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Normal Approximation to the Distribution of the Estimated Yield Index S_{pk}

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Abstract. Process yield is the most common criterion used in the manufacturing industry for measuring process performance. A measurement index, called S_{pk}, has been proposed to calculate the yield for normal processes. The measurement index S_{pk} establishes the relationship between the manufacturing specifications and the actual process performance, which provides an exact measure on process yield. Unfortunately, the sampling distribution of the estimated S_{pk} is mathematically intractable. Therefore, process performance testing cannot be performed. In this paper; we consider a normal approximation to the distribution of the estimated S_{pk}, and investigate its accuracy computationally. We compare the critical values calculated from the approximate distribution with those obtained using the standard simulation technique, for various commonly used quality requirements. Extensive computational results are provided and analyzed. The investigation is useful to the practitioners for making decisions in testing process performance based on the yield.

Key words: critical value, process yield

1. Introduction

Process yield has longtime been a standard criterion used in the manufacturing industry as a common measure on process performance. Process yield is currently defined as the percentage of processed product unit passing the inspection. That is, the product characteristic must fall within the manufacturing tolerance. For product units rejected (nonconformities), additional costs would be incurred to the factory for scrapping or repairing the product. All passed product units are equally accepted by the producer, which requires the factory no additional cost. For processes with two-sided manufacturing specifications, the process yield can be calculated as $\%\text{Yield} = F(\text{USL}) - F(\text{LSL})$, where USL and LSL are the upper and the lower specification limits, respectively, and $F(\cdot)$ is the cumulative distribution function of the process characteristic. If the process characteristic follows the normal distribution, then the process yield can be alternatively expressed as $\text{Yield}\% = \Phi(\text{USL} - \mu)/\sigma] - \Phi[(\text{LSL} - \mu)/\sigma]$, where μ is the process mean, σ is the process standard deviation, and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

Table I. Some S_{pk} values and the corresponding nonconformities.

S_{pk}	Yield	PPM
1.00	0.9973002039	2699.796
1.10	0.9990331517	966.848
1.20	0.9996817828	318.217
1.30	0.9999038073	96.193
1.33	0.9999339267	66.073
1.40	0.9999733085	26.691
1.50	0.9999932047	6.795
1.60	0.9999984133	1.587
1.67	0.9999994557	0.544
1.70	0.9999996603	0.340
1.80	0.9999999334	0.067
1.90	0.9999999880	0.012
2.00	0.9999999980	0.002

Based on the expression of process yield, Boyles (1994) considered the yield measurement index S_{pk} for normal processes (defined in the following). The index S_{pk} establishes the relationship between the manufacturing specifications and the actual process performance, which provides an exact measure on the process yield. If $S_{pk} = c$, then the process yield can be expressed as $\%\text{Yield} = 2\Phi(3c) - 1$. Obviously, there is a one-to-one correspondence between S_{pk} and the process yield. Thus, S_{pk} provides an exact (rather than approximate) measure of the process yield. Table I summarizes the process yield, nonconformities (in PPM) as a function of the measurement index S_{pk}, for $S_{pk} = 1.00(0.1)2.00$, including the most commonly-used performance requirements, 1.00, 1.33, 1.50, 1.67, and 2.00. For example, if a process has yield measure $S_{pk} = 1.33$, then the corresponding nonconformities is roughly 66 PPM (parts per million).

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\text{USL}-\mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu-\text{LSL}}{\sigma}\right)\right\}.$$

2. Approximate Distribution of the Estimated S_{pk}

To estimate the yield measurement index S_{pk}, we consider the following natural estimator $\hat{S}_{pk}$, where $\bar{X} = (\sum_{i=1}^{n} X_i)/n$, and $S = [(n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2]^{1/2}$ are

the sample mean, and the sample standard deviation, the conventional estimators of μ and σ, respectively, which may be obtained from a stable process.

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\text{USL} - \bar{X}}{S}\right) + \frac{1}{2}\Phi\left(\frac{\bar{X} - \text{LSL}}{S}\right)\right\}.$$

The exact distribution of S_{pk} is mathematically intractable. Therefore, testing of the process performance cannot be performed. On the other hand, Lee et al. (2002) obtained a normal approximation to the distribution of S_{pk} using Taylor expansion technique. By taking the first order of the Taylor expansion, it is shown that the estimator can be expressed approximately as:

$$\hat{S}_{pk} \cong S_{pk} + \frac{1}{6\sqrt{n}}\frac{W}{\phi(3S_{pk})},$$

where

$$\begin{aligned} W &= \frac{\sqrt{n}(S^2 - \sigma^2)}{\sigma^2}\left\{\frac{\text{USL} - \mu}{2\sigma}\phi\left(\frac{\text{USL} - \mu}{\sigma}\right) + \frac{\mu - \text{LSL}}{2\sigma}\phi\left(\frac{\mu - \text{LSL}}{\sigma}\right)\right\} \\ &\quad - \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\left\{\phi\left(\frac{\text{USL} - \mu}{\sigma}\right) - \phi\left(\frac{\mu - \text{LSL}}{\sigma}\right)\right\} \\ &= \frac{\sqrt{n}(S^2 - \sigma^2)}{\sigma^2}\left\{\frac{d - (\mu - m)}{2\sigma}\phi\left(\frac{d - (\mu - m)}{\sigma}\right)\right. \\ &\quad \left. + \frac{d + (\mu - m)}{2\sigma}\phi\left(\frac{d + (\mu - m)}{\sigma}\right)\right\} \\ &\quad - \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\left\{\phi\left(\frac{d - (\mu - m)}{\sigma}\right) - \phi\left(\frac{d + (\mu - m)}{\sigma}\right)\right\}. \end{aligned}$$

We note that $W = (\sqrt{n/2})[a(S^2 - \sigma^2)/\sigma^2] - \sqrt{n}[b(\bar{X} - \mu)/\sigma]$ for $\mu < m$, and $W = (\sqrt{n/2})[a(S^2 - \sigma^2)/\sigma^2] + \sqrt{n}[b(\bar{X} - \mu)/\sigma]$ for $\mu > m$, where a and b are functions of μ and σ, as defined in the following, and ϕ is the probability density function of the standard normal distribution $N(0, 1)$. Thus, the statistic W is normally distributed as $N(0, a^2 + b^2)$, and the natural estimator $\hat{S}_{pk}$ is approximately distributed as $N(\mu_S, \sigma_S^2)$, where $E(\hat{S}_{pk}) = \mu_S = S_{pk}$ and $\text{Var}(\hat{S}_{pk}) = \sigma_S^2 = (a^2+b^2)\{36n[\phi(3S_{pk})]^2\}^{-1}$, which can be expressed as functions of the widely used precision index $C_p = (\text{USL} - \text{LSL})/6\sigma$, and the accuracy index $C_a = 1 - |\mu - m|/d$.

The parameter C_p is defined as $C_p = (\text{USL} - \text{LSL})/6\sigma$, a function of the process standard deviation, which measures the overall process variation relative to the specification tolerance therefore only reflecting process potential. The parameter C_a is defined as $C_a = 1 - |\mu - m|/d$ (see Pearn et al. (1998)), a function of the process mean, which measures the degree of process centering,

where $m = (\text{USL} + \text{LSL})/2$ is the mid- point between the upper and the lower specification limits, and $d = (\text{USL} - \text{LSL})/2$ is half of the length of the specification interval. The parameter C_a alerts the user if the process mean deviates from its target value. In fact, a mathematical relationship among the three measurements can be established as $\Phi(3S_{pk}) = \{\Phi(3C_pC_a) + \Phi[3C_p(2 - C_a)]\}/2$.

$$
\begin{aligned}
a &= \frac{1}{\sqrt{2}}\left\{\frac{\text{USL}-\mu}{\sigma}\phi\left(\frac{\text{USL}-\mu}{\sigma}\right) + \frac{\mu-\text{LSL}}{\sigma}\phi\left(\frac{\mu-\text{LSL}}{\sigma}\right)\right\} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{d-(\mu-m)}{\sigma}\phi\left(\frac{d-(\mu-m)}{\sigma}\right) + \frac{d+(\mu-m)}{\sigma}\phi\left(\frac{d+(\mu-m)}{\sigma}\right)\right\} \\
&= \frac{1}{\sqrt{2}}\{3C_p(2-C_a)\phi(3C_p(2-C_a)) + 3C_pC_a\phi(3C_pC_a)\}. \\
b &= \phi\left(\frac{\text{USL}-\mu}{\sigma}\right) - \phi\left(\frac{\mu-\text{LSL}}{\sigma}\right) \\
&= \phi\left(\frac{d-(\mu-m)}{\sigma}\right) - \phi\left(\frac{d+(\mu-m)}{\sigma}\right) \\
&= \phi\{3C_p(2-C_a)\} - \phi(3C_pC_a).
\end{aligned}
$$

The probability density function of the approximate distribution, $N(\mu_S, \sigma_S^2)$, can be expressed as:

$$
\begin{aligned}
f(x) = \sqrt{\frac{18n}{\pi}}\frac{\phi(3S_{pk})}{\sqrt{a^2+b^2}}\exp\Bigg[-\frac{18n(\phi(3S_{pk}))^2}{a^2+b^2} \\
\times(x - S_{pk})^2\Bigg], \quad -\infty < x < \infty.
\end{aligned}
$$

3. Calculation of the Critical Values

The formula of the normal approximation obtained for the distribution of $\hat{S}_{pk}$ is rather complicate, and the calculation is cumbersome to deal with. For the approximation to be useful to the practitioners, we calculate the critical values c_o computationally using the Maple-V programming software (see Appendix). Since the critical values c_o is a function of the parameters C_p and C_a, we have considered the factor of the two parameters in the calculations to ensure that the critical values obtained are reliable.

3.1. NORMAL APPROXIMATION

Tables II(a)–II(e) display the parameters of the process characteristics used in the critical value calculations, covering the most commonly used performance requirements $S_{pk} = 1.00$ (capable), 1.33 (satisfactory), 1.50 (good), 1.67 (excellent), and 2.00 (super). Table II(a) summarizes the process characteristics with $C_p =$

Table 2a. $S_{pk} = 1.00$ with $C_p = 1.0(0.1)2.0$, the corresponding μ, σ, C_a, and the calculated a, b, and $n\text{Var}(\hat{S}_{pk})$. Note: $\phi(3S_{pk}) = 0.004431848$.

C_p	C_a	μ	σ	a	b	$n\text{Var}(\hat{S}_{pk})$
1.0	1.000000000	15.00000000	1.666666667	0.018802740	0.000000000	0.500000072
1.1	0.845650984	15.77174508	1.515151515	0.016792051	−0.007843357	0.485784252
1.2	0.772993431	16.13503285	1.388888889	0.016414728	−0.008282188	0.478071906
1.3	0.713386252	16.43306874	1.282051282	0.016369929	−0.008317128	0.476815000
1.4	0.662422888	16.68788556	1.190476190	0.016266525	−0.008319133	0.476704578
1.5	0.618261111	16.90869445	1.111111111	0.016366355	−0.008319213	0.476698592
1.6	0.579619785	17.10190108	1.041666667	0.016366350	−0.008319216	0.476698431
1.7	0.545524504	17.27237748	0.980392157	0.016366349	−0.008319216	0.476698383
1.8	0.515217587	17.42391207	0.925925926	0.016366349	−0.008319216	0.476698383
1.9	0.488100872	17.55949564	0.877192982	0.016366349	−0.008319216	0.476698383
2.0	0.463695828	17.68152086	0.833333333	0.016366349	−0.008319216	0.476698383

Table 2b. $S_{pk} = 1.33$ with $C_p = 1.33$, $1.4(0.1)2.3$, the corresponding μ, σ, C_a, and the calculated a, b, and $n\text{Var}(\hat{S}_{pk})$. Note: $\phi(3S_{pk}) = 0.000139285$.

C_p	C_a	μ	σ	a	b	$n\text{Var}(\hat{S}_{pk})$
1.33	1.000000000	15.00000000	1.250000000	0.000785945	0.000000000	0.884449647
1.4	0.912324580	15.43837710	1.190476190	0.000738619	−0.000246892	0.868419183
1.5	0.849520868	15.75239566	1.111111111	0.000725580	−0.000266999	0.855878025
1.6	0.796341133	16.01829434	1.041666667	0.000724513	−0.000267996	0.854426361
1.7	0.749494830	16.25252585	0.980392157	0.000724459	−0.000268034	0.854343495
1.8	0.707856167	16.46071917	0.925925926	0.000724458	−0.000268034	0.854341420
1.9	0.670600579	16.64699711	0.877192982	0.000724457	−0.000268034	0.854339345
2.0	0.637070550	16.81464725	0.833333333	0.000724457	−0.000268034	0.854339345
2.1	0.606733857	16.96633072	0.793650794	0.000724457	−0.000268034	0.854339345
2.2	0.579155045	17.10422478	0.757575758	0.000724457	−0.000268034	0.854339345
2.3	0.553974391	17.23012805	0.724637681	0.000724457	−0.000268034	0.854339345

1.0(01)2.0, the corresponding C_a, and (μ, σ) for $S_{pk} = 1.00$. Table II(b) summarizes the process characteristics with $C_p = 1.33$, $1.4(0.1)2.3$, the corresponding C_a, and (μ, σ) for $S_{pk} = 1.33$. Table II(c) summarizes the process characteristics with $C_p = 1.5(0.1)2.5$, the corresponding C_a, and (μ, σ) for $S_{pk} = 1.50$. Table II(d) summarizes the process characteristics $C_p = 1.67$, $1.7(0.1)2.6$, the corresponding C_a, and (μ, σ) for $S_{pk} = 1.67$. Table II(e) summarizes the process characteristics $C_p = 2.0(0.1)3.0$, the corresponding C_a, and (μ, σ) for $S_{pk} = 2.00$.

Table 2c. $S_{pk} = 1.50$ with $C_p = 1.5(0.1)2.5$, the corresponding μ, σ, C_a, and the calculated a, b, and $n\text{Var}(\hat{S}_{pk})$. Note: $\phi(3S_{pk}) = 0.000015984$.

C_p	C_a	μ	σ	a	b	$n\text{Var}(\hat{S}_{pk})$
1.5	1.000000000	15.00000000	1.111111111	0.000101720	0.000000000	1.124965647
1.6	0.906849563	15.46575219	1.041666667	0.000095910	−0.000030236	1.099522577
1.7	0.853029665	15.73485168	0.980392157	0.000095365	−0.000030966	1.093046152
1.8	0.805624734	15.97187633	0.925925926	0.000095336	−0.000030991	1.092613277
1.9	0.763223120	16.18388440	0.877192982	0.000095335	−0.000030991	1.092592546
2.0	0.725061959	16.37469021	0.833333333	0.000095335	−0.000030991	1.092592546
2.1	0.690535199	16.54732401	0.793650794	0.000095335	−0.000030991	1.092592546
2.2	0.659147235	16.70426383	0.757575758	0.000095335	−0.000030991	1.092592546
2.3	0.630488660	16.84755670	0.724637681	0.000095335	−0.000030991	1.092592546
2.4	0.604218299	16.97890851	0.694444444	0.000095335	−0.000030991	1.092592546
2.5	0.580049567	17.09975217	0.666666667	0.000095335	−0.000030991	1.092592546

Table 2d. $S_{pk} = 1.67$ with $C_p = 1.7, 1.7(0.1)2.6$, the corresponding μ, σ, C_a, and the calculated a, b, and $n\text{Var}(\hat{S}_{pk})$. Note: $\phi(3S_{pk}) = 0.000001414$.

C_p	C_a	μ	σ	a	b	$n\text{Var}(\hat{S}_{pk})$
1.67	1.000000000	15.00000000	1.000000000	0.000010019	0.000000000	1.394592848
1.7	0.960124663	15.19937669	0.980392157	0.000009753	−0.000002168	1.386824835
1.8	0.902865766	15.48567117	0.925925926	0.000009519	−0.000002740	1.363175026
1.9	0.855248895	15.72375553	0.877192982	0.000009505	−0.000002757	1.360773093
2.0	0.812484428	15.93757786	0.833333333	0.000009504	−0.000002757	1.360508999
2.1	0.773794664	16.13102668	0.793650794	0.000009504	−0.000002757	1.360508999
2.2	0.738622179	16.30688911	0.757575758	0.000009504	−0.000002757	1.360508999
2.3	0.706508171	16.46745915	0.724637681	0.000009504	−0.000002757	1.360508999
2.4	0.677070331	16.61464835	0.694444444	0.000009504	−0.000002757	1.360508999
2.5	0.649987517	16.75006242	0.666666667	0.000009504	−0.000002757	1.360508999
2.6	0.624987997	16.87506002	0.641025641	0.000009504	−0.000002757	1.360508999

Table III displays the critical values c_o computed from the normal approximation. We note that for given fixed performance requirement S_{pk}, the differences among those calculated critical values corresponding to various parameter values of C_p and C_a, are sufficiently small and can be neglected. To justify this result, we also calculated the main factor in the variance of the normal approximation, $(a^2 + b^2)\{36[\phi(3S_{pk})]^2\}^{-1}$ which determines the magnitude of the critical value c_o, with different parameter values of C_p and C_a. The results are tabulated in

Table 2e. $S_{pk} = 2.00$ with $C_p = 2.0(0.1)3.0$, the corresponding μ, σ, C_a, and the calculated a, b, and $n\text{Var}(\hat{S}_{pk})$. Note: $\phi(3S_{pk}) = 0.000000006$.

C_p	C_a	μ	σ	a	b	$n\text{Var}(\hat{S}_{pk})$
2.0	1.000000000	15.00000000	0.833333333	0.000000052	0.000000000	2.086419753
2.1	0.934480725	15.32759638	0.793650794	0.000000050	−0.000000012	2.040123457
2.2	0.891884461	15.5405777	0.757575758	0.000000050	−0.000000012	2.040123457
2.3	0.853105312	15.73447344	0.724637681	0.000000050	−0.000000012	2.040123457
2.4	0.817559242	15.91220379	0.694444444	0.000000050	−0.000000012	2.040123457
2.5	0.784856872	16.07571564	0.666666667	0.000000050	−0.000000012	2.040123457
2.6	0.754670069	16.22664966	0.641025641	0.000000050	−0.000000012	2.040123457
2.7	0.726719326	16.36640337	0.617283951	0.000000050	−0.000000012	2.040123457
2.8	0.700765064	16.49617468	0.595238095	0.000000050	−0.000000012	2.040123457
2.9	0.676600752	16.61699624	0.574712644	0.000000050	−0.000000012	2.040123457
3.0	0.654047393	16.72976304	0.555555556	0.000000050	−0.000000012	2.040123457

Tables II(a)–II(e), which indicate that the factor $(a^2 + b^2)\{36[\phi(3S_{pk})]\}^{-1}$ is insensitive to the value changes of C_p and C_a in all cases, except for $S_{pk} = C_p$ ($C_a = 1.00$, the process is perfectly centered). Consequently, the critical values c_o may be considered as a constant, which is independent of the process characteristics C_p and C_a for fixed performance requirement S_{pk}. Such behavior of the normal approximation can be expected, as it is a linear approximation obtained from taking the first term in the Taylor expansion.

3.2. SIMULATION TECHNIQUE

To assess the accuracy of the normal approximation, we also obtain the critical values using the standard simulation technique to compare with the critical values obtained from the normal approximation, under the same performance requirements. We note that the natural estimator S_{pk} can be rewritten and expressed as a function of the two parameters C_p and C_a. In fact, a mathematical relationship among the three measurements can be established as $\Phi(3S_{pk}) = \{\Phi(3C_pC_a) + \Phi[3C_p(2 - C_a)]\}/2$.

$$
\begin{aligned}
S_{pk} &= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\text{USL} - \mu}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{\mu - \text{LSL}}{\sigma}\right)\right\} \\
&= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{d - |\mu - m|}{\sigma}\right) + \frac{1}{2}\Phi\left(\frac{d + |\mu - m|}{\sigma}\right)\right\} \\
&= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{1 - |\mu - m|/d}{\sigma/d}\right) + \frac{1}{2}\Phi\left(\frac{1 + |\mu - m|/d}{\sigma/d}\right)\right\}
\end{aligned}
$$

Table III. Approximate c_o for various S_{pk}, $n = 5(5)200$, and $\alpha = 0.05, 0.025, 0.01$.

n	1.00			1.33			1.50			1.67			2.00		
	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
5	1.52	1.62	1.74	2.02	2.15	2.31	2.28	2.43	2.60	2.54	2.71	2.90	3.04	3.24	3.47
10	1.37	1.44	1.52	1.82	1.91	2.02	2.05	2.16	2.28	2.29	2.40	2.54	2.74	2.88	3.04
15	1.30	1.36	1.43	1.73	1.81	1.90	1.95	2.04	2.14	2.17	2.27	2.38	2.60	2.72	2.85
20	1.26	1.31	1.37	1.68	1.74	1.82	1.89	1.97	2.05	2.11	2.19	2.29	2.52	2.62	2.74
25	1.23	1.28	1.33	1.64	1.70	1.77	1.85	1.92	1.99	2.06	2.13	2.22	2.47	2.55	2.66
30	1.21	1.25	1.30	1.61	1.67	1.73	1.82	1.88	1.95	2.03	2.09	2.17	2.43	2.51	2.60
35	1.20	1.23	1.28	1.59	1.64	1.70	1.80	1.85	1.92	2.00	2.06	2.14	2.39	2.47	2.56
40	1.18	1.22	1.26	1.58	1.62	1.68	1.78	1.83	1.89	1.98	2.04	2.11	2.37	2.44	2.52
45	1.17	1.21	1.25	1.56	1.61	1.66	1.76	1.81	1.87	1.96	2.02	2.08	2.35	2.41	2.49
50	1.16	1.20	1.23	1.55	1.59	1.64	1.75	1.79	1.85	1.95	2.00	2.06	2.33	2.39	2.47
55	1.16	1.19	1.22	1.54	1.58	1.63	1.74	1.78	1.83	1.93	1.98	2.04	2.31	2.37	2.44
60	1.15	1.18	1.21	1.53	1.57	1.61	1.73	1.77	1.82	1.92	1.97	2.03	2.30	2.36	2.43
65	1.14	1.17	1.20	1.52	1.56	1.60	1.72	1.76	1.81	1.91	1.96	2.01	2.29	2.34	2.41
70	1.14	1.17	1.20	1.52	1.55	1.59	1.71	1.75	1.80	1.90	1.95	2.00	2.28	2.33	2.39
75	1.13	1.16	1.19	1.51	1.54	1.58	1.70	1.74	1.79	1.89	1.94	1.99	2.27	2.32	2.38
80	1.13	1.16	1.18	1.50	1.54	1.58	1.70	1.73	1.78	1.89	1.93	1.98	2.26	2.31	2.37
85	1.13	1.15	1.18	1.50	1.53	1.57	1.69	1.73	1.77	1.88	1.92	1.97	2.25	2.30	2.36
90	1.12	1.15	1.17	1.49	1.52	1.56	1.68	1.72	1.76	1.88	1.91	1.96	2.25	2.29	2.35
95	1.12	1.14	1.17	1.49	1.52	1.55	1.68	1.71	1.75	1.87	1.91	1.95	2.24	2.28	2.34
100	1.12	1.14	1.16	1.49	1.51	1.55	1.67	1.71	1.75	1.86	1.90	1.95	2.23	2.28	2.33
105	1.11	1.14	1.16	1.48	1.51	1.54	1.67	1.70	1.74	1.86	1.90	1.94	2.23	2.27	2.32
110	1.11	1.13	1.16	1.48	1.51	1.54	1.67	1.70	1.74	1.86	1.89	1.93	2.22	2.26	2.31
115	1.11	1.13	1.15	1.47	1.50	1.53	1.66	1.69	1.73	1.85	1.89	1.93	2.22	2.26	2.31
120	1.11	1.13	1.15	1.47	1.50	1.53	1.66	1.69	1.73	1.85	1.88	1.92	2.21	2.25	2.30
125	1.10	1.12	1.15	1.47	1.50	1.53	1.66	1.69	1.72	1.84	1.88	1.92	2.21	2.25	2.29
130	1.10	1.12	1.14	1.47	1.49	1.52	1.65	1.68	1.72	1.84	1.87	1.91	2.20	2.24	2.29
135	1.10	1.12	1.14	1.46	1.49	1.52	1.65	1.68	1.71	1.84	1.87	1.91	2.20	2.24	2.28

Table III. Continued.

n	1.00			1.33			1.50			1.67			2.00		
	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
140	1.10	1.12	1.14	1.46	1.49	1.52	1.65	1.68	1.71	1.83	1.87	1.90	2.20	2.23	2.28
145	1.10	1.12	1.14	1.46	1.48	1.51	1.65	1.67	1.71	1.83	1.86	1.90	2.19	2.23	2.27
150	1.10	1.11	1.13	1.46	1.48	1.51	1.64	1.67	1.70	1.83	1.86	1.89	2.19	2.23	2.27
155	1.09	1.11	1.13	1.45	1.48	1.51	1.64	1.67	1.70	1.83	1.86	1.89	2.19	2.22	2.26
160	1.09	1.11	1.13	1.45	1.48	1.50	1.64	1.66	1.70	1.82	1.85	1.89	2.18	2.22	2.26
165	1.09	1.11	1.13	1.45	1.47	1.50	1.64	1.66	1.69	1.82	1.85	1.88	2.18	2.22	2.26
170	1.09	1.11	1.13	1.45	1.47	1.50	1.63	1.66	1.69	1.82	1.85	1.88	2.18	2.21	2.25
175	1.09	1.11	1.12	1.45	1.47	1.50	1.63	1.66	1.69	1.82	1.85	1.88	2.18	2.21	2.25
180	1.09	1.10	1.12	1.45	1.47	1.49	1.63	1.66	1.68	1.82	1.84	1.88	2.17	2.21	2.25
185	1.09	1.10	1.12	1.44	1.47	1.49	1.63	1.65	1.68	1.81	1.84	1.87	2.17	2.20	2.24
190	1.08	1.10	1.12	1.44	1.46	1.49	1.63	1.65	1.68	1.81	1.84	1.87	2.17	2.20	2.24
195	1.08	1.10	1.12	1.44	1.46	1.49	1.63	1.65	1.68	1.81	1.84	1.87	2.17	2.20	2.24
200	1.08	1.10	1.12	1.44	1.46	1.49	1.62	1.65	1.67	1.81	1.83	1.86	2.16	2.20	2.23

$$= \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi[3C_pC_a] + \frac{1}{2}\Phi[3C_p(2-C_a)]\right\}.$$

The simulation was carried out using the SAS programming software, with $N = 10000$ replications for each sample size of n, on the same set of performance requirements $S_{pk} = 1.00, 1.33, 1.50, 1.67$, and 2.00, displayed in Tables II(a)–II(e). The simulation results indicate that the critical values are more sensitive to the two parameters C_p and C_a than those from the normal approximation. For example, given fixed $S_{pk} = 1.67$ with $n = 20$, $c_o = 2.24$ for $C_p = 1.67$, and $c_o = 2.30$ for $C_p = 1.9$. For practical purpose, we may take the maximal values of c_o among those parameters of C_p and C_a we investigated, to obtain conservative bounds on the critical values for reliability purpose. This approach ensures that the decisions made based on the critical values have the risk of wrongly concluding an incapable process as capable, no greater than the preset type I error α. Table IV summarizes the critical values c_o (the maximal ones among those with different C_p and C_a) obtained from the simulation.

4. Accuracy of the Normal Approximation

Table V displays a comparison of the critical values c_o generated by the normal approximation and the simulation technique for various selected sample sizes $n = 5(5)50$, 60, 65, 75, 90, 110, 130, and 150, with risk $\alpha = 0.05$. It is noted that the normal approximation significantly under- approximates the critical values, particularly, for small sample sizes $n < 40$, as the magnitude of the underestimation exceeds 0.10. Therefore, for short run applications (such as accepting a supplier providing short production runs in QS-9000 certification), one should avoid using the normal approximation. It is also noted that the underestimation can be as large as 0.07 for $n = 50$, 0.03 for $n = 110$, and 0.02 for $n = 150$. Therefore, in real applications a sample of size greater than 150 is recommended.

4.1. CONVERGENCE OF THE APPROXIMATION

Table VI displays the sample sizes required for the normal approximation to converge to S_{pk} within a sampling error less than 0.10, 0.09, 0.08, 0.07, 0.06, 0.05, 0.04, 0.03, 0.02, 0.01, 0.00 respectively (with accuracy up to the second decimal point, or 5×10^{-3}). For example, for $S_{pk} = 1.33$ with risk $\alpha = 0.05$, a sample size of $n \geq 95909$ ensures that the sampling error is no greater than 5×10^{-3} which is negligible. Thus, if $\hat{S}_{pk} > 1.33$, then we may conclude that the actual performance $S_{pk} > 1.33$. The investigation is not for practical purpose. But, the computations illustrate the behavior, and the rate of convergence for the normal approximation.

Table IV. Simulated c_o for various S_{pk}, n, $n = 5(5)200$, and $\alpha = 0.05, 0.025, 0.01$.

n	1.00			1.33			1.50			1.67			2.00		
	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
5	2.06	2.32	2.55	3.10	3.77	4.86	3.51	4.26	5.72	3.93	4.88	6.24	4.81	5.91	7.44
10	1.62	1.78	2.05	2.18	2.45	2.80	2.48	2.74	3.12	2.74	3.03	3.46	3.26	3.66	4.19
15	1.46	1.57	1.75	1.94	2.10	2.33	2.19	2.37	2.59	2.44	2.64	2.88	2.91	3.14	3.47
20	1.37	1.45	1.57	1.82	1.94	2.11	2.05	2.19	2.37	2.30	2.47	2.68	2.74	2.92	3.15
25	1.31	1.38	1.48	1.75	1.84	1.98	1.98	2.09	2.24	2.20	2.33	2.49	2.63	2.77	2.97
30	1.28	1.34	1.43	1.70	1.79	1.90	1.93	2.02	2.15	2.14	2.25	2.40	2.57	2.71	2.86
35	1.25	1.31	1.38	1.67	1.74	1.84	1.89	1.98	2.08	2.10	2.19	2.32	2.51	2.62	2.76
40	1.23	1.28	1.35	1.64	1.72	1.80	1.85	1.93	2.04	2.06	2.15	2.25	2.47	2.57	2.71
45	1.22	1.26	1.32	1.61	1.68	1.76	1.82	1.90	1.98	2.02	2.12	2.22	2.43	2.53	2.66
50	1.20	1.24	1.30	1.60	1.66	1.73	1.80	1.87	1.96	2.01	2.08	2.18	2.40	2.40	2.61
55	1.19	1.23	1.28	1.58	1.65	1.71	1.79	1.86	1.93	1.99	2.06	2.14	2.38	2.47	2.57
60	1.18	1.22	1.27	1.57	1.63	1.69	1.77	1.83	1.91	1.98	2.04	2.13	2.36	2.44	2.55
65	1.17	1.20	1.26	1.56	1.61	1.67	1.76	1.82	1.89	1.96	2.02	2.10	2.34	2.42	2.51
70	1.16	1.20	1.24	1.55	1.60	1.66	1.77	1.80	1.87	1.95	2.01	2.08	2.33	2.40	2.50
75	1.15	1.19	1.23	1.54	1.59	1.65	1.74	1.79	1.85	1.94	2.00	2.07	2.31	2.39	2.48
80	1.15	1.18	1.22	1.53	1.58	1.63	1.73	1.78	1.93	1.93	1.98	2.05	2.31	2.37	2.46
85	1.14	1.17	1.21	1.53	1.57	1.63	1.72	1.77	1.83	1.92	1.97	2.04	2.30	2.36	2.43
90	1.14	1.17	1.21	1.52	1.56	1.61	1.71	1.76	1.81	1.91	1.96	2.02	2.28	2.34	2.42
95	1.14	1.17	1.20	1.51	1.55	1.61	1.71	1.75	1.80	1.90	1.95	2.01	2.27	2.34	2.42
100	1.13	1.16	1.20	1.50	1.54	1.59	1.70	1.74	1.80	1.89	1.94	2.00	2.27	2.33	2.41
105	1.13	1.16	1.19	1.50	1.54	1.58	1.70	1.74	1.79	1.89	1.93	1.99	2.26	2.32	2.38
110	1.13	1.15	1.19	1.50	1.53	1.58	1.69	1.73	1.78	1.89	1.93	1.98	2.25	2.31	2.37
115	1.12	1.15	1.18	1.49	1.53	1.57	1.69	1.73	1.77	1.88	1.99	1.97	2.25	2.30	2.36
120	1.12	1.14	1.18	1.49	1.52	1.57	1.68	1.72	1.76	1.87	1.91	1.96	2.24	2.29	2.36
125	1.12	1.14	1.18	1.49	1.52	1.56	1.68	1.71	1.76	1.86	1.90	1.95	2.24	2.29	2.34
130	1.12	1.14	1.17	1.48	1.52	1.56	1.68	1.71	1.75	1.86	1.90	1.95	2.23	2.28	2.34
135	1.11	1.14	1.17	1.48	1.51	1.55	1.67	1.70	1.75	1.86	1.90	1.94	2.23	2.27	2.33

Table IV. Continued.

n	1.00			1.33			1.50			1.67			2.00		
	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01	0.05	0.025	0.01
140	1.11	1.13	1.16	1.48	1.51	1.55	1.67	1.70	1.74	1.86	1.89	1.94	2.22	2.27	2.33
145	1.11	1.13	1.16	1.48	1.51	1.54	1.66	1.70	1.74	1.85	1.89	1.93	2.22	2.26	2.32
150	1.11	1.13	1.15	1.47	1.50	1.54	1.66	1.69	1.73	1.85	1.89	1.93	2.21	2.25	2.31
155	1.10	1.12	1.15	1.47	1.50	1.53	1.66	1.69	1.73	1.84	1.88	1.92	2.21	2.25	2.30
160	1.10	1.12	1.15	1.47	1.50	1.53	1.65	1.69	1.72	1.84	1.88	1.92	2.21	2.25	2.30
165	1.10	1.12	1.14	1.46	1.49	1.52	1.65	1.68	1.72	1.84	1.87	1.91	2.20	2.24	2.29
170	1.10	1.12	1.14	1.46	1.49	1.52	1.65	1.68	1.71	1.84	1.87	1.91	2.20	2.24	2.29
175	1.10	1.11	1.14	1.46	1.49	1.52	1.65	1.68	1.71	1.83	1.87	1.91	2.20	2.24	2.28
180	1.10	1.11	1.14	1.46	1.48	1.52	1.65	1.67	1.71	1.83	1.86	1.90	2.19	2.23	2.28
185	1.09	1.11	1.13	1.46	1.48	1.52	1.64	1.67	1.71	1.83	1.86	1.90	2.19	2.23	2.28
190	1.09	1.11	1.13	1.45	1.48	1.51	1.64	1.67	1.70	1.83	1.86	1.89	2.19	2.22	2.27
195	1.09	1.11	1.13	1.45	1.47	1.51	1.64	1.67	1.70	1.82	1.85	1.89	2.18	2.22	2.27
200	1.09	1.11	1.13	1.45	1.47	1.51	1.64	1.67	1.70	1.82	1.85	1.89	2.18	2.22	2.26

Table V. A comparison of c_o between the normal approximation and the simulation with $\alpha = 0.05$.

n	1.00	1.33	1.50	1.67	2.00					
5	1.52	2.06	2.02	3.10	2.28	3.51	2.54	3.93	3.04	4.81
10	1.37	1.62	1.82	2.18	2.05	2.48	2.29	2.74	2.74	3.26
15	1.30	1.46	1.73	1.94	1.95	2.19	2.17	2.44	2.60	2.91
20	1.26	1.37	1.68	1.82	1.89	2.05	2.11	2.30	2.52	2.74
25	1.23	1.31	1.64	1.75	1.85	1.98	2.06	2.20	2.47	2.63
30	1.21	1.28	1.61	1.70	1.82	1.93	2.03	2.14	2.43	2.57
35	1.20	1.25	1.59	1.67	1.80	1.89	2.00	2.10	2.39	2.51
40	1.18	1.23	1.58	1.64	1.78	1.85	1.98	2.06	2.37	2.47
45	1.17	1.22	1.56	1.61	1.76	1.82	1.96	2.03	2.35	2.43
50	1.16	1.20	1.55	1.60	1.75	1.80	1.95	2.01	2.33	2.40
60	1.15	1.18	1.53	1.57	1.73	1.77	1.92	1.98	2.30	2.36
65	1.14	1.17	1.52	1.56	1.72	1.76	1.91	1.96	2.29	2.34
75	1.13	1.15	1.51	1.54	1.70	1.74	1.89	1.94	2.27	2.31
90	1.12	1.14	1.49	1.52	1.68	1.71	1.88	1.91	2.25	2.28
110	1.11	1.13	1.48	1.50	1.67	1.69	1.86	1.89	2.22	2.25
130	1.10	1.12	1.47	1.48	1.65	1.68	1.84	1.86	2.20	2.23
150	1.10	1.11	1.46	1.47	1.64	1.66	1.83	1.85	2.19	2.21

Table VI. Sample sizes required for the normal approximation to converge.

	α	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
1.00	0.05	124	151	188	241	321	449	670	1108	2174	6053	54220
	0.025	176	214	267	342	456	637	951	1573	3086	8594	76984
	0.01	248	301	375	482	642	897	1340	2216	4347	12108	108455
1.33	0.05	220	266	332	426	568	793	1185	1959	3845	10707	95909
	0.025	312	377	471	605	806	1126	1682	2782	5458	15202	136176
	0.01	439	531	664	853	1135	1586	2369	3919	7690	21417	191846
1.50	0.05	279	338	422	542	722	1009	1507	2492	4890	13619	121994
	0.025	396	480	599	770	1025	1432	2139	3538	6943	19336	173212
	0.01	558	676	844	1084	1444	2017	3014	4985	9781	27241	244024
1.67	0.05	346	419	523	672	895	1250	1868	3089	6061	16881	151213
	0.025	491	595	743	954	1270	1775	2652	4386	8606	23968	214699
	0.01	692	838	1046	1344	1789	2500	3736	6179	12124	33766	302470
2.00	0.05	496	601	750	964	1283	1793	2679	4430	8692	24211	216878
	0.025	704	853	1065	1368	1822	2545	3803	6290	12342	34376	307933
	0.01	992	1201	1500	1927	2566	3585	5357	8862	17388	48429	433819

5. An Application Example

In the following, we present an example of the LCM (liquid crystal module) manufacturing process. The example we investigated was taken from a manufacturing factory making the Liquid Crystal Display Module (LCD Module). The LCD Module is one of the key components used in many high-tech electronic commercial devices for the display function, such as the cellular phone, the PDA (personal digital assistant), the pocket calculator, digital watch, automobile accessory visual displays, and many others. Three key components make the LCD Module functions properly Those include the liquid crystal display, the back lighting, and the peripheral (interface) system.

The mounting technology for the chip-on-glass makes the exposed particle overturned, with the side of circuits facing downward. Then, the electricity conduction is joined between the IC and the panel of the liquid-crystal display through the mounting material. Currently, the mounting technology of the chip-on-glass is the best manufacturing technology for the LCD Module in terms of the mounting density. It is important to note that different mounting material requires different mounting technology of the chip-on-glass.

ACF (Anisotropic Conductive Film) is one of the several developed mounting materials, which is now the most widely used material for the chip-on-glass. For the main bonding process, the bonding precision is an essential process parameter we focused on in our study. We investigated a particular model of the LCD Module product with the upper and the lower specification limits set to USL $= 15\ \mu$m, LSL $= -15\ \mu$m, and the target value is set to $T = 0$. If the characteristic data does not fall within the tolerance (LCL, UCL), the lifetime or reliability of the LCD Module will be discounted. To ensure the production quality, the yield index for a particular model we investigated was set to $S_{pk} \geq 1.50$. If the capability requirement fails to be met, the LCD Module product would be seriously affected on its reliability or lifetime.

To test the process performance, we consider the hypothesis testing H_0: $S_{pk} \leq 1.33$ (process is incapable), versus the alternative H_1: $S_{pk} > 1.33$ (process is capable) with type I error $\alpha = 0.05$. According to the process control plan of the COG the random sample data in consecutive four day are collected, which are displayed in Table VII. The 160 sample observations are obtained through the inspection, using the microscope by visual, which were collected eight pieces per every two hours. These 160 observations were justified taken from a stable process (in control), and the characteristic distribution is shown to be approximate normal. The calculated mean and the standard deviation of the 160 sample observations are summarized in the following. Checking Table III we obtain the critical value 1.45 for risk $\alpha = 0.05$.

$$\bar{X} = 0.1754, \quad S = 3.1570,$$

$$\Phi\left(\frac{\text{USL} - \bar{X}}{S}\right) = 0.9999,$$

Table VII. The collected sample data with 160 observations.

−1.40	1.70	−0.17	−3.70	0.40	−1.30	0.48	3.10	0.38	3.00	2.10	0.8	2.20	5.60	−4.20	1.90
5.90	−2.60	−3.40	2.10	2.50	−0.25	−2.20	1.90	1.20	−0.90	−0.82	−5.10	−2.20	3.40	6.70	
3.90	−1.30	7.50	1.90	−2.30	2.40	−4.40	0.09	1.30	−4.90	−3.80	2.30	0.03	−2.80	2.60	−4.60
−5.00	−4.70	0.66	−0.17	1.50	−1.20	0.20	3.00	1.30	−3.80	1.50	−0.43	−0.96	−1.90	−8.40	0.05
−0.69	−3.70	−5.40	1.40	2.00	5.00	0.25	−0.84	4.80	2.70	−1.60	4.40	5.60	−3.30	−4.70	0.81
8.00	3.80	2.30	3.00	−1.20	0.70	−2.20	−1.30	1.40	−2.60	3.40	1.00	−0.07	0.79	3.10	4.10
−0.27	3.40	3.00	2.20	−2.20	−2.30	−3.70	3.00	−6.30	0.72	2.40	1.50	−0.42	−5.20	−1.60	−3.90
−0.40	−3.90	−1.70	0.79	7.70	3.30	−1.40	0.59	1.50	0.44	3.10	1.30	1.60	−2.20	3.20	−3.80
−1.30	3.80	0.06	1.60	−3.80	−5.40	−2.20	0.33	−2.70	−1.70	0.82	3.20	3.50	3.00	6.50	2.50
−0.20	1.80	0.33	−0.31	0.54	0.84	−1.70	0.37	−11.0	3.50	1.60	−3.90	−3.20	0.11	3.90	1.70

$$\Phi\left(\frac{\bar{X}-\mathrm{LSL}}{S}\right) = 0.9999,$$

$$\hat{S}_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{1}{2}\Phi\left(\frac{\mathrm{USL}-\bar{X}}{S}\right)+\frac{1}{2}\Phi\left(\frac{\bar{X}-\mathrm{LSL}}{S}\right)\right\} = 1.5814.$$

Since the calculated $\hat{S}_{pk}$ from the sample data, 1.58, is greater than the critical value 1.45, then we may conclude, with 95% confidence the process meets the performance requirement $S_{pk} > 1.33$. The probability of wrongly judging an incapable process as a capable one is no greater than 5%.

6. Conclusions

Process yield is the most common criterion used in the manufacturing industry for measuring process performance. A measurement index, called S_{pk}, has been proposed to calculate the yield for normal processes. The index S_{pk} establishes the relationship between the manufacturing specification and the actual process performance, which provides an exact measure on process yield. Unfortunately, the distributional properties of the estimated S_{pk} are mathematically intractable. In this paper, we considered a normal approximation to the distribution of the estimated S_{pk}, and investigated its accuracy computationally. We compared the critical values calculated from the approximate distribution with those obtained using the standard simulation technique, for some commonly used quality requirements. Computational results are provided and analyzed. The results indicated that a sample size of $n > 150$ is required for the approximation to be accurate. The investigation is useful to the practitioners for making reliable decisions in testing process performance based on the yield.

Appendix

```
with(stats):
n := : alpha := : cp := : ca := :
x1 := 3*cp*(2 − ca):
x2 := 3*cp*ca:
p1 := statevalf[cdf, normald[0, 1]J(xl):
p2 := statevalf[cdf, normald[0, 1]](x2):
p3 := (p1 + p2)/2:
spk := (1/3)*statevalf[icdf; normald[0, 1]](p3):
f1 := statevalf[pdf, normald[0, 1]](x1):
f2 := statevalf[pdf; normald[0, 1]](x2):
f3 := statevalf[pdf, normald[0, 1]](3*spk):
a := 2^(−0.5)*(x1*f1 + x2*f2):
b := f1 − f2:
var := ((a^2 + b^2)/n)*(6*f3)^(−2):
```

c := statevalf[icdf, normald[0, 1]](1 − alpha)*var^0.5: c0 := spk + c;

References

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