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On some super fault-tolerant Hamiltonian graphs

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Abstract

A k-regular *Hamiltonian* and *Hamiltonian connected* graph G is *super fault-tolerant Hamiltonian* if G remains Hamiltonian after removing at most $k-2$ nodes and/or edges and remains Hamiltonian connected after removing at most $k-3$ nodes and/or edges. A super fault-tolerant Hamiltonian graph has a certain optimal flavor with respect to the *fault-tolerant Hamiltonicity* and *Hamiltonian connectivity*. In this paper, we investigate a construction scheme to construct super fault-tolerant Hamiltonian graphs. In particularly, *twisted-cubes*, *crossed-cubes*, and *Möbius cubes* are all special cases of this construction scheme. Therefore, they are all super fault-tolerant Hamiltonian graphs.

Keywords: k-regular; k-Hamiltonian; k-Hamiltonian connected; Fault-tolerant; Super fault-tolerant Hamiltonian

1. Introduction

For the graph definitions and notations we follow [4]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a,b) | (a,b)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set*. The *degree* of a node v, denoted by $\deg(v)$, is the number of edges incident to v. A graph G is *k-regular* if $\deg(v) = k$ for every node in G. Two nodes a and b are *adjacent* if $(a,b) \in E$.

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A *path* is a sequence of adjacent edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the nodes $v_0, v_1, \ldots, v_m$ are distinct. We also write the path $\langle v_0, v_1, v_2, \ldots, v_m\rangle$ as $\langle v_0, P(v_0, v_i), v_i, v_{i+1}, \ldots, v_j, P(v_j, v_t), v_t, \ldots, v_m\rangle$ where $P(v_0, v_i) = \langle v_0, v_1, \ldots, v_i\rangle$ and $P(v_j, v_t) = \langle v_j, v_{j+1}, \ldots, v_t\rangle$. For our purpose in this paper, a path may contain only one node. A path is a *Hamiltonian path* if its nodes are distinct and they span V. A *cycle* is a path with at least three nodes such that the first node is the same as the last one. A cycle is a *Hamiltonian cycle* if it traverses every node of G exactly once. A graph G is *Hamiltonian* if it has a Hamiltonian cycle, and G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two nodes of G.

The architecture of an interconnection network is usually represented by a graph. There are a lot of mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum for all conditions. One has to design a suitable network depending on the requirements of their properties. The Hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have relatively high probability of failure. There are many researches on the ring embedding problems in faulty interconnection networks [2,10,12,14,15].

Since node faults and edge faults may happen when a network is used, it is practically meaningful to consider faulty networks. A graph G is called *l-fault-tolerant Hamiltonian* (*l-fault-tolerant Hamiltonian connected* respectively) or simply l-Hamiltonian (l-Hamiltonian connected respectively) if it remains Hamiltonian (Hamiltonian connected respectively), after removing at most l nodes and/or edges. The *fault-tolerant Hamiltonicity*, $\mathscr{H}_{\rm f}(G)$, is defined to be the maximum integer l such that $G - F$ remains Hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is Hamiltonian, and undefined if otherwise. Obviously, $\mathscr{H}_{\rm f}(G) \leqslant \delta(G) - 2$, where $\delta(G) = \min\{\deg(v) | v \in V(G)\}$. A regular graph G is *optimal fault-tolerant Hamiltonian* if $\mathscr{H}_{\rm f}(G) = \delta(G) - 2$. *Twisted-cubes*, *crossed-cubes*, and *Möbius cubes* are proved to be optimal fault-tolerant Hamiltonian [7–9]. All these families of graphs have some good properties in common, including that they can all be recursively constructed. In establishing their fault-tolerant Hamiltonicity, another parameter called *fault-tolerant Hamiltonian connectivity* is used. The fault-tolerant Hamiltonian connectivity, $\mathscr{H}^{\kappa}_{\rm f}(G)$, is defined to be the maximum integer l such that $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leqslant l$ if G is Hamiltonian connected, and undefined if otherwise. Obviously, $\mathscr{H}^{\kappa}_{\rm f}(G) \leqslant \delta(G) - 3$. A regular graph G is *optimal fault-tolerant Hamiltonian connected* if $\mathscr{H}^{\kappa}_{\rm f}(G) = \delta(G) - 3$. Again, twisted-cubes, crossed-cubes, and Möbius cubes are proved to be optimal fault-tolerant Hamiltonian connected [7–9]. We call those regular graphs *super fault-tolerant Hamiltonian* if $\mathscr{H}_{\rm f}(G) = \delta(G) - 2$ and $\mathscr{H}^{\kappa}_{\rm f}(G) = \delta(G) - 3$.

All the proofs of super fault-tolerant Hamiltonicity of the aforementioned families of graphs are done by induction. We observe that there are certain common phenomena behind the recursive structures so that we may construct other super fault-tolerant Hamiltonian graphs. In this paper, we try to investigate these phenomena and establish some construction schemes of super fault-tolerant Hamiltonian graphs.

The rest of this article is organized as follows. In Section 2, some terminologies and notations are introduced. Section 3 describes four lemmas which we shall use in our main results. The main results are proved in Section 4. Finally, the conclusion is given in Section 5.

2. Terminology and notation

Before introducing the terminologies and notations, let us briefly explain our motivation. The *hypercube* is a popular network because of its attractive properties, including *regularity*, *symmetry*, *small diameter*, *strong connectivity*, *recursive construction*, *partitionability*, and *relatively low link complexity* [3,11,13]. There are some variations of the hypercube appearing in literature; such as twisted-cubes, crossed-cubes, Möbius cubes, and so on. These variations preserve most of the good topological properties of the hypercube, and even better. For example, the *diameter* of these variation cubes is around half of that of the hypercube. Recently, twisted-cubes, crossed-cubes, and Möbius cubes are proved to be super fault-tolerant Hamiltonian graphs. We note, however, the hypercube is a *bipartite graph*,and therefore its fault-tolerant Hamiltonicity is zero. Basically, all those variations of the hypercube, including the hypercube itself, are recursively constructed in a very similar way as follows: An n-dimension cube $Q_n = (V, E)$ is constructed from two identical $(n-1)$-dimension cubes, Q^0_{n-1} and Q^1_{n-1}. The node set is $V(Q_n) = V(Q^0_{n-1}) \cup V(Q^1_{n-1})$, and the edge set is $E(Q_n) = E(Q^0_{n-1}) \cup E(Q^1_{n-1}) \cup M$ where M is a set of edges connecting the nodes of Q^0_{n-1} and Q^1_{n-1} in a one to one fashion.

For different lower dimensional cube Q_{n-1} and different matching M, we obtain different higher dimensional cubes Q_n, such as twisted-cubes, crossed-cubes, Möbius cubes, and hypercubes, with variant fault-tolerant Hamiltonicity. This motivates us to study some construction schemes of super fault-tolerant Hamiltonian graphs, and it leads to the following definition. Let G_1 and G_2 be two graphs with the same number of nodes. Let M be an arbitrary *perfect matching* between the nodes of G_1 and G_2; i.e., M is a set of edges connecting the nodes of G_1 and G_2 in a one to one fashion. For convenience, G_1 and G_2 are called *components*. Formally, we define graph $G(G_1, G_2; M)$, which has node set $V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2)$, and edge set $E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M$. See Fig. 1. What we have in mind

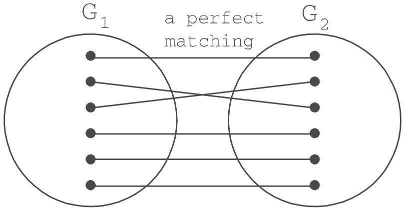

Fig. 1. $G(G_1, G_2; M)$.

is the following: Let G_1 and G_2 be two k-regular super fault-tolerant Hamiltonian graphs with the same number of nodes, and let M be an arbitrary perfect matching. Then $G(G_1, G_2; M)$ is $(k+1)$-regular. The degree of $G(G_1, G_2; M)$, as compared with G_1 and G_2, is increased by 1. We expect that its fault-tolerant Hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ and fault-tolerant Hamiltonian connectivity $\mathscr{H}_{\mathrm{f}}^{\kappa}(G)$ are also increased by 1. This is indeed the case under the condition that $k \geqslant 5$. Then $G(G_1, G_2; M)$ is also a super fault-tolerant Hamiltonian graph.

For ease of exposition, we make some convention about our notations. Consider the graph $G(G_1, G_2; M)$. For each component G_i, we use small letters with subscript i to denote the nodes in G_i, e.g., u_i, v_i, etc. Thus, u_1 is a node in G_1, and u_2 is a node in G_2. A perfect matching M connecting the nodes of G_1 and G_2 in pairs, such pairs of nodes are called *matching nodes*, and these edges are called *matching edges*. We shall use the same letter with different subscripts to denote matching nodes of each other; e.g., u_1 and u_2 are the matching nodes of each other in components G_1 and G_2.

We need some more terms. We shall consider graphs with some faults. Our objective is to find a fault free Hamiltonian cycle (Hamiltonian path respectively). In this paper, each fault can be a faulty node or a faulty edge. If a node v is not faulty, we say v is a *healthy node*. We call an edge e (respectively a matching edge e) *healthy* if both edge e and its two endpoints are not faulty. We use F_i to denote the set of faults in G_i for $i = 1, 2$. Let $f_i = |F_i|$ for $i = 1, 2$. Given two distinct healthy nodes x and y, we use *x, y-Hamiltonian path* to call a fault free Hamiltonian path joining x and y, HP_i to denote a fault free Hamiltonian path in $G_i - F_i$ for $i = 1, 2$. A fault free x, y-Hamiltonian path in $G_i - F_i$ can be written as $\langle x, \mathrm{HP}_i, y\rangle$ for $i = 1, 2$. In addition, path $\langle x, \mathrm{HP}_i, y\rangle$ is a cycle if $x = y$.

3. Preliminaries

Consider an interconnection network G, and suppose that there are faults in it. Let F_G be the set of faults in G, and f_G be the number of faults in G. Suppose that G is k-Hamiltonian (k-Hamiltonian connected respectively) and $f_G \leqslant k$. Let

u be a healthy node in G. It is clear that some of the edges incident to u is on a Hamiltonian cycle (Hamiltonian path respectively) in $G - F_G$, but not every edge incident to u is on some Hamiltonian cycle (Hamiltonian path respectively) in $G - F_G$. In the following two lemmas, we prove that at least a fix number of edges incident to node u are on some Hamiltonian cycles (Hamiltonian paths respectively) in $G - F_G$.

Lemma 1. *Let G be a k-Hamiltonian graph, F_G be a set of faults in G with $|F_G| \leqslant k$, and u be a healthy node in G. Then there are at least $k - f_G + 2$ edges incident to node u, such that each one of them is on some Hamiltonian cycle in $G - F_G$.*

Proof. We know that G is k-Hamiltonian, and there are f_G faults in G. Hence, $G - F_G$ is still Hamiltonian even if we add $k - f_G$ more faults to $G - F_G$. Suppose $f_G < k$. Let HC be a Hamiltonian cycle in $G - F_G$, and let e be an edge on HC and incident to node u. Deleting edge e, $G - F_G - \{e\}$ still contains a Hamiltonian cycle. Repeating this process $k - f_G$ times, we find $k - f_G + 2$ edges incident to node u and each one of them is on some Hamiltonian cycle in $G - F_G$. □

Lemma 2. *Let G be a k-Hamiltonian connected graph, F_G be a set of faults in G with $|F_G| \leqslant k$, and $\{x, y, u\}$ be three distinct healthy nodes in G. Then there are at least $k - f_G + 2$ edges incident to node u, such that each one of them is on some x, y-Hamiltonian path in $G - F_G$.*

Proof. It is known that G is k-Hamiltonian connected, and there are f_G faults in G. Thus, $G - F_G$ is still Hamiltonian connected even if we add $k - f_G$ more faults to $G - F_G$. Suppose $f_G < k$. Let HP be an x, y-Hamiltonian path in $G - F_G$, and let e be an edge on HP and incident to node u. Deleting edge e, $G - F_G - \{e\}$ still contains an x, y-Hamiltonian path. Repeating this process $k - f_G$ times, we find $k - f_G + 2$ edges incident to node u and each one of them is on some x, y-Hamiltonian path in $G - F_G$. □

Lemma 3. *Let G_1 and G_2 be two k-regular graphs with the same number of nodes. If the total number of faults in $G(G_1, G_2; M)$ is no greater than k, there exists at least one healthy matching edge between G_1 and G_2.*

Lemma 4. *Let G_1 and G_2 be two k-regular graphs with the same number of nodes, and let x and y be two healthy nodes in $G(G_1, G_2; M)$. If the total number of faults in $G(G_1, G_2; M)$ is no greater than $k - 2$, there exists at least one healthy matching edge between G_1 and G_2 whose endpoints are neither x nor y.*

The above two lemmas result immediately from the fact that $|V(G_1)| = |V(G_2)| \geqslant k+1$.

4. Super fault-tolerant Hamiltonian graphs

Let G_1 and G_2 be two k-regular super fault-tolerant Hamiltonian graphs. The following two theorems state that the fault-tolerant Hamiltonicity $\mathscr{H}_{\mathrm{f}}(G)$ and fault-tolerant Hamiltonian connectivity $\mathscr{H}_{\mathrm{f}}^{\kappa}(G)$ of the graph $G(G_1, G_2; M)$, as compared with G_1 and G_2, are increased by 1. Hence, $G(G_1, G_2; M)$ is a super fault-tolerant Hamiltonian graph. We make one simple observation first.

Observation 1. *To prove that a graph G is l-Hamiltonian (respectively l-Hamiltonian connected), it suffices to show that $G - F_G$ is Hamiltonian (respectively Hamiltonian connected) for any faulty set $F_G \subset V(G) \cup E(G)$ with $|F_G| = l$. If the total number of faults $|F_G|$ is strictly less than l, we may arbitrarily designate $l - |F_G|$ healthy edges as faulty to make exactly l faults.*

Theorem 1. *Assume $k \geqslant 4$. Let G_1 and G_2 be two k-regular super fault-tolerant Hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k-1)$-Hamiltonian.*

Proof. $G(G_1, G_2; M)$ is $(k+1)$-regular. To prove that $G(G_1, G_2; M)$ is $(k-1)$-Hamiltonian, it suffices to show that $G(G_1, G_2; M) - F_{(1\ldots2)}$ is Hamiltonian for any faulty set $F_{(1\ldots2)} \subset V(G) \cup E(G)$ with $|F_{(1\ldots2)}| = k-1$.

Case 1. All $k-1$ faults are in the same component.

We may assume without loss of generality that all faults are in G_1. Since G_1 is $(k-2)$-Hamiltonian and $f_1 = k-1$, $G_1 - F_1$ has a Hamiltonian path $\langle u_1, \mathrm{HP}_1, v_1\rangle$. Let (u_1, u_2) and (v_1, v_2) be two matching edges between G_1 and G_2. In G_2, there exists a u_2, v_2-Hamiltonian path $\langle u_2, \mathrm{HP}_2, v_2\rangle$ since $f_2 = 0$ and G_2 is $(k-3)$-Hamiltonian connected. Therefore, $\langle u_1, \mathrm{HP}_1, v_1, v_2, \mathrm{HP}_2, u_2, u_1\rangle$ forms a fault free Hamiltonian cycle in this case. See Fig. 2.

Case 2. Not all $k-1$ faults are in the same component.

Without loss of generality, we may assume that $f_2 \leqslant f_1 \leqslant k-2$. In this case, we claim that $G_2 - F_2$ is Hamiltonian connected if $k \geqslant 4$. Suppose not, then

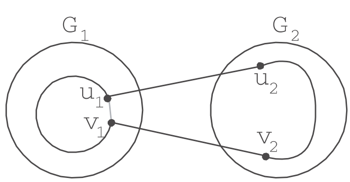

Fig. 2. Case 1: All $k-1$ faults are in the same component.

$f_2 \geqslant k-2$, $k-2 \leqslant f_2 \leqslant f_1 \leqslant k-2$, so $k-2=f_2=f_1=k-2$ and $(k-2)+(k-2) \leqslant k-1$. Therefore, $k \leqslant 3$, which is a contradiction.

By Lemma 3, there exists a healthy matching edge between G_1 and G_2, say (u_1, u_2). Claim that we can find a node v_1 incident to u_1 such that (u_1, v_1) is on a Hamiltonian cycle in $G_1 - F_1$, and the matching edge (v_1, v_2) incident to v_1 is healthy. Then, the case is proved since $G_2 - F_2$ is Hamiltonian connected.

Now, in $G_1 - F_1$, by Lemma 1, of all the healthy nodes incedent to u_1, there are at least $(k-2) - f_1 + 2 = k - f_1$ edges which are on some Hamiltonian cycle in $G_1 - F_1$. Of all these $k - f_1$ edges, there is at least one edge, say (u_1, v_1), such that v_1, v_2, and (v_1, v_2) are healthy. Were it not true, $G(G_1, G_2; M)$ would contain at least $f_1 + (k - f_1) = k$ faults, contradicting the fact that the total number of faults is $k-1$. Therefore, we have a fault free Hamiltonian cycle $\langle u_1, \mathrm{HP}_1, v_1, v_2, \mathrm{HP}_2, u_2, u_1 \rangle$ in this case. See Fig. 3. This completes the proof of this theorem. □

The fault-tolerant Hamiltonian connectivity $\mathscr{H}_f^\kappa(G)$ of $G(G_1, G_2; M)$ is also increased by 1, as stated in the following theorem.

Theorem 2. *Assume $k \geqslant 5$. Let G_1 and G_2 be two k-regular super fault-tolerant Hamiltonian graphs and $|V(G_1)| = |V(G_2)|$. Then graph $G(G_1, G_2; M)$ is $(k-2)$-Hamiltonian connected.*

Proof. Let $F_{(1\ldots2)}$ be a set of faults, $F_{(1\ldots2)} \subset V(G) \cup E(G)$ and $|F_{(1\ldots2)}| = k-2$. Let x and y be two healthy nodes in $G(G_1, G_2; M)$, we shall find a fault free Hamiltonian path joining x and y. The proof is classified into two cases.

Case 1. x and y are not in the same component.

Without loss of generality, we may assume x is in G_1, and y is in G_2. This case can be further divided into two subcases.

Subcase 1.1. All $k-2$ faults are in the same component.

Without loss of generality, we may assume that all $k-2$ faults are in G_1. So there is a fault free Hamiltonian cycle in $G_1 - F_1$. On this fault free cycle, there are two nodes incident to x. One of these two nodes is not matched with y, say u_1. Now, we delete edge (x, u_1) and add matching edge (u_1, u_2). In G_2, there is a u_2, y-Hamiltonian path $\langle u_2, \mathrm{HP}_2, y \rangle$ because G_2 is Hamiltonian connected.

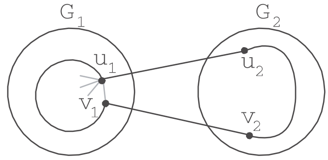

Fig. 3. Case 2: Not all $k-1$ faults are in the same component.

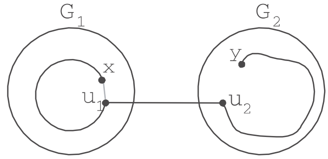

Fig. 4. Case 1.1: All $k-2$ faults are in the same component.

Thus, $\langle x, \mathrm{HP}_1, u_1, u_2, \mathrm{HP}_2, y\rangle$ forms a fault free x, y-Hamiltonian path in this subcase. See Fig. 4.

Subcase 1.2. Not all $k-2$ faults are in the same component.

Since the total number of faults $f_{(1\dots2)}$ is $k-2$ and not all faults are in the same component, we may assume that $f_2 \leqslant f_1 \leqslant k-3$. By Lemma 4, we can find a healthy matching edge (u_1, u_2) between G_1 and G_2, where $u_1 \neq x$ and $u_2 \neq y$. Since $f_2 \leqslant f_1 \leqslant k-3$, $G_1 - F_1$ and $G_2 - F_2$ are Hamiltonian connected, there is one x, u_1-Hamiltonian path $\langle x, \mathrm{HP}_1, u_1\rangle$ in $G_1 - F_1$ and one u_2, y-Hamiltonian path $\langle u_2, \mathrm{HP}_2, y\rangle$ in $G_2 - F_2$. So $\langle x, \mathrm{HP}_1, u_1, u_2, \mathrm{HP}_2, y\rangle$ is a fault free x, y-Hamiltonian path in this subcase. See Fig. 5.

Case 2. x and y are in the same component.

Without loss of generality, we may assume x and y are in G_1. We shall divide this case into three subcases.

Subcase 2.1. All $k-2$ faults are in G_1.

G_1 is $(k-3)$-Hamiltonian connected and $f_1 = k-2$. Let g be a faulty edge or a faulty node. In $G_1 - (F_1 - \{g\})$, there is a Hamiltonian path $\langle x, P(x,y), y\rangle$ joining x and y. Removing the fault g, this Hamiltonian path is separated into two subpaths, say $\langle x, P(x,u_1), u_1\rangle$ and $\langle v_1, P(v_1,y), y\rangle$, which cover all the nodes of $G_1 - F_1$. Then we add two matching edges (u_1, u_2) and (v_1, v_2). In G_2, there exists a u_2, v_2-Hamiltonian path $\langle u_2, \mathrm{HP}_2, v_2\rangle$ since $f_2 = 0$. Thus, we have a fault free x, y-Hamiltonian path $\langle x, P(x,u_1), u_1, u_2, \mathrm{HP}_2, v_2, v_1, P(v_1,y), y\rangle$ in this subcase. See Fig. 6.

Subcase 2.2. All $k-2$ faults are in G_2.

Let x_2 be the matching node of x in G_2, and y_2 be the matching node of y in G_2. This subcase can be further divided into two subcases:

Subcase 2.2.1. At least one of x_2 and y_2 is healthy.

Without loss of generality, we may assume y_2 is healthy. We add the matching edge (y, y_2). In $G_2 - F_2$, there exists a Hamiltonian cycle since

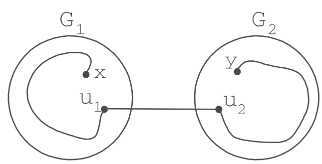

Fig. 5. Case 1.2: Not all $k-2$ faults are in the same component.

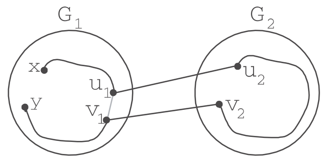

Fig. 6. Case 2.1: All $k-2$ faults are in G_1.

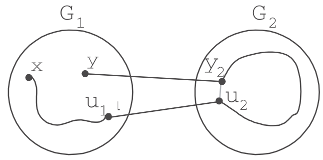

Fig. 7. Subcase 2.2.1: At least one of x_2 and y_2 is healthy.

$f_2 = k-2$. On this fault free cycle, there are two nodes incident to node y_2. At least one of these two nodes is not adjacent to x, say u_2. We then add the matching edge (u_1, u_2), and delete edge (u_2, y_2). In $G_1 - \{y\}$, we claim that there exists a fault free x, u_1-Hamiltonian path $\langle x, P(x, u_1), u_1 \rangle$. Suppose not, then $k-3 < 1$, so $k < 4$. It is a contradiction. Therefore, we have a fault free x, y-Hamiltonian path $\langle x, P(x, u_1), u_1, u_2, \mathrm{HP}_2, y_2, y \rangle$ in this subcase. See Fig. 7.

Subcase 2.2.2. Both x_2 and y_2 are faulty.

In G_1, the number of healthy edges incident to y is k and $f_2 = k-2$. We can find a healthy node u_1 incident to y such that $u_1 \neq x$ and u_2 is healthy, where u_2 is the matching node of u_1 in G_2. In $G_2 - F_2$, there exists a Hamiltonian cycle since $f_2 = k-2$. Let v_2 be a node on this cycle incident to u_2. Node v_2 is not matched with x and y since x_2 and y_2 are faulty in this subcase. Then we add the matching edge (v_1, v_2) and delete edge (u_2, v_2). In this subcase, we claim that $G_1 - \{u_1, y\}$ has a fault free x, v_1-Hamiltonian path $\langle x, P(x, v_1), v_1 \rangle$ for $k \geqslant 5$. Suppose not, $k-3 < 2$, and $k < 5$. It is a contradiction. Thus, $\langle x, P(x, v_1), v_1, v_2, \mathrm{HP}_2, u_2, u_1, y \rangle$ forms a fault free x, y-Hamiltonian path in this subcase. See Fig. 8.

Subcase 2.3. Neither all $k-2$ faults are in G_1 nor all $k-2$ faults are in G_2.

Since $f_{(1\ldots2)} = k-2$ and not all faults are in one component, we have $f_1 \leqslant k-3$ and $f_2 \leqslant k-3$. Consequently, both $G_1 - F_1$ and $G_2 - F_2$ are Hamiltonian

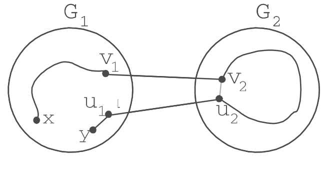

Fig. 8. Subcase 2.2.2: Both x_2 and y_2 are faulty.

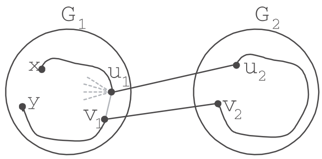

Fig. 9. Subcase 2.3: Neither all $k-2$ faults are in G_1 nor all $k-2$ faults are in G_2.

connected. By Lemma 4, there is at least one healthy matching edge between G_1 and G_2, say (u_1, u_2), such that $u_1 \notin \{x, y\}$. In $G_1 - F_1$, by Lemma 2, there are at least $(k-3) - f_1 + 2 = k - 1 - f_1$ edges incident to node u_1, such that each one of them is on some x, y-Hamiltonian path in $G_1 - F_1$. Among these $k - 1 - f_1$ edges, we claim that there is at least one, say (u_1, v_1), such that v_1, v_2, and (v_1, v_2) are healthy. If this is not true, $f_{(1\ldots2)} = f_1 + (f_{(1\ldots2)} - f_1) \geqslant f_1 + (k - 1 - f_1) = k - 1$, which contradicts the fact that $f_{(1\ldots2)} = k - 2$. We then delete edge (u_1, v_1) and add both edges (u_1, u_2) and (v_1, v_2). In $G_2 - F_2$, there is a u_2, v_2-Hamiltonian path $\langle u_2, \mathrm{HP}_2, v_2\rangle$ as a result of $f_2 \leqslant k - 3$. Therefore, $\langle x, P(x, u_1), u_1, u_2, \mathrm{HP}_2, v_2, v_1, P(v_1, y), y\rangle$ forms a fault free x, y-Hamiltonian path in this subcase, where $\langle x, P(x, u_1), u_1, v_1, P(v_1, y), y\rangle$ is a Hamiltonian path in $G_1 - F_1$. See Fig. 9. Thus, this theorem is proved. □

Corollary 1. *Assume that G_1 and G_2 are k-regular super fault-tolerant Hamiltonian where $k \geqslant 5$ and $|V(G_1)| = |V(G_2)|$. Then $G(G_1, G_2; M)$ is $(k+1)$-regular super fault-tolerant Hamiltonian.*

In the following, we briefly introduce the definitions of the twisted-cubes, the crossed-cubes, and the Möbius cubes. It is straightforward to see that these cubes are all special cases of the construction scheme proposed in the previous section.

In [1], the twisted n-cube TQ_n is defined for odd values of n. The vertex set of the twisted n-cube TQ_n is the set of all binary strings of length n. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ be any vertex in TQ_n. For $0 \leqslant i \leqslant n-1$, let the *ith parity function* be $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive or operation. We can recursively define TQ_n as follows: A twisted 1-cube, TQ_1, is a complete graph with two vertices 0 and 1. Suppose that $n \geqslant 3$. We can decompose the vertices of TQ_n into four sets, $\mathrm{TQ}^{0,0}_{n-2}, \mathrm{TQ}^{0,1}_{n-2}, \mathrm{TQ}^{1,0}_{n-2}$, and $\mathrm{TQ}^{1,1}_{n-2}$ where $\mathrm{TQ}^{i,j}_{n-2}$ consists of those vertices u with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i, j) \in \{(0,0), (0,1), (1,0), (1,1)\}$, the induced subgraph of $\mathrm{TQ}^{i,j}_{n-2}$ in TQ_n is isomorphic to TQ_{n-2}. The edges that connect these four subtwisted cubes can be described as follows: Any vertex $u_{n-1}u_{n-2}\ldots u_1u_0$ with $P_{n-3}(u) = 0$ is connected to $\bar{u}_{n-1}\bar{u}_{n-2}\ldots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$; and to $u_{n-1}\bar{u}_{n-2}\ldots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\ldots u_1u_0$ if $P_{n-3}(u) = 1$.

From the definition, both the subgraph induced by $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and the subgraph induced by $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ are isomorphic to $TQ_{n-2} \times K_2$, where K_2 is the complete graph with two vertices. Moreover, the edges joining $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ form a perfect matching of TQ_n. Recursively applying Theorems 1 and 2, we can easily prove that TQ_n is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leqslant 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

Now, we introduce the definition of the crossed-cubes. Two two-digit binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted by $x \sim y$, if and only if $(x,y) \in \{(00,00),(10,10),(01,11),(11,01)\}$. An n-dimension crossed-cube CQ_n [6] is a graph $CQ_n = (V,E)$ that is recursively constructed as follows: CQ_1 is a complete graph with two vertices labeled by 0 and 1. CQ_n consists of two identical $(n-1)$-dimension crossed-cubes, CQ^0_{n-1} and CQ^1_{n-1}. The vertex $u = 0u_{n-2}\dots u_0 \in V(CQ^0_{n-1})$ and vertex $v = 1v_{n-2}\dots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if (1) $u_{n-2} = v_{n-2}$ if n is even; and (2) for $0 \leqslant i < \lfloor n-1/2 \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

According to the definition, CQ_n can be viewed as $G(CQ_{n-1}, CQ_{n-1}; M)$ for some perfect matching M. Again, recursively applying Theorems 1 and 2, we can easily prove that CQ_n is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leqslant 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

The Möbius cube [5], $MQ_n = (V,E)$, of dimension n has 2^n vertices. Each vertex is labeled by a unique n-bit binary string as its address and has connections to n other distinct vertices. The vertex with address $X = x_{n-1}x_{n-2}\dots x_0$ connects to n other vertices Y_i, $0 \leqslant i \leqslant n-1$, where the address of Y_i satisfies (1) $Y_i = (x_{n-1}\dots x_{i+1}\overline{x_i}\dots x_0)$ if $x_{i+1} = 0$; or (2) $Y_i = (x_{n-1}\dots x_{i+1}\overline{x_i\dots x_0})$ if $x_{i+1} = 1$.

From the above definition, X connects to Y_i by complementing the bit x_i if $x_{i+1} = 0$, or by complementing all bits of $x_i \dots x_0$ if $x_{i+1} = 1$. For the connection between X and Y_{n-1}, we can assume that the unspecified x_n is either 0 or 1, which gives slightly different topologies. If x_n is 0, we call the network generated the "0-Möbius cube", denoted by $0\text{-}MQ_n$; and if x_n is 1, we call the network generated the "1-Möbius cube", denoted by $1\text{-}MQ_n$.

According to the above definition, $0\text{-}MQ_{n+1}$ and $1\text{-}MQ_{n+1}$ can be recursively constructed from a $0\text{-}MQ_n$ and a $1\text{-}MQ_n$ by adding a perfect matching. Recursively applying Theorems 1 and 2, we can easily prove that MQ_n is super fault-tolerant Hamiltonian for $n > 5$. As for $n \leqslant 5$, it can be checked by a computer program that it is super fault-tolerant Hamiltonian.

5. Conclusion

The fault-tolerant Hamiltonicity and the fault-tolerant Hamiltonian connectivity are essential parameters of an interconnection network. In this paper, we propose a family of k-regular, $(k-2)$-Hamiltonian, and $(k-3)$-Hamiltonian connected graphs. These graphs are maximally fault-tolerant, and we call them super fault-tolerant Hamiltonian graphs.

One of the contributions of this paper is the following. We propose a construction scheme to construct, with flexibility, many k-regular super fault-tolerant Hamiltonian graphs for $k \geqslant 6$. As for small values of k, $k \leqslant 5$, there are some examples in literature, such as twisted-cubes, crossed-cubes, and Möbius cubes, etc.

There are many popular interconnection networks which are k-regular graphs. Some of them, e.g., twisted-cubes, crossed-cubes, and Möbius cubes, can be recursively constructed using our construction schemes. And therefore, they are in fact a subclass of our proposed family of graphs. Then, we know that they are super fault-tolerant Hamiltonian as long as the case is true for initial cases $k \leqslant 5$. For small values of k, $k \leqslant 5$, we may use a computer program to check that it is $(k-2)$-Hamiltonian and $(k-3)$-Hamiltonian connected.

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