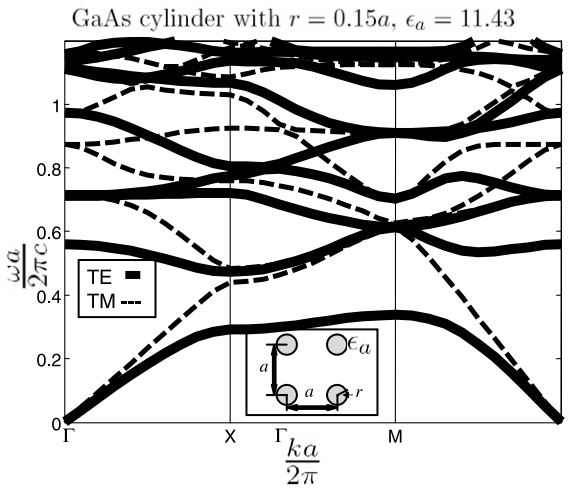

Fig. 1. Photonic band structure for a square lattice with GaAs cylindrical rods in a background of air. The length of side of the unit cell of the square lattice (shown in the inset) is a. The dielectric constant of GaAs is $\epsilon_a = 11.43$. The radius of the rod is $0.15a$. The solid lines and the dashed lines denote bands of the TE and TM modes, respectively.

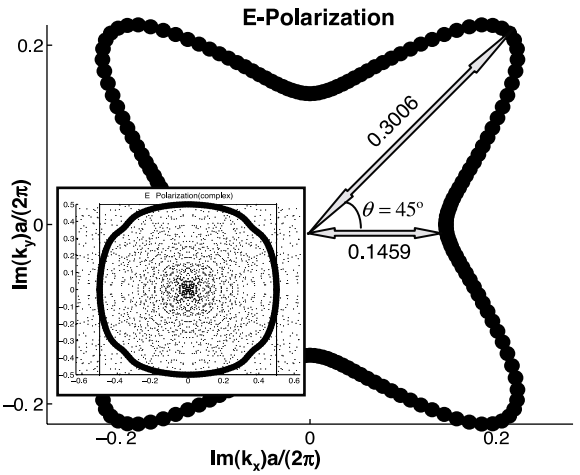

Fig. 3. The contour map of $a/\lambda_p(\theta)$ for E-polarization for $\omega = 0.4(2\pi c/a)$ in polar coordinates, where θ is measured from a horizontal line anti-clockwise and the square marks the first Brillouin Zone. The desired result is the deepest penetration at each angle, which is proportional to the imaginary part of k, so this part of k is plotted as a thick curve in the inset, in which polar coordinates are used and the length scale is $2\pi/a$. Both the dots and the thick curve denote the real part of eigenvalue k.

numerical values of real part of k calculated from Eq. (5) seem to be spread randomly as shown in dots. In fact, there exists some regular behavior. Let us think a simple situation, if the medium is air, $\vec{k}$ obeys $|\vec{k} + \vec{G}|^2 = (\omega/c)^2$. This is equivalent to $k = \hat{k}\cdot\vec{G} \pm \sqrt{|\hat{k}\cdot\vec{G}|^2 + (\omega/c)^2 - G^2}$ and most of the real part of k equals to $\hat{k}\cdot\vec{G}$. Hence, the values of real part of k for $\theta = 0°$ and $45°$ are concentrated on origin, origin and corner, respectively. For other angles, the behavior is more complicated, we don't want to discuss them deeply. Now, let us come back the main figure, it indicates that the attenuated length is $\lambda_p = 6.8540a$ along $\theta = 0°$ and $\lambda_p = 3.3272a$ along $\theta = 45°$ (diagonal direction of square lattice). For instance, when two parallel line defects that point along the edge of square lattice are separated by a distance of $7a$ or more, no coupling or cross talk occurs. Similarly, when two parallel line defects in the diagonal direction are separated by a distance of $4a$ or more from each other, then again no coupling or cross talk occurs. Hence, the map of the attenuated length in $\vec{k}$-space presents data that are valuable in designing several waveguides. The data also support the designing of waveguides without leakage from the photonic crystal.

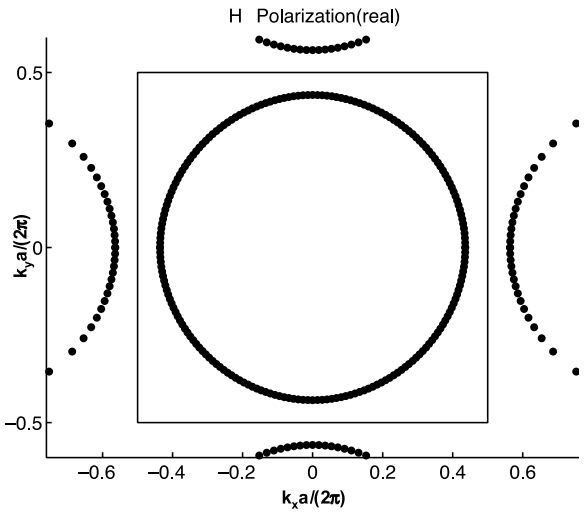

Fig. 2. Constant frequency $\omega = 0.4(2\pi c/a)$ map of H-polarization is for real k values. The square marks the first Brillouin Zone.

4. Conclusion

That more than a decade after Ho et al. [7] proposed the plane wave method to calculate the photonic band structure, no method has been proposed for directly calculating the wave number eigenvalue equation in that method is surprising. This study introduced a simple transformation of the frequency eigenvalue equation into a wave number eigenvalue equation. Such a transformation can be extended to a 3D photonic crystal. The proposed method for obtaining a constant frequency contour and a map of the length of attenuation in k-space for a 2D photonic crystal, have also been illustrated. The constant frequency contour provides valuable information for understanding some newly observed phenomena exhibited by photonic crystals. The

map of attenuated length is useful in designing the waveguides made from photonic crystal.

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