Fault Hamiltonicity and Fault Hamiltonian Connectivity of the Arrangement Graphs

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Abstract—The arrangement graph $A_{n,k}$ is a generalization of the star graph. There are some results concerning fault Hamiltonicity and fault Hamiltonian connectivity of the arrangement graph. However, these results are restricted in some particular cases and, thus, are less completed. In this paper, we improve these results and obtain a stronger and simpler statement. Let $n-k \geq 2$ and $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$. We prove that $A_{n,k}-F$ is Hamiltonian if $|F| \leq k(n-k)-2$ and $A_{n,k}-F$ is Hamiltonian connected if $|F| \leq k(n-k)-3$. These results are optimal.

Index Terms—Hamiltonian cycle, Hamiltonian connected, fault tolerance, arrangement graph.

1 Introduction

The interconnection network has been an important research area for parallel and distributed computer systems. Designing an interconnection network is multi-objected and complicated. For simplifying this task, we usually use a graph to represent the network's topology, where vertices represent processors and edges represent links between processors. The hypercube [15] and the star graph [1], [2] are two examples. The hypercube possesses many good properties and is implemented as many multiprocessor systems. Akers et. al. [1] proposed the star graph, which has smaller degree, diameter, and average distance than the hypercube while reserving symmetry properties and desirable fault-tolerant characteristics. As a result, the star graph has been recognized as an alternative to the hypercube. However, the hypercube and the star are less flexible in adjusting their sizes.

The arrangement graph [6] was proposed by Day and Tripathi as a generalization of the star graph. It is more flexible in its size than the star graph. Given two positive integers n and k with $n > k$, the (n,k)-arrangement graph $A_{n,k}$ is the graph (V,E), where $V = \{p \mid p \text{ is an arrangement of } k \text{ elements out of the } n \text{ symbols: } 1, 2 \cdots, n\}$ and $E = \{(p,q) \mid p, q \in V \text{ and } p, q \text{ differ in exactly one position }\}$. A more precise definition and an example will be given in the next section. $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $\frac{n!}{(n-k)!}$ vertices. $A_{n,1}$ is isomorphic to the complete graph K_n and $A_{n,n-1}$ is isomorphic to the n-dimensional star graph. Moreover, $A_{n,k}$ is vertex symmetric and edge symmetric [6].

Hamiltonicity is an important property for network topologies. Thus, the existence of a Hamiltonian cycle is a desired property for a new proposed topology. Hamiltonian connectivity is a related property of Hamiltonicity, namely, there is a Hamiltonian path between any two vertices of a graph. Since processors or links may fail sometimes, fault Hamiltonicity and fault Hamiltonian connectivity are concerned in many studies on network topologies, such as hypercubes [4], [11], twisted cubes [10], deBruijn networks [14], and star graphs [16], [9]. We say that a graph G can tolerate f faults when embedding a Hamiltonian cycle if there is a Hamiltonian cycle in $G-F$ for any $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$. We use *f-Hamiltonian* to denote this property of G. Similarly, we use *f-Hamiltonian connected* to denote the property that there is a Hamiltonian path between any two vertices in $G-F$ for any $F \subseteq V(G) \cup E(G)$ with $|F| \leq f$.

There are also some studies concerning fault Hamiltonicity and fault Hamiltonian connectivity of the arrangement graph. Hsieh et al. [8] studied the existence of Hamiltonian cycles in faulty arrangement graphs. It is proven that $A_{n,k}-F$ is Hamiltonian if it satisfies one of the following conditions:

1. ($k = 2$ and $n-k \geq 3$, or $k \geq 3$, $n-k \geq 4 + \lceil \frac{k}{2} \rceil$), and $F \subseteq E(A_{n,k})$ with $|F| \leq k(n-k)-2$,
2. $k \geq 2$, $n-k \geq 2 + \lceil \frac{k}{2} \rceil$, and $F \subseteq E(A_{n,k})$ with $|F| \leq k(n-k-3)-1$,
3. $k \geq 2$, $n-k \geq 3$, and $F \subseteq E(A_{n,k})$ with $|F| \leq k$,
4. $n-k \geq 3$, and $F \subseteq V(A_{n,k})$ with $|F| \leq k-3$, or
5. $n-k \geq 3$, and $F \subseteq E(A_{n,k}) \cup V(A_{n,k})$ with $|F| \leq k$.

Lo and Chen [12] studied the *edge* fault Hamiltonian connectivity of the arrangement graph. They restricted the fault distribution and then showed that $A_{n,k}$ is $k(n-k)-2$ edge fault Hamiltonian connected. However, these results are more restricted and less complete.

In this paper, we improve these results to get a much stronger and simpler statement. We prove that $A_{n,k}$ is $(k(n-k)-2)$-Hamiltonian and $(k(n-k)-3)$-Hamiltonian connected if $n-k \geq 2$, where the faults can be vertices and edges. For $n-k=1$, $A_{n,n-1}$ is isomorphic to the n-dimensional star graph, which is bipartite and thus cannot tolerate any vertex fault when embedding Hamiltonian cycles and paths. Observing that a regular graph of degree d is at most $(d-2)$-Hamiltonian and $(d-3)$-Hamiltonian connected, our results are optimal. In the following section, we discuss some basic properties of the arrangement

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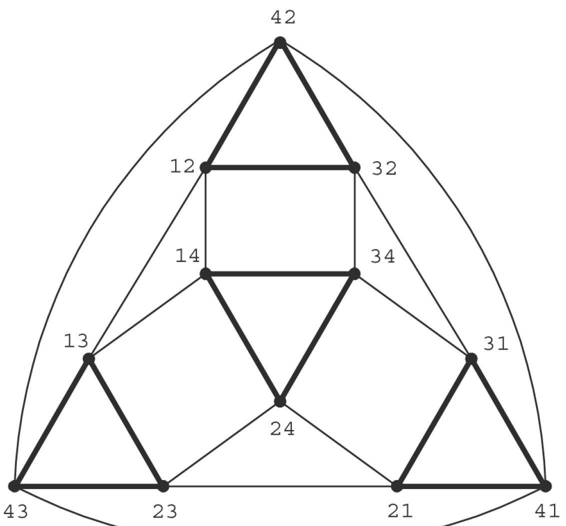


Fig. 1. $A_{4,2}$.

graphs. In Section 3, we prove our main theorem. Since the proof of the main theorem is rather long, several steps are broken into lemmas. We prove these lemmas in Sections 4, 5, and in the Appendix.

2 Some Properties of the Arrangement Graphs

In this paper, we concentrate on loopless undirected graphs. For the graph definition and notation, we follow [3]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b) \text{ is an unordered pair of } V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices, a and b, are *adjacent* if $(a, b) \in E$. A *path* is represented by $\langle v_0, v_1, v_2 \cdots, v_k\rangle$. We also write the path $\langle v_0, v_1, v_2 \cdots, v_k\rangle$ as $\langle v_0, P_1, v_i, v_{i+1} \cdots, v_j, P_2, v_t, \cdots, v_k\rangle$, where P_1 is the path $\langle v_0, v_1, \cdots, v_i\rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \cdots, v_t\rangle$. A path is a *Hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *Hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Let n and k be two positive integers with $n > k$. And, let $\langle n\rangle$ and $\langle k\rangle$ denote the sets $\{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, k\}$, respectively. Then, the vertex set of $A_{n,k}$, $V(A_{n,k})$ is $\{p \mid p = p_1p_2 \cdots p_k$ with $p_i \in \langle n\rangle$ for $1 \le i \le k$ and $p_i \ne p_j$ if $i \ne j\}$ and the edge set of $A_{n,k}$, $E(A_{n,k})$, is $\{(p, q) \mid p, q \in V(A_{n,k})$ and, for some $i \in \langle k\rangle$, $p_i \ne q_i$ and $p_j = q_j$ for all $j \ne i\}$. Fig. 1 illustrates $A_{4,2}$.

For consistency, we always use F to denote the faulty set of $A_{n,k}$ in the following discussion, where $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$. Let $G = (V', E')$ be a subgraph of $A_{n,k}$. We use $F(G)$ to denote the set $(V' \cup E') \cap F$. We say that an edge (u, v) is *fault-free* if $u, v,$ and (u, v) are not in F. Assume that t is any index in $\langle k\rangle$. Let f_t denote the function defined on $V(A_{n,k})$ into itself by assigning $i_1i_2 \dots i_k$ to $j_1j_2 \dots j_k$, where $j_t = i_k$, $j_k = i_t$, and $j_r = i_r$ if $r \ne t, k$. The following lemma is easily derived by the definition of the arrangement graphs.

Lemma 1. *f_t is an automorphism of $A_{n,k}$ for any $t \in \langle k\rangle$.*

Let i and j be two positive integers with $1 \le i, j \le n$. And, let $V(A_{n,k}^{(j:i)}) = \{p \mid p = p_1p_2 \cdots p_k$ and $p_j = i\}$. Obviously, $\{V(A_{n,k}^{(j:i)}) \mid 1 \le i \le n\}$ forms a partition of $V(A_{n,k})$. Let $A_{n,k}^{(j:i)}$ denote the subgraph of $A_{n,k}$ induced by $V(A_{n,k}^{(j:i)})$. It is easy to see that each $A_{n,k}^{(j:i)}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. Assume that $t \le k$. Let $j_1, j_2, \dots j_t$ be t distinct indices of $\langle k\rangle$ and $i_1, i_2, \dots, i_t$ be t distinct elements of $\langle n\rangle$. We use $A_{n,k}^{(j_1,j_2,\dots,j_t:i_1,i_2,\dots,i_t)}$ to denote the graph induced by $\bigcap_{l=1}^{t} V(A_{n,k}^{(j_l:i_l)})$. We have the following lemmas.

Lemma 2. *Suppose that $k \ge 2$, $n - k \ge 2$, and $|F| \le k(n-k) - 2$. Then, there exists an index $j \in \langle k\rangle$ such that $|F(A_{n,k}^{(j:i)})| \le (k-1)(n-k) - 1$ for every $i \in \langle n\rangle$.*

Proof. Suppose that the lemma is not true. Then, for each index $j \in \langle k\rangle$, there exists $i \in \langle n\rangle$ such that $|F(A_{n,k}^{(j:i)})| \ge (k-1)(n-k)$. Assume that $|F(A_{n,k}^{(j:i_j)})| \ge (k-1)(n-k)$ for $j \in \langle k\rangle$ and $\{i_1, i_2, \cdots, i_k\} \le \langle n\rangle$.

Suppose $i_1, i_2, \cdots, i_k$ are not distinct k numbers. Without loss of generality, assume that $i_1 = i_2$. Since $F(A_{n,k}^{(1:i_1)}) \cap F(A_{n,k}^{(2:i_1)}) = \emptyset$,

$$|F| \ge |F(A_{n,k}^{(1:i_1)})| + |F(A_{n,k}^{(2:i_1)})| \ge 2(k-1)(n-k) \ge k(n-k).$$

We get a contradiction.

Now, suppose $i_1, i_2, \cdots, i_k$ are distinct. Without loss of generality, assume that $i_j = j$ for each $j \in \langle k\rangle$. We prove that $|F(A_{n,k}^{(1,2,\dots,t:1,2,\dots,t)})| \ge (k-t)(n-k) + 2(t-1)$ for any t with $1 \le t \le k$ by induction: Since $|F(A_{n,k}^{(1:1)})| \ge (k-1)(n-k)$, the statement holds for $t = 1$. Assume that the statement holds for any t' with $1 \le t' < t \le k$. By the inclusion-exclusion principle,

$$\begin{aligned}
|F(A_{n,k}^{(1,2,\dots,t:1,2,\dots,t)})| &= |F(A_{n,k}^{(1,2,\dots,t-1:1,2,\dots,t-1)})| + |F(A_{n,k}^{(t:t)})| \\
&\quad - |F(A_{n,k}^{(1,2,\dots,t-1:1,2,\dots,t-1)} \cup F(A_{n,k}^{(t:t)})| \\
&\ge |F(A_{n,k}^{(1,2,\dots,t-1:1,2,\dots,t-1)})| + |F(A_{n,k}^{(t:t)})| - |F| \\
&\ge ((k-t+1)(n-k) + 2(t-2)) + (k-1)(n-k) \\
&\quad - (k(n-k) - 2) \\
&= (k-t)(n-k) + 2(t-1).
\end{aligned}$$

Thus, the statement holds for any t with $1 \le t \le k$, i.e., $|F(A_{n,k}^{(1,2,\dots,k:1,2,\dots,k)})| \ge 2(k-1) \ge 2$. However, the subgraph $A_{n,k}^{(1,2,\dots,k:1,2,\dots,k)}$ of $A_{n,k}$ consists of only the single vertex $p = 12 \cdots k$. So, we get a contradiction. And, hence, the lemma is proven. □

Using a similar argument, we have the following lemma.

Lemma 3. *Suppose that $k \ge 3$, $n - k \ge 2$, and $|F| \le k(n-k) - 3$. Then, there exists an index $j \in \langle k\rangle$ such that $|F(A_{n,k}^{(j:i)})| \le (k-1)(n-k) - 2$ for every $i \in \langle n\rangle$.*

For simplicity, if there is no ambiguity, we use $A_{n,k}^{i}$ to denote $A_{n,k}^{(k:i)}$ and $E^{i,j}$ to denote the set of edges between $A_{n,k}^{i}$

and $A^j_{n,k}$. Assume that I is any subset of $\{1, 2, \ldots, n\}$. We use $A^I_{n,k}$ to denote the subgraph of $A_{n,k}$ induced by $\bigcup_{i\in I} V(A^i_{n,k})$. The following proposition follows directly from the definition of the arrangement graphs.

Proposition 1. *Let i and j be two distinct elements of $\langle n\rangle$.*

1. $|E^{i,j}| = \frac{(n-2)!}{(n-k-1)!}$.
2. *If (u, v) and (u', v') are distinct edges in $E^{i,j}$, then*
   a. $\{u, v\} \cap \{u', v'\} = \emptyset$ *and*
   b. $(u, u') \in E(A^i_{n,k})$ *if and only if* $(v, v') \in E(A^j_{n,k})$.

Let $u \in V(A^i_{n,k})$ for some $i \in \langle n\rangle$. We use $N^I(u)$ to denote the set of all neighbors of u which are in $A^I_{n,k}$. Particularly, we use $N^*(u)$ as an abbreviation of $N^{\langle n\rangle-\{i\}}(u)$. We call vertices in $N^*(u)$ the *outer neighbors* of u. Obviously, $|N^{\{i\}}(u)| = (k-1)(n-k)$ and $|N^*(u)| = (n-k)$. We say that u is adjacent to $A^j_{n,k}$ if u has an outer neighbor in $A^j_{n,k}$. Then, we define the *adjacent set* $AS(u)$ of u as $\{j \mid u \text{ is adjacent to } A^j_{n,k}\}$. And, we have the following proposition:

Proposition 2. *For $n > k > 1$, if u and v are two distinct vertices in $A^i_{n,k}$ with $d(u, v) \le 2$, then $AS(u) \ne AS(v)$.*

Proof. Let $u = u_1u_2\cdots u_k$ and $v = v_1v_2\cdots v_k$. If $d(u, v) = 1$, there is an index $j \in \langle k-1\rangle$ such that $u_j \ne v_j$. Then, $v_j \in AS(u)$, but $v_j \notin AS(v)$. The statement follows.

If $d(u, v) = 2$, there is a vertex $w \in V(A^i_{n,k})$ such that $d(u, w) = d(v, w) = 1$. Let $w = w_1w_2\cdots w_k$. And, let j_1 and j_2 be two indices such that $w_{j_1} \ne u_{j_1}$ and $v_{j_2} \ne w_{j_2}$. Obviously, $j_1 \ne j_2$. Otherwise, $d(u, v) = 1$. So, w_{j_1} is not in $\{u_1, u_2, \cdots, u_k\}$ but in $\{v_1, v_2, \cdots, v_k\}$. By definition, w_{j_1} is in $AS(u)$ but not in $AS(v)$. Thus, the result also follows. □

Let F be a faulty set. The *good edge set* $GE^{i,j}(F)$ is the set of edges $(u, v) \in E^{i,j}$ such that $\{u, v, (u, v)\} \cap F = \emptyset$. Then, we have following statement.

Proposition 3. *Let $n > k > 3$, $n - k \ge 2$, $I \subseteq \langle n\rangle$, and $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$. Then,*

1. *If $|F(A^I_{n,k})| \le k(n-k) - 3$, then $|GE^{i,j}(F)| \ge 3$ for every $i \ne j \in I$, and*
2. *If $|F(A^I_{n,k})| = k(n-k) - 2$, then there exists only one (i, j) with $|GE^{i,j}(F)| = 2$ if (i, j) is the only pair such that $|GE^{i,j}(F)| < 3$.*

Proof. First, consider that $|F(A^I_{n,k})| \le k(n-k) - 3$. Suppose that $|GE^{i,j}(F)| < 3$ for some $i, j \in \langle n\rangle$. Since $|E^{i,j}| = (n-2)!/(n-k-1)! \ge (n-2)(n-k) \ge k(n-k)$, $|F(A^{\{i,j\}}_{n,k})| > k(n-k) - 3$. We get a contradiction.

Now, consider that $|F(A^I_{n,k})| = k(n-k) - 2$. If $|I| \le 2$, the statement follows. Assume that $|I| \le 3$. Suppose that there are two pairs $\{i, j\} \ne \{i', j'\}$ such that $|GE^{i,j}(F)| < 3$ and $|GE^{i',j'}(F)| < 3$. If $\{i, j\} \cap \{i', j'\} = \emptyset$, $F(A^{\{i,j\}}_{n,k}) \cap F(A^{\{i',j'\}}_{n,k}) = \emptyset$ and

$$|F(A^I_{n,k})| \ge (|E^{i,j}| - 2) + (|E^{i',j'}| - 2) > k(n-k) - 2.$$

So, $\{i, j\} \cap \{i', j'\} \ne \emptyset$. Assume that $i = i'$ and $j \ne j'$. Let $V_j = \{v \in V(A^i_{n,k}) \mid v \text{ be adjacent to } A^j_{n,k}\}$, $V_{j'} = \{v \in V(A^i_{n,k}) \mid v \text{ be adjacent to } A^{j'}_{n,k}\}$, and $\hat{V} = V_j \cap V_{j'}$. Then,

$$\begin{aligned}|F(A^I_{n,k})| &\ge (|E^{i,j}| - 2) + (|E^{i,j'}| - 2) - |\hat{V} \cap F| \\ &\ge 2(k(n-k) - 2) - |\hat{V} \cap F|.\end{aligned}$$

So, $|\hat{V} \cap F| = k(n-k) - 2$, i.e., $F \subseteq \hat{V}$. Note that the number of faulty edges inside subgraphs do not affect the number of fault-free edges between subgraphs.

However,

$$\begin{aligned}&|V_j - \hat{V}| = |\{v = v_1\cdots v_k \mid v_k = i, j \notin \{v_1, \cdots, v_{k-1}\}, \\ &j' \in \{v_1, \cdots, v_{k-1}\}\}| = (k-1)(n-3)\cdots(n-k) \\ &\ge (k-1)(n-k) \ge 4.\end{aligned}$$

So, $|GE^{i,j}(F)| \ge 4$. We get a contradiction. Thus, there is only one pair $\{i, j\}$ such that $|GE^{i,j}(F)| < 3$. Then, suppose that $|GE^{i,j}(F)| \le 1$.

$$|F(A^I_{n,k})| \ge |E^{i,j}| - 1 \ge k(n-k) - 1.$$

So, $|GE^{i,j}(F)| = 2$. Hence, the statement follows. □

Lemma 4. *Suppose that*

1. $k \ge 3$ *and* $n - k \ge 2$,
2. $I \subseteq \langle n\rangle$ *with* $|I| \ge 2$,
3. $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$, *and*
4. $A^l_{n,k} - F$ *is Hamiltonian connected for each* $l \in I$ *and* $|F(A^I_{n,k})| \le k(n-k) - 3$.

Then, for any $x \in V(A^i_{n,k})$ and $y \in V(A^j_{n,k})$ with $i \ne j \in I$, there is a Hamiltonian path of $A^I_{n,k} - F$ joining x and y.

Proof. Since $|F(A^I_{n,k})| \le k(n-k) - 3$, by Proposition 3, $|GE^{i_1,i_2}(F)| \ge 3$ for every $i_1 \ne i_2 \in I$. We prove this lemma by induction on $|I|$. Suppose that $|I| = 2$. Then, $I = \{i, j\}$ for some i, j. Since $|GE^{i,j}(F)| \ge 3$, there exists an edge $(u, v) \in GE^{i,j}(F)$ such that $u \ne x \in V(A^i_{n,k})$ and $v \ne y \in V(A^j_{n,k})$. Then, by assumption that each $A^l_{n,k} - F$ is Hamiltonian connected, there is a Hamiltonian path P_1 of $A^i_{n,k} - F$ from x to u and a Hamiltonian path P_2 of $A^j_{n,k} - F$ from v to y. Thus, $\langle x, P_1, u, v, P_2, y\rangle$ forms a Hamiltonian path of $A^I_{n,k} - F$ from x to y.

Now, assume that the lemma is true for all I' with $2 \le |I'| < I$. Thus, there is an $i' \in I$ with $i' \ne i, j$. Since $|GE^{i',j}(F)| \ge 3$, we can find an edge $(u, v) \in GE^{i',j}(F)$ with $u \in V(A^{i'}_{n,k})$ and $v \ne y \in V(A^j_{n,k})$. Then, there is a Hamiltonian path P_1 of $A^{I-\{j\}}_{n,k} - F$ from x to u and a Hamiltonian path P_2 of $A^j_{n,k} - F$ from v to y. Thus, $\langle x, P_1, u, v, P_2, y\rangle$ forms a Hamiltonian path of $A^I_{n,k} - F$ from x to y. □

The following lemma is proven by Ore [13].

Lemma 5. *A graph G is Hamiltonian if G has at least $C^{n-1}_2 + 2$ edges and Hamiltonian connected if G has at least $C^{n-1}_2 + 3$ edges.*

Lemma 6. *Assume that $n \ge 3$. Then, K_n is $(n-3)$-fault Hamiltonian and $(n-4)$-fault Hamiltonian connected.*

Proof. Let $F = F_v \cup F_e$ for $F_v \subseteq V(K_n)$ and $F_e \subseteq E(K_n)$ such that $|F| \leq n - i$ for $i = 2$ or 3. Then, $K_n - F$ is isomorphic to $K_{n-|F_v|} - F'$ for some $F' \subseteq E(K_{n-|F_v|})$ with $|F'| \leq |F_e|$. So, the number of edges in $K_{n-|F_v|} - F'$ is at least $C^{n-|F_v|-1}_{2} + i$. Hence, the proof of this lemma follows from Lemma 5. □

3 Main Theorem

Lemma 7. *Let $G = (V, E)$ be a loopless undirected graph and δ be the minimum degree of G. Then, G is at most $\delta - 2$ Hamiltonian if $\delta \geq 2$ and $\delta - 3$ Hamiltonian connected if $\delta \geq 3$.*

Proof. Let $u \in V(G)$ be a vertex of degree δ. Removing $(\delta - 1)$ edges connecting to u results in the isolation of u. Clearly, the remaining graph is not Hamiltonian. Then, consider removing $(\delta - 2)$ edges which connect to u. Let v_1 and v_2 be the remaining vertices connecting to u. Since $\delta \geq 3$, $|V(G)| \geq 4$. Thus, it is impossible that there is a Hamiltonian path in the remaining graph between v_1 and v_2 since u connects to only v_1 and v_2. Hence, the lemma follows. □

Theorem 1. *Let n and k be two positive integers with $n - k \geq 2$. Then, $A_{n,k}$ is $k(n-k) - 2$ Hamiltonian and $k(n-k) - 3$ Hamiltonian connected.*

Proof. Our proof is by induction on k. However, the proof of the induction is rather long. We break the whole proof into lemmas and prove them in the following sections.

The induction bases are $A_{n,1}$ and $A_{n,2}$. Since $A_{n,1}$ is K_n, by Lemma 6, the theorem is true for $A_{n,1}$. The case of $A_{n,2}$ is stated in the following lemma and its proof is in the Appendix.

Lemma 8. *$A_{n,2}$ is $2(n-2) - 2$ Hamiltonian and $2(n-2) - 3$ Hamiltonian connected if $n \geq 4$.*

We use the following two lemmas in the induction steps for the cases $k \geq 3$:

Lemma 9. *Suppose that, for some $k \geq 3$ and $n - k \geq 2$, $A_{n-1,k-1}$ is $(k-1)(n-k) - 2$ Hamiltonian and $(k-1)(n-k) - 3$ Hamiltonian connected. Then, $A_{n,k}$ is $k(n-k) - 2$ Hamiltonian.*

Lemma 10. *Suppose that, for some $k \geq 3$ and $n - k \geq 2$, $A_{n-1,k-1}$ is $(k-1)(n-k) - 2$ Hamiltonian and $(k-1)(n-k) - 3$ Hamiltonian connected. Then, $A_{n,k}$ is $k(n-k) - 3$ Hamiltonian connected.*

The proofs of the two lemmas are in Sections 4 and 5, respectively. With these lemmas, the theorem is proven. □

Hence, our results are optimal.

4 Proof of Lemma 9

Assume that F is any faulty set of $A_{n,k}$ with $|F| \leq k(n-k) - 2$. By Lemma 2, there exists an index $j \in \langle k \rangle$ such that $|F(A^{(j:i)}_{n,k})| \leq (k-1)(n-k) - 1$ for every $i \in \langle n \rangle$. By Lemma 1, we may assume that $j = k$. So, $|F(A^i_{n,k})| \leq (k-1)(n-k) - 1$ for every $i \in \langle n \rangle$. Without loss of generality, we further assume that

$$|F(A^1_{n,k})| \geq |F(A^2_{n,k})| \geq \cdots \geq |F(A^n_{n,k})|.$$

For convenience, we use $N^*_F(u)$ to denote the set of outer neighbors of u in $A_{n,k} - F$, i.e., $N^*_F(u) = \{v \mid (u, v) \in E(A_{n,k}) - F \text{ and } v \in N^*(u) - F\}$.

Case 1: $|F(A^1_{n,k})| \leq (k-1)(n-k) - 3$. Then, by induction hypothesis, $A^i_{n,k} - F$ is still Hamiltonian connected for every $i \in \langle n \rangle$. Consider two subcases:

Subcase 1.1: $|GE^{i,j}(F)| \geq 3$ for every $i \neq j \in \langle n \rangle$. If $F = \emptyset$, the lemma follows. If $F \neq \emptyset$, there is an index i such that $|F(A^1_{n,k})| + \sum_{l \neq i} |E^{i,l} \cap F| \geq 1$. Obviously, there are two distinct indices, j and l, for $j, l \neq i$ such that $|GE^{i,j}(F)| \geq 3$ and $|GE^{i,l}(F)| \geq 3$. Thus, we can find two edges $(u, x) \in GE^{i,j}(F)$ and $(v, y) \in GE^{i,l}(F)$ such that $u \neq v \in V(A^i_{n,k})$, $x \in V(A^j_{n,k})$, and $y \in V(A^l_{n,k})$. Then, there is a Hamiltonian path P_1 of $A^i_{n,k} - F$ from u to v and, by Lemma 4, there is a Hamiltonian path P_2 of $A^{\langle n \rangle - \{i\}}_{n,k} - F$ from y to x. Therefore, $\langle u, P_1, v, y, P_2, x, u \rangle$ is a Hamiltonian cycle of $A_{n,k} - F$.

Subcase 1.2: $|GE^{i,j}(F)| < 3$ for some $i \neq j \in \langle n \rangle$. By Proposition 3, there is only one pair i, j with $|GE^{i,j}(F)| = 2$ and $|GE^{i',j'}(F)| \geq 3$ for any $\{i', j'\} \neq \{i, j\}$. So, we can find two edges, $(u, x) \in GE^{i,j}(F)$ and $(v, y) \in GE^{i,l}(F)$, such that $u \neq v \in V(A^i_{n,k})$, $x \in V(A^j_{n,k})$, and $y \in V(A^l_{n,k})$ for $l \neq i, j$. Then, there is a Hamiltonian path P_1 of $A^i_{n,k} - F$ from u to v and, by Lemma 4, there is a Hamiltonian path P_2 of $A^{\langle n \rangle - \{i\}}_{n,k} - F$ from y to x. Therefore, $\langle u, P_1, v, y, P_2, x, u \rangle$ is a Hamiltonian cycle of $A_{n,k} - F$. See Fig. 2 for an illustration.

Case 2: $|F(A^1_{n,k})| = (k-1)(n-k) - 2$. So, $A^1_{n,k} - F$ is still Hamiltonian. Let C be a Hamiltonian cycle of $A^1_{n,k} - F$. Consider two cases:

Subcase 2.1: $|F(A^2_{n,k})| < (k-1)(n-k) - 2$. Then, $A^i_{n,k} - F$ is still Hamiltonian connected for every $i \in \langle n \rangle - \{1\}$. Since there are at most $(n-k)$ faults outside $A^1_{n,k}$ and

$$\begin{aligned} |V(A^1_{n,k}) - F| &\geq (n-1)(n-2) - (k-1)(n-k) + 2 \\ &\geq (n-1-k+1)(n-2) \geq 3(n-k), \end{aligned}$$

there exists a vertex u on C such that $N^*_F(u) = N^*(u)$. Consider the two neighbors of u on C, say v and v'. Clearly, $|N^*_F(v) \cup N^*_F(v')| \geq 1$. Thus, we may assume that there is an edge $(v, y) \in GE^{1,i}(F)$ for some $i \in \langle n \rangle - \{1\}$. (If $N^*_F(v) = \emptyset$, we use v' in place of v.) Then, there is an edge $(u, x) \in GE^{1,j}(F)$ with $j \neq i$ since $N^*_F(u) \geq 2$. Since $|F(A^{\langle n \rangle - \{1\}}_{n,k})| < k(n-k) - 2$, by Lemma 4, there is a Hamiltonian path $\langle x, P_1, y \rangle$ of $A^{\langle n \rangle - \{1\}}_{n,k} - F$ between x and y. Let $C = \langle u, v, P_2, v', u \rangle$. Then, $\langle u, x, P_1, y, v, P_2, v', u \rangle$ forms a Hamiltonian cycle of $A_{n,k} - F$. See Fig. 3a for an illustration.

Subcase 2.2: $|F(A^2_{n,k})| = (k-1)(n-k) - 2$. Then, $n = 5$, $k = 3$, and no fault is outside $A^1_{5,3}$ and $A^2_{5,3}$. So, $A^2_{5,3} - F$ is still Hamiltonian and $A^3_{5,3} - F$, $A^4_{5,3} - F$, and $A^5_{5,3} - F$ are Hamiltonian connected. Let C_2 be a Hamiltonian cycle of $A^2_{5,3} - F$. Since $|GE^{1,2}(F)| \geq 2$, let $(u, v) \in GE^{1,2}(F)$ for some $u \in V(A^1_{5,3})$ and $v \in V(A^2_{5,3})$. And, let $C = \langle u, u', P_1, u'', u \rangle$ and $C_2 = \langle v, P_2, v', v \rangle$. Since $N^*(v') = 2$, there exists $(v', y) \in GE^{2,i}(F)$ for some $y \in V(A^i_{5,3})$ with $i \neq 1$. By Proposition 2, $AS(u') \neq AS(u'')$ and then $|AS(u') \cup AS(u'')| \geq 3$. So, we can assume that there exists $(u') \in GE^{1,j}(F)$ for some $x \in V(A^j_{5,3})$ with $j \neq 2, i$. By Lemma 4, there is a Hamiltonian

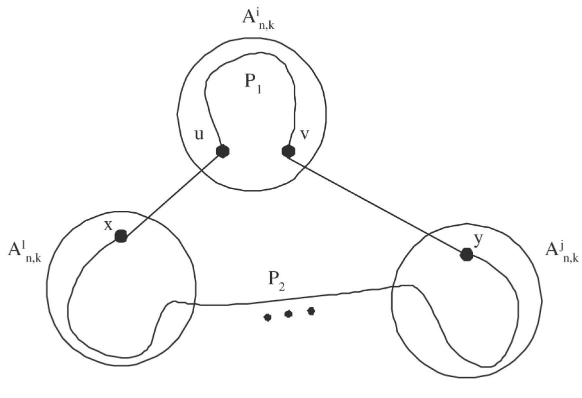


Fig. 2. Lemma 9, case 1.

path $\langle y, P_3, x\rangle$ of $A_{5,3}^{\{3,4,5\}}$ between x and y. So, $\langle u, v, P_2, v', y, P_3, x, u', P_1, u'', u\rangle$ forms a Hamiltonian cycle of $A_{5,3} - F$. See Fig. 3b for an illustration.

Case 3: $|F(A^1_{n,k})| = (k-1)(n-k) - 1$. Then, there are at most $(n-k) - 1$ faults outside $A^1_{n,k}$. So, $|F(A^i_{n,k})| \leq (n-k) - 1 \leq 2(n-k) - 3 \leq (k-1)(n-k) - 3$ for every $i \in \langle n\rangle - \{1\}$ and, by induction hypothesis, $A^i_{n,k} - F$ is still Hamiltonian connected. Let $f \in F(A^1_{n,k})$. f is either a vertex or an edge. Since $|F(A^1_{n,k}) - f| = (k-1)(n-k) - 2$, there is a Hamiltonian cycle C of $A^1_{n,k} - (F - f)$. Then, we consider the following three cases:

1. f is not on C. Let u, v be any two adjacent vertices on C.
2. f is an edge on C. Let u, v be the two vertices linked by f.
3. f is a vertex on C. Let u, v be the two vertices which are adjacent to f on C.

Thus, we have a Hamiltonian path $\langle u, P_1, v\rangle$ of $A^1_{n,k} - F$. Since $|N^*(u)| = |N^*(v)| = (n-k)$ and

$$|AS(u) \cup AS(v)| = n - k + 1,$$

there must exist two edges $(u, x) \in GE^{1,i}(F)$ and $(v, y) \in GE^{1,j}(F)$ with $i \neq j \in \langle n\rangle - \{1\}$. By Lemma 4, there is a Hamiltonian path $\langle y, P_2, x\rangle$ of $A^{\langle n\rangle - \{1\}}_{n,k}$ between x and y. So, $\langle u, P_1, v, y, P_2, x, u\rangle$ is a Hamiltonian cycle of $A_{n,k} - F$. See Fig. 4 for an illustration

This completes the induction proof. And, hence, the lemma follows. □

5 Proof of Lemma 10

Assume that F is any faulty set of $A_{n,k}$ with $|F| \leq k(n-k) - 3$. By Lemma 3, there exists an index $j \in \langle k\rangle$ such that $|F(A^{(j:i)}_{n,k})| \leq (k-1)(n-k) - 2$ for every $i \in \langle n\rangle$. By Lemma 1, we may assume that $j = k$. So, $|F(A^i_{n,k})| \leq (k-1)(n-k) - 2$ for every $i \in \langle n\rangle$. Without loss of generality, we further assume that $|F(A^1_{n,k})| \geq |F(A^2_{n,k})| \geq \cdots \geq |F(A^n_{n,k})|$. Let $x \in A^i_{n,k}$ and $y \in A^j_{n,k}$ with $i, j \in \langle n\rangle$ be two arbitrary vertices. We shall construct a Hamiltonian path of $A_{n,k} - F$ between x and y. For convenience, again we use $N^*_F(u)$ to denote the set of outer neighbors of u in $A_{n,k} - F$, i.e., $N^*_F(u) = \{v \mid (u, v) \in E(A_{n,k}) - F \text{ and } v \in N^*(u) - F\}$.

Case 1: $|F(A^1_{n,k})| \leq (k-1)(n-k) - 3$. Then, by induction hypothesis, $A^i_{n,k} - F$ is still Hamiltonian connected for every $i \in \langle n\rangle$. Consider two subcases:

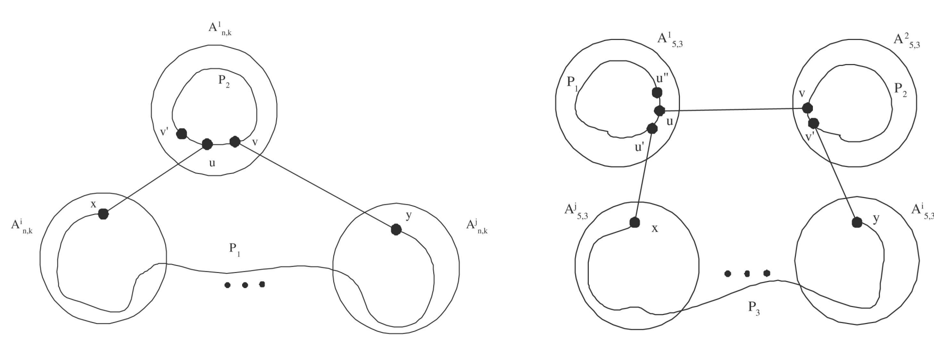
Subcase 1.1: $i \neq j$. Since $|F| \leq k(n-k) - 3$, by Lemma 4, there is a Hamiltonian path of $A_{n,k} - F$ joining x to y.

Subcase 1.2: $i = j$. By induction hypothesis, there is a Hamiltonian path P of $A^i_{n,k} - F$ from x to y. Let l be the number of vertices on P. Then,

$$\begin{aligned} l &= \frac{(n-1)!}{(n-k)!} - |F \cap V(A^i_{n,k})| \geq (n-1)(n-2) - |F(A^1_{n,k})| \\ &\geq (n-1)k - (k-1)(n-k) + 3 \\ &\geq k(n-1) - k(n-k) = k(k-1) \geq 2k. \end{aligned}$$

We claim that there exist two adjacent vertices u and v on P such that $|N^*_F(u)| \geq 1$ and $|N^*_F(v)| \geq 2$. Suppose that the statement is false. Then, for every two adjacent vertices u' and v' on P, $|N^*_F(u')| + |N^*_F(v')| \leq \max\{2, n-k\} = n-k$. Thus, $|F| \geq \lfloor l/2\rfloor (n-k) \geq k(n-k)$. We get a contradiction. Therefore, there exist two neighbors a and b of u and v, respectively, such that $(u, a) \in GE^{i,i'}(F)$ and $(v, b) \in GE^{i,j'}(F)$ with $i' \neq j' \in \langle n\rangle - \{i\}$. Since $|F| \leq k(n-k) - 3$, by Lemma 4, there is a Hamiltonian path $\langle a, P_1, b\rangle$ of $A^{\langle n\rangle - \{i\}}_{n,k} - F$. The Hamiltonian path P of $A^i_{n,k} - F$ can be written as $\langle x, P_2, u, v, P_3, y\rangle$ for some subpaths P_1 and P_2. Then, $\langle x, P_2, u, a, P_1, b, v, P_3, y\rangle$ is a Hamiltonian path of $A_{n,k} - F$ from x to y. See Fig. 5 for an illustration.

Case 2: $|F(A^1_{n,k})| = (k-1)(n-k) - 2$. So, $A^1_{n,k} - F$ is still Hamiltonian and there are at most $(n-k) - 1$ faults outside $A^1_{n,k}$. Let C be a Hamiltonian cycle of $A^1_{n,k} - F$. Now, consider the following four subcases:



(a) (b)

Fig. 3. Lemma 9, subcase 2.1 and subcase 2.2.

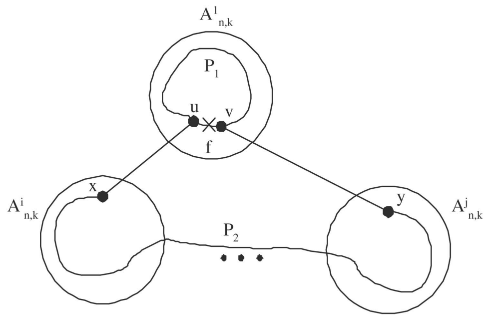


Fig. 4. Lemma 9, case 3.

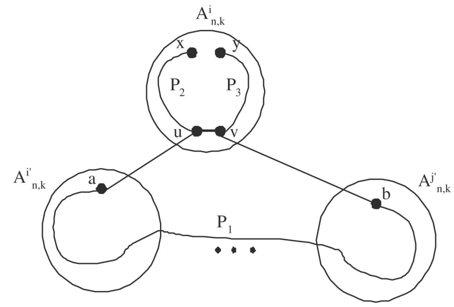


Fig. 5. Lemma 10, subcase 1.2.

Subcase 2.1: $i = j = 1$. If x, y are adjacent on C, let $C = \langle x, P, y, x\rangle$ and then the proof is similar to Subcase 1.2. Suppose x, y are not adjacent on C. Assume that $C = \langle x, v, P_1, y, u, P_2, x\rangle$. Since there are at most $(n-k)-1$ faults outside $A^1_{n,k}$, both $|N^*_F(u)|$ and $|N^*_F(v)|$ are at least 1 and one of them is at least 2. So, we can find two edges $(u, a) \in GE^{1,i'}(F)$ and $(v, b) \in GE^{1,j'}(F)$ with $i' \neq j' \in \langle n\rangle - \{1\}$. By Lemma 4, there is a Hamiltonian path P_3 of $A^{\langle n\rangle - \{1\}}_{n,k} - F$ from a to b. Therefore, $\langle x, P_2, u, a, P_3, b, v, P_1, y\rangle$ is a Hamiltonian path of $A_{n,k} - F$ between x and y. See Fig. 6a for an illustration.

Subcase 2.2: $i = 1$ and $j \neq 1$. Let $C = \langle x, u', P_1, u, x\rangle$. Since $|AS(u) \cup AS(u')| \geq n - k + 1$ and there are at most $(n - k - 1)$ faults outside $A^1_{n,k}$, we conclude that one of u and u' has an outer neighbor in $A_{n,k} - F$ which is not in $A^j_{n,k}$. We may assume that $(u, v) \in GE^{1,j'}(F)$ with $j' \neq j$. By Lemma 4, there is a Hamiltonian path $\langle v, P_2, y\rangle$ of $A^{\langle n\rangle - \{1\}}_{n,k} - F$. Then, $\langle x, u', P_1, u, v, P_2, y\rangle$ is a Hamiltonian path of $A_{n,k} - F$. See Fig. 6b for an illustration.

Subcase 2.3: $i = j \neq 1$. Consider two cases:

1. $n - k \geq 3$. The number of vertices on C which are adjacent to $A^i_{n,k}$ is at least

$$|E^{1,i}| - |F(A^1_{n,k})| = (n-2)\cdots(n-k) - (k-1)(n-k) + 2 \geq (n-k)(n-k-1) + 2.$$

Since there are at most $(n - k - 1)$ faults outside $A^1_{n,k}$ and

$$(n-k)(n-k-1) + 2 - 2(n-k-1) = (n-k-1)(n-k-2) + 2 \geq 4,$$

there is one pair of adjacent vertices u and v on C such that $|N^*_F(u)| = |N^*_F(v)| = n - k$ and u is adjacent to $A^i_{n,k}$ in $A_{n,k} - F$. Let $(u, u') \in GE^{1,i}(F)$ for some $u' \in V(A^i_{n,k})$. And, let $C = \langle u, v, P_1, u\rangle$. Without loss of generality, assume that $u' \neq y$. Since $A^i_{n,k} - F$ is Hamiltonian connected, let $P = \langle x, P_2, u', v', P_3, y\rangle$ be a Hamiltonian path of $A^i_{n,k} - F$ between x and y with $u' \neq v'$. Since there are two neighbors of u' on P, we may assume that $|N^*_F(v')| \geq 2$ without loss of generality. Let $(v', a) \in GE^{i,i'}(F)$ for some $a \in V(A^{i'}_{n,k})$ with $i' \neq 1$. Since $N^*_F(v) = n - k \geq 3$, there is an edge $(v, b) \in GE^{1,j'}(F)$ for some $b \in V(A^{j'}_{n,k})$ with $j' \neq i, i'$. By Lemma 4, there is a Hamiltonian path $\langle b, P_4, a\rangle$ of $A^{\langle n\rangle - \{1,i\}}_{n,k} - F$. So, $\langle x, P_2, u', u, P_1, v, b, P_4, a, v', P_3, y\rangle$ forms a Hamiltonian path of $A_{n,k} - F$ between x and y. See Fig. 7 for an illustration.
2. $n - k = 2$. Then, there is at most one fault f outside $A^1_{n,k}$. Consider the vertices on C which are adjacent to $A^i_{n,k}$. Clearly, the number of them is at least 2 and the number of other vertices on C is also at least 2. Thus, there are two pairs of vertices on C, say $\{u, v\}$ and $\{r, s\}$, such that u, r are adjacent to $A^i_{n,k}$ and v, s are not. Since there is at most one fault outside $A^1_{n,k}$, we may assume that $N^*_F(u) = N^*(u)$ and $N^*_F(v) = N^*(v)$. Let $C = \langle u, P_1, v, u\rangle$. And, let $(u, u') \in GE^{1,i}(F)$ for some $u' \in V(A^1_{n,k})$. Without loss of generality, assume that $u' \neq y$. Then, let $\langle x, P_2, u', v', P_3, y\rangle$ be a Hamiltonian path of $A^i_{n,k} - F$. If $N^*_F(v') \neq N^*(v')$, i.e., f is adjacent or incident to v', then we may use $\{r, s\}$ in place of $\{u, v\}$. Thus, we assume that $|N^*_F(v')| = |N^*(v')| = 2$ and then there is an edge $(v', a) \in GE^{i,i'}(F)$ for some $a \in V(A^{i'}_{n,k})$ with $i' \neq i$. Since $i \neq AS(v)$ and $|N^*_F(v)| = 2$, there is an edge $(v, b) \in GE^{1,j'}(F)$ for some $b \in V(A^{j'}_{n,k})$ with $j' \neq i, i'$. By Lemma 4, there is a Hamiltonian path $\langle b, P_4, a\rangle$ of $A^{\langle n\rangle - \{1,i\}}_{n,k} - F$. So, $\langle x, P_2, u', u, P_1, v, b, P_4, a, v', P_3, y\rangle$ forms a Hamiltonian path of $A_{n,k} - F$ between x and y.

Subcase 2.4: $i, j, 1$ are distinct. Since

$$|V(A^1_{n,k}) - F| \geq (n-1)(n-2) - (k-1)(n-k) + 2 \geq (n-1-k+1)(n-2) \geq 3(n-k),$$

we can find two adjacent vertices u and v on C such that $N^*_F(u) = N^*(u)$ and $N^*_F(v) = N^*(v)$. Let $C = \langle u, P_1, v, u\rangle$. Consider two cases:

1. $N^*(u)$ or $N^*(v)$ is $\{x, y\}$. Assume that $N^*(u) = \{x, y\}$. Then, $n - k = 2$ and there is at most one fault outside $A^1_{n,k}$. Assume that there is no fault in $A^i_{n,k}$. Let $F' = F(A^{\langle n\rangle - \{1\}}_{n,k}) \cup \{x\}$. (If there is a fault in $A^i_{n,k}$, we can change x to y.) Then, $|F'| = 2 \leq k(n-k) - 3$ and $A^l_{n,k} - F'$ is still Hamiltonian connected for every $l \in \langle n\rangle - \{1\}$. By Proposition 2, there is a neighbor of v, say w, such that $w \in A^{i'}_{n,k}$ with $i' \notin \{1, j\}$. Clearly, $w \neq x$. By Lemma 4, there is a Hamiltonian path $\langle w, P_2, y\rangle$ of $A^{\langle n\rangle - \{1\}}_{n,k} - F'$. So, $\langle x, u, P_1, v, w, P_2, y\rangle$ is a Hamiltonian path of $A_{n,k} - F$. See Fig. 8a for illustration.

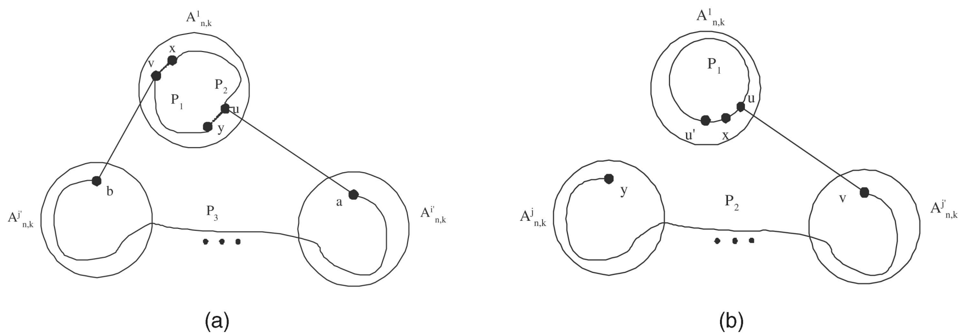


Fig. 6. Lemma 10, subcase 2.1 and subcase 2.2.

2. None of $N^*(u)$ and $N^*(v)$ is $\{x, y\}$. Then, there is an index $i' \in AS(u) \cup AS(v)$ such that $i' \notin \{1, i, j\}$. Assume that there is an edge $(u, a) \in GE^{1,i'}(F)$ for some $a \in V(A^{i'}_{n,k})$. Consider $N^*_F(v)$. Suppose that there is an outer neighbor b of v in $A^i_{n,k}$ or $A^j_{n,k}$. We may assume that $b \in V(A^i_{n,k})$. Then, there is a Hamiltonian path $\langle x, P_2, b\rangle$ of $A^i_{n,k} - F$ and a Hamiltonian path $\langle a, P_3, y\rangle$ of $A^{\langle n\rangle - \{1,i\}}_{n,k} - F$. And, we have a Hamiltonian path $\langle x, P_2, b, v, P_1, u, a, P_3, y\rangle$ of $A_{n,k} - F$. Suppose that there is no outer neighbor of v in $A^i_{n,k}$ or $A^j_{n,k}$. Since $|AS(v)| \geq 2$ and $i, j \notin AS(v)$, there is an edge $(v, b) \in GE^{1,j'}(F)$ for some $b \in V(A^{j'}_{n,k})$ with $j' \notin \{1, i, j, i'\}$. Then, there is a Hamiltonian path $\langle x, P_2, b\rangle$ of $A^{\{i,j'\}}_{n,k} - F$ and a Hamiltonian path $\langle a, P_3, y\rangle$ of $A^{\langle n\rangle - \{1,i,j'\}}_{n,k} - F$. So, $\langle x, P_2, b, v, P_1, u, a, P_3, y\rangle$ is a Hamiltonian path of $A_{n,k} - F$. See Fig. 8b for an illustration.

Hence, the lemma follows. □

6 Conclusion and Discussion

There are many studies concerning fault-tolerant Hamiltonicity and fault-tolerant Hamiltonian connectivity. We find that the current results on the arrangement graphs [8], [9] are not optimal. In this paper, we obtain an optimal result that the arrangement graph $A_{n,k}$ for $n - k \geq 2$ is $(k(n-k) - 2)$-Hamiltonian and $(k(n-k) - 3)$-Hamiltonian connected. For the case $n - k = 1$, since $A_{n,n-1}$ is isomorphic to an n-dimensional star graph which is bipartite, it cannot tolerate any vertex fault as far as fault Hamiltonicity and fault Hamiltonian connectivity are concerned.

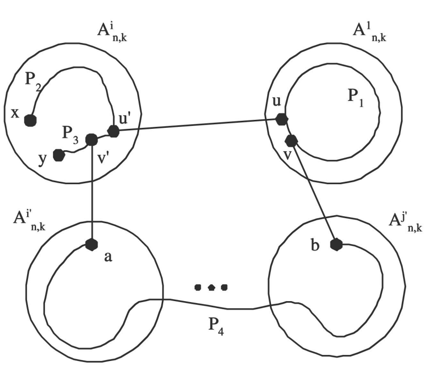


Fig. 7. Lemma 10, subcase 2.3.

In order to establish the fault-tolerant Hamiltonian property, we find that it is difficult to construct a fault-free Hamiltonian cycle directly. Therefore, we propose a new idea for proving this result. We propose to use fault Hamiltonian connectivity as a tool to attack the fault Hamiltonicity of the arrangement graph. We prove that it is f fault-tolerant Hamiltonian by simultaneously proving it is $f - 1$ fault-tolerant Hamiltonian connected. This strategy makes the proof tractable and systematic. It would be useful to apply this strategy to other interconnection networks for the same type of problems.

Appendix

Proof of Lemma 8

In this section, we concentrate our discussion on $A_{n,2}$. As the degree of $A_{n,2}$ is relatively larger than that of $A_{n,k}$ for $k \geq 3$, it can tolerate more faults. This makes the proof complex and we must use some special properties of $A_{n,2}$. In fact, when n is small, such as $4, 5, 6$, the resource (vertices or edges) which we can use is few. To prove these cases is very tedious. With a long and detailed discussion, we have completed the theoretical proof for a small value of n ($n = 4, 5, 6$). Nevertheless, we do not present them in this paper for reducing complexity. However, we can also verify these small cases directly using the computer by translating the original proof into a program.

Now, we give a quick view of $A_{n,2}$ and then present some of its special properties. $A_{n,2}$ consists of $n(n-1)$ vertices and $n(n-1)(n-2)$ edges. Indeed, it admits a vertex decomposition into n subgraphs, each isomorphic to K_{n-1}. Each vertex is labeled by a 2-digit string and, clearly, connects to $2(n-2)$ neighbors, where half of them are outer neighbors. So, $AS(u) = n - 2$ and there is only one subgraph not adjacent to u for any vertex $u \in V(A_{n,2})$. (Recall that $AS(u)$ is the adjacent set of u.) On the other hand, $|E^{i,j}| = n - 2$ for any $i, j \in \langle n\rangle$, so there is only one vertex in $A^i_{n,2}$ not adjacent to $A^j_{n,2}$. These two properties are important for our proof. In the following, we first show two useful propositions. Then, we divide the proof of Lemma 8 into two parts: first for Hamiltonicity and second for Hamiltonian connectivity.

Given any $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$, by Lemma 1, we may assume that

$$\max_{i \in \langle n\rangle}\{|F(A^{1:i}_{n,2})\} \geq |F(A^1_{n,2})| \geq |F(A^2_{n,2}| \geq \cdots \geq |F(A^n_{n,2})|.$$

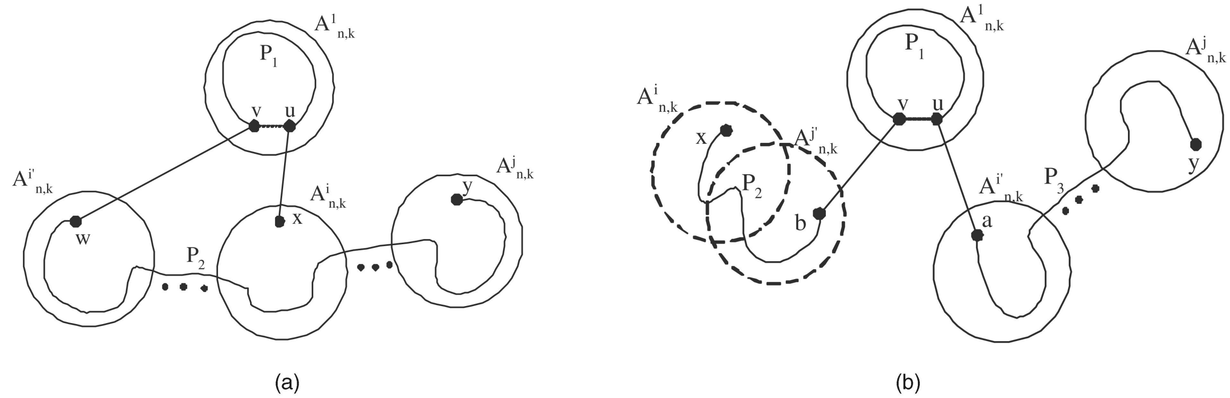


Fig. 8. Lemma 10, subcase 2.4.

Recall that $A^i_{n,2}$ is the abbreviation of $A^{2:i}_{n,2}$. So, we have the following proposition:

Proposition 4 (Fault distribution in $A_{n,2}$). *Let $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$ with $|F| \leq 2n-6$ and*

$$\max_{i\in\langle n\rangle}\{|F(A^{1:i}_{n,2})|\} \geq |F(A^1_{n,2})| \geq |F(A^2_{n,2}| \geq \cdots \geq |F(A^n_{n,2})|.$$

Then, $|F(A^1_{n,2})| \leq n-3$. In other words,

$$|F(A^2_{n,2})| \leq |F| - 2|F(A^1_{n,2})| + 2.$$

Proof. Let j be the index such that

$$|F(A^{1:j}_{n,2})| = \max_{i\in\langle n\rangle}\{|F(A^{1:i}_{n,2})|\}.$$

Suppose that $|F(A^1_{n,2})| \geq n-2$. The intersection of $A^{1:j}_{n,2}$ and $A^1_{n,2}$ contains only the vertex $j1$ for $j \neq 1$ or is empty for $j=1$, so $|F(A^{1:j}_{n,2}) \cap F(A^1_{n,2})| \leq 1$. Thus, $|F| \geq |F(A^{1:j}_{n,2})| + |F(A^1_{n,2})| - 1 \geq 2n-5$, which contradicts the given condition. So, $|F(A^1_{n,2})| \leq n-3$.

Then, consider $|F(A^2_{n,2})|$. Let $l = |F(A^1_{n,2})|$. Since $|F(A^{1:j}_{n,2}) \cap F(A^2_{n,2})| \leq 1$,

$$\begin{aligned} F(A^2_{n,2}) &\leq |F| - |F(A^{1:j}_{n,2}) \cup F(A^1_{n,2}) - F(A^2_{n,2})| \\ &\leq |F| - (2l-1) + |F(A^{1:j}_{n,2}) \cap F(A^2_{n,2})| \\ &\quad + |F(A^1_{n,2}) \cap F(A^2_{n,2})| \\ &\leq |F| - (2l-1) + 1 + 0 = |F| - 2l + 2. \end{aligned}$$

Hence, the statement follows. □

The following proposition uses a method similar to that used in Lemma 4 which helps us to construct a Hamiltonian path between any two given vertices in a given index set I of subgraphs of $A_{n,2}$. Notice that all subgraphs of $A_{n,2}$ are isomorphic to K_{n-1}, which is $(n-3)$ Hamiltonian and $(n-4)$ Hamiltonian connected. For convenience, we introduce a new notation, $EF(A^i_{n,2})$, called *extended faulty set of* $A^i_{n,2}$. $EF(A^i_{n,2})$ is defined to be the set $F(A^i_{n,2}) + \sum_{l\neq i}(E^{i,l} \cap F)$.

Lemma 11. *Let $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$ and $I \subseteq \langle n\rangle$ with $n \geq 7$ and $|I| \geq 2$. Let $x \in V(A^i_{n,2})$ and $y \in V(A^j_{n,2})$ with $i \neq j \in I$. Then, there is a Hamiltonian path of $A^I_{n,2} - F$ between x and y if $|F(A^I_{n,2})| \leq n-7+|I|$ and $|F(A^l_{n,2})| \leq n-5$ for each $l \in I$.*

Proof. Since each subgraph of $A_{n,2}$ is isomorphic to K_{n-1}, by the given condition, $A^l_{n,2} - F$ contains at least four vertices and, by Lemma 6, is still Hamiltonian connected for each $l \in I$. We prove this proposition by induction on $|I|$:

Case 1: $|I| = 2$. Then, $I = \{i,j\}$ and $|F| \leq n-5$. Since $|E^{i,j}| = n-2$, $|GE^{i,j}(F)| \geq 3$. There is an edge $(u,v) \in GE^{i,j}(F)$ for $u \neq x \in V(A^i_{n,2})$ and $v \neq y \in V(A^j_{n,2})$. Since $A^i_{n,2} - F$ and $A^j_{n,2} - F$ are still Hamiltonian connected, there are a Hamiltonian path $\langle x, P_1, u\rangle$ of $A^i_{n,2} - F$ and a Hamiltonian path $\langle v, P_2, y\rangle$ of $A^j_{n,2} - F$. Thus, $\langle x, P_1, u, v, P_2, y\rangle$ forms a Hamiltonian path of $A^I_{n,2} - F$ between x and y.

Case 2: $|I| = d \geq 3$. Assume that, for any $I' \subseteq \langle n\rangle$ with $2 \leq |I'| < d$, the statement is true. Consider the following two cases:

Subcase 2.1: $|F(A^{I-\{i,j\}}_{n,2})| \leq n+d-8$. Without loss of generality, assume that $|EF(A^i_{n,2})| \geq |EF(A^j_{n,2})|$. Thus, $|F(A^{I-\{i\}}_{n,2})| = |F(A^I_{n,2})| - |EF(A^i_{n,2})| \leq n+d-8$. (Remember that $|F(A^I_{n,2})| \leq n+d-7$.) Since $|F(A^i_{n,2})| \leq n-5$ and $|E^{i,l}| = n-2$ for $l \neq i$, there is an index $i' \neq j$ such that $|GE^{i,i'}(F)| \geq 2$. Otherwise, $|GE^{i,i'}(F)| \leq 1$ for all $i' \neq i, j$ and

$$\begin{aligned} |EF(A^i_{n,2})| &= |F(A^i_{n,2})| + \left|\sum_{l\neq i}(E^{i,l} \cap F)\right| \\ &\geq n-5+(n-3-(n-5))*(d-2) > n+d-7 \end{aligned}$$

for $d \geq 3$ which gives a contradiction. Thus, there is an edge $(u,v) \in GE^{i,i'}(F)$ for $u \neq x \in V(A^i_{n,2})$ and $v \in V(A^{i'}_{n,2})$. Since $A^i_{n,2} - F$ is Hamiltonian connected, there are a Hamiltonian path $\langle x, P_1, u\rangle$ of it and, by induction hypothesis, a Hamiltonian path $\langle v, P_2, y\rangle$ of $A^{I-\{i\}}_{n,2} - F$. So, the path $\langle x, P_1, u, v, P_2, y\rangle$ forms a Hamiltonian path of $A^I_{n,2}$ between x and y.

Subcase 2.2: $|F(A^{I-\{i,j\}}_{n,2})| = n+d-7$. Then, $|EF(A^i_{n,2})| = |EF(A^j_{n,2})| = 0$. If $I = \{i,j,i'\}$, i.e., $d=3$, then $|F(A^{i'}_{n,2})| = n-4$, which contradicts the given condition. So, $|I| = d \geq 4$. Since $n \geq 7$, there is an index $i' \in I - \{i,j\}$ such that $|EF(A^{i'}_{n,2})| \geq 2$. So,

$$\left|F\left(A^{I-\{i,i'\}}_{n,2}\right)\right| \leq n+d-9.$$

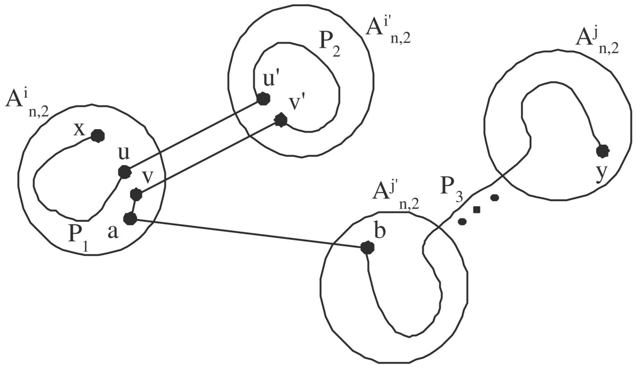


Fig. 9. Lemma 11, subcase 2.2.

However, $|F(A_{n,2}^{\{i,i'\}})| \leq n-5$. So, $|GE^{i,i'}(F)| \geq 3$ and then there are two edges, $(u,u'),(v,v') \in GE^{i,i'}(F)$, where $u,v \in V(A_{n,2}^{i})$ and $u',v' \in V(A_{n,2}^{i'})$ such that $u,v \neq x$. Let $j' \notin \{i,i',j\}$. Since $|GE^{i,j'}(F)| \geq 3$, there is an edge $(a,b) \in GE^{i,j'}(F)$, where $a \in V(A_{n,2}^{i})$ and $b \in V(A_{n,2}^{j'}) - F$ such that $a \neq x, u$. Note that v and a can be the same vertex. Since $A_{n,2}^{i} - \{v,a\}$ is still Hamiltonian connected, there is a Hamiltonian path $\langle x, P_1, u\rangle$ of $A_{n,2}^{i} - \{v,a\}$. Since $A_{n,2}^{i'} - F$ is Hamiltonian connected, there is a Hamiltonian path $\langle u', P_2, v'\rangle$ of it. By induction hypothesis, there is a Hamiltonian path $\langle b, P_3, y\rangle$ of $A_{n,2}^{I-\{i,i'\}} - F$. Thus, $\langle x, P_1, u, u', P_2, v', v, (a,)b, P_3, y\rangle$ forms a Hamiltonian path between x and y. See Fig. 9 for an illustration.

Hence, the lemma follows. □

Now, we prove Lemma 8. We divide the proof into two parts: first for Hamiltonicity and second for Hamiltonian connectivity. For the readability of the proof, we introduce the general steps of our proof for constructing a Hamiltonian cycle or path for each case here:

1. Find some key vertices in a particular order.
2. Construct some critical subpaths between key vertices.
3. Use Lemma 11 to construct the remaining part.
4. Concatenate these subpaths to form a Hamiltonian cycle or path.

Part 1. $A_{n,2}$ *is* $2(n-2)-2$ *Hamiltonian for* $n \geq 7$.

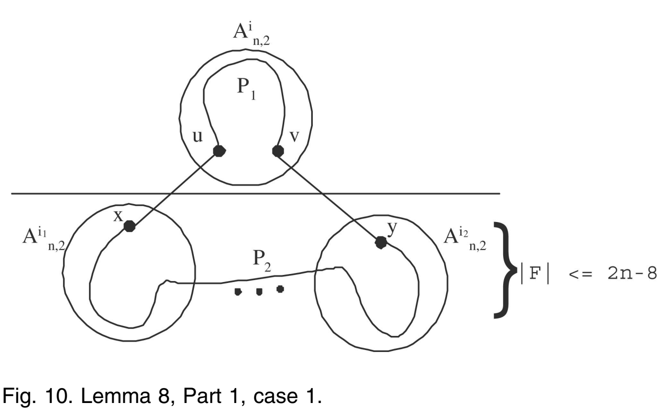


Fig. 10. Lemma 8, Part 1, case 1.

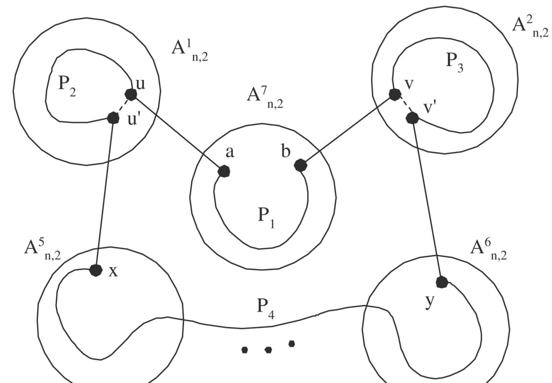


Fig. 11. Lemma 8, Part 1, subcase 2.2.

Proof. Let $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$ be an arbitrary faulty set of $A_{n,2}$ with $|F| \leq 2n-6$. By Proposition 4, we may assume that $\max_{i\in\langle n\rangle}\{|F(A_{n,2}^{1:i})\} \geq |F(A_{n,2}^{1})| \geq |F(A_{n,2}^{2}| \geq \cdots \geq |F(A_{n,2}^{n})|$ and $|F(A_{n,2}^{1})| \leq n-3$. Consider the following three cases:

Case 1. $|F(A_{n,2}^{1})| \leq n-5$. Then, $A_{n,2}^{i} - F$ is still Hamiltonian connected for each $i \in \langle n\rangle$. If $|F| = 0$, the statement follows. If $0 < |F| \leq 2n-7$, let i be the index such that $|EF(A_{n,2}^{i})| = 1$. If $|F| = 2n-6$, then $|F| > n$ for $n \geq 7$ and, by pigeonhole principle, there is an index i such that $|EF(A_{n,2}^{i})| \geq 2$. So, $|F(A_{n,2}^{\langle n\rangle-\{i\}})| \leq 2n-8$. Now, consider the edges connecting to $A_{n,2}^{i}$. Since $|E^{i,l}| = n-2$ for $l \neq i$ and $|F(A_{n,2}^{i})| \leq n-5$, there are two distinct indices $i_1, i_2 \neq i$ such that $|GE^{i,i_1}(F)| \geq 2$ and $|GE^{i,i_2}(F)| \geq 2$. Otherwise,

$$|F| \geq (n-5) + (n-3-(n-5))(n-2) > 2n-6.$$

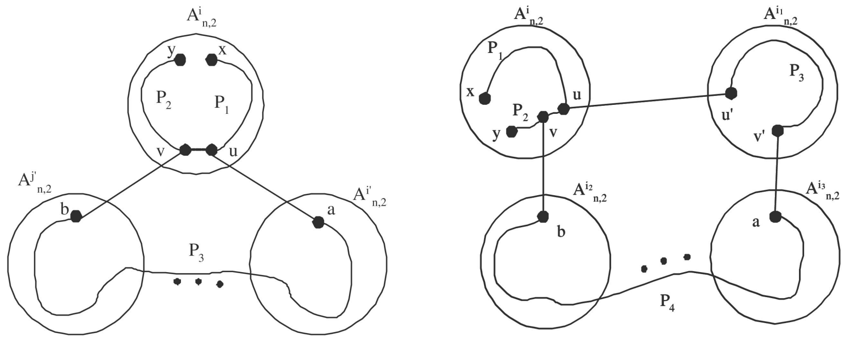
So, there are two edges $(u,x) \in GE^{i,i_1}(F)$ and $(v,y) \in GE^{i,i_2}(F)$ such that $u \neq v \in V(A_{n,2}^{i})$, $x \in V(A_{n,2}^{i_1})$, and $y \in V(A_{n,2}^{i_2})$. Since $A_{n,2}^{i} - F$ is Hamiltonian connected, there is a Hamiltonian path $\langle u, P_1, v\rangle$ of it. Let $I = \langle n\rangle - \{i\}$. Since $|F(A_{n,2}^{I})| \leq 2n-8 = n+|I|-7$, by Lemma 11, there is a Hamiltonian path $\langle y, P_2, x\rangle$ of $A_{n,2}^{I} - F$. So, $\langle u, P_1, v, y, P_2, x, u\rangle$ forms a Hamiltonian cycle of $A_{n,2} - F$. See Fig. 10 for an illustration.

Case 2. $|F(A_{n,2}^{1})| = n-4$. Then, $A_{n,2}^{1} - F$ is still Hamiltonian and there are $n-2$ faults outside $A_{n,2}^{1}$. Let C_1 be a Hamiltonian cycle of $A_{n,2}^{1} - F$. We observe that $|F(A_{n,2}^{2})| \leq |F(A_{n,2}^{1})|$, so $|F(A_{n,2}^{2})| \leq n-4$. Consider the following two subcases:

Subcase 2.1. $|F(A_{n,2}^{2})| \leq n-5$. Then, for every $i \neq 1$, $A_{n,2}^{i} - F$ is still Hamiltonian connected and $|V(A_{n,2}^{i}) - F| \geq 4$. Since $|V(A_{n,2}^{1}) - F| \geq 3$, there are two adjacent vertices u and v on C_1 such that $|N_F^{*}(u)| \geq 1$ and $|N_F^{*}(v)| \geq 2$. Otherwise,

$$|F| \geq n-4+3(n-3) > 2n-6.$$

Thus, there are two edges, $(u,x) \in GE^{1,i}(F)$ and $(v,y) \in GE^{1,j}(F)$, with $i \neq j \in \langle n\rangle - \{1\}$. Let $I = \langle n\rangle - \{1\}$. Then, $|F(A_{n,2}^{I})| \leq n-2 \leq n+|I|-7$. By Lemma 11, there is a Hamiltonian path P_1 of $A_{n,2}^{I} - F$ joining x to y. Assume that $C_1 = \langle v, P_2, u, v\rangle$. Then,



(a) (b)

Fig. 12. Lemma 8, Part 2, subcase 1.2.

$\langle u, x, P_1, y, v, P_2, u\rangle$ forms a Hamiltonian cycle of $A_{n,2} - F$.

Subcase 2.2. $|F(A^2_{n,2})| = n - 4$. So, $A^2_{n,2} - F$ is still Hamiltonian and $A^i_{n,2}$ is Hamiltonian connected for every $i \in \langle n\rangle - \{1, 2\}$. Let C_2 be a Hamiltonian cycle of $A^2_{n,2} - F$. Clearly, there are at most two faults outside $A^{\{1,2\}}_{n,2}$ and, so, there are at least three indices is such that $|F(A^i_{n,2})| + |E^{1,i} \cap F| + |E^{2,i} \cap F| = 0$. Assume that $4, 5, 7$ are such indices. Since $|V(A^1_{n,2} - F)| \geq 3$ and $|V(A^2_{n,2} - F)| \geq 3$, $|GE^{1,7}(F)| \geq 2$ and $|GE^{2,7}(F)| \geq 2$. There are two vertices $a \neq b$ in $A^7_{n,2}$ such that $(u, a) \in GE^{1,7}(F)$ and $(v, b) \in GE^{2,7}(F)$ for some $u \in V(A^1_{n,2})$ and $v \in V(A^2_{n,2})$. Clearly, there is a Hamiltonian path P_1 of $A^7_{n,2}$ joining b to a. Consider the two neighbors of u on C_1. There must be one of them adjacent to $A^5_{n,2}$. Let u' be such a vertex and $(u', x) \in E^{1,5}$ for some $x \in V(A^5_{n,2})$. Obviously, $(u_1, x) \in GE^{1,5}(F)$. Similarly, there is a vertex v' adjacent to v on C_2 and an edge $(v', y) \in GE^{2,6}(F)$ for some $y \in V(A^6_{n,2})$. Let $C_1 = \langle u, P_2, u', u\rangle$ and $C_2 = \langle v, v', P_3, v\rangle$, respectively. Let $I = \langle n\rangle - \{1, 2, 7\}$. Then $|F(A^I_{n,2})| \leq 2 \leq n + |I| - 7$. By Lemma 11, there is a Hamiltonian path P_4 of $A^I_{n,2} - F$ joining x to y. Thus, $\langle u, P_2, u', x, P_4, y, v', P_3, v, b, P_1, a, u\rangle$ forms a Hamiltonian cycle of $A_{n,2} - F$. See Fig. 11 for an illustration.

Case 3. $|F(A^1_{n,2})| = n - 3$. Then, $A^1_{n,2} - F$ may not be Hamiltonian. However, similar to Case 3 of Lemma 9, there is a Hamiltonian path $\langle u, P_1, v\rangle$ of $A^1_{n,2} - F$ for some vertices $u, v \in V(A^1_{n,2}) - F$. By Proposition 4, $|F(A^2_{n,2})| \leq 2n - 6 - 2(n - 3) + 2 = 2 \leq n - 5$ for $n \geq 7$ and, then, $A^i_{n,2} - F$ are Hamiltonian connected for each $i \neq 1$. Since $|N^*(u)| = |N^*(v)| = n - 2$, $|N^*_F(u)| \geq 1$ and $|N^*_F(v)| \geq 2$. Otherwise,

$$|F| \geq n - 3 + \max\{n - 2, 2(n - 3)\} > 2n - 6.$$

So, there exist two edges $(u, x) \in GE^{1,i}(F)$ and $(v, y) \in GE^{1,j}(F)$ with $i \neq j \in \langle n\rangle - \{1\}$. Since $|F(A^{\langle n\rangle - \{1\}}_{n,2})| \leq n - 3$, by Lemma 11, there is a Hamiltonian path $\langle y, P_2, x\rangle$ of $A^{\langle n\rangle - \{1\}}_{n,2} - F$. So, $\langle u, P_1, v, y, P_2, x, u\rangle$ forms a Hamiltonian path of $A_{n,2} - F$.

This completes the proof of **Part 1.** □

Part 2. *$A_{n,2}$ is $2(n - 2) - 3$ Hamiltonian connected for $n \geq 7$.*

Proof. Let $F \subseteq V(A_{n,2}) \cup E(A_{n,2})$ be an arbitrary faulty set of $A_{n,2}$ with $|F| \leq 2n - 7$. By Proposition 4, we may assume that

$$\max_{i \in \langle n\rangle}\{|F(A^{1:i}_{n,2})\} \geq |F(A^1_{n,2})| \geq |F(A^2_{n,2}| \geq \cdots \geq |F(A^n_{n,2})|$$

and $|F(A^1_{n,2})| \leq n - 3$. Let $x \in V(A^i_{n,2})$ and $y \in V(A^j_{n,2})$ for some $i, j \in \langle n\rangle$. We claim that there is a Hamiltonian path of $A_{n,2} - F$ from x to y. Consider the following cases:

Case 1. $|F(A^1_{n,2})| \leq n - 5$. Then, for every $i \in \langle n\rangle$, $A^i_{n,2} - F$ is Hamiltonian connected and $|V(A^i_{n,2}) - F| \geq 4$. Consider the following two subcases:

Subcase 1.1. $i \neq j$. By Lemma 11, there is a Hamiltonian path of $A_{n,2} - F$ from x to y.

Subcase 1.2. $i = j$. Let P be a Hamiltonian path of $A^i_{n,2} - F$ from x to y. First, consider that $|F(A^{\langle n\rangle - \{i\}}_{n,2})| \leq 2n - 8$. Since $|V(A^i_{n,2}) - F| \geq 4$, there are two vertices u and v adjacent on P such that $|N^*_F(u)| \geq 1$ and $|N^*_F(v)| \geq 2$. Otherwise,

$$|F| \geq \max\{2(n - 2), 4(n - 3)\} > 2n - 7.$$

Thus, there are two edges, $(u, a) \in GE^{i,i'}(F)$ and $(v, b) \in GE^{i,j'}(F)$, for some $a \in V(A^{i'}_{n,2})$ and $b \in V(A^{j'}_{n,2})$ with $i' \neq j'$. Notice that u, v, x, and y are not necessarily distinct. Then, let $P = \langle x, P_1, u, v, P_2, y\rangle$, where P_1 or P_2

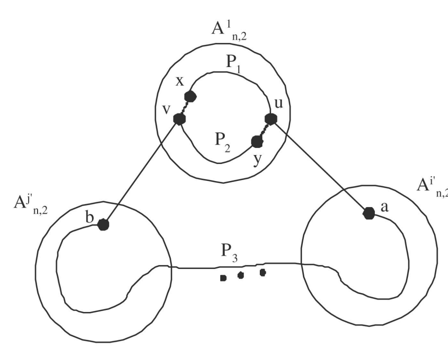


Fig. 13. Lemma 8, Part 2, subcase 2.1.

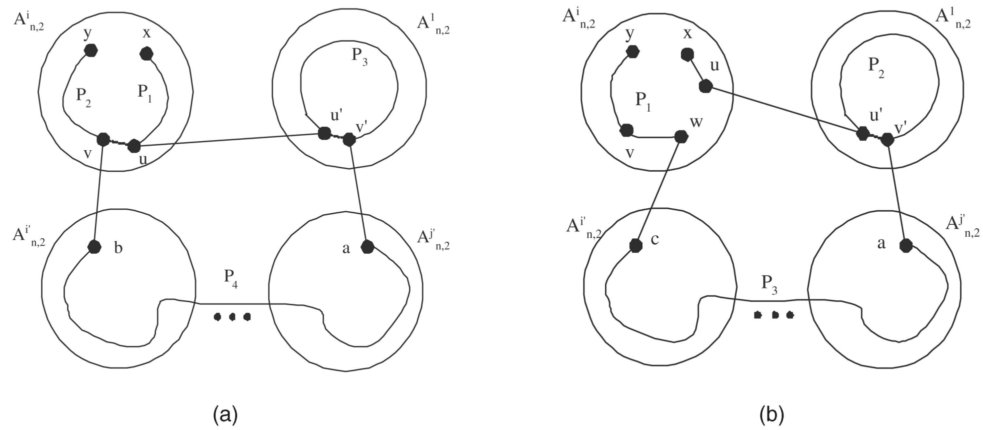


Fig. 14. Lemma 8, Part 2, situations 1 and 2 of subcase 2.2.

may be of length 0. Let $I = \langle n\rangle - \{i\}$. Then, $|F(A^I_{n,2})| \le 2n-8 = n+|I|-7$. By Lemma 11, there is a Hamiltonian path P_3 of $A^I_{n,2}-F$ from a to b. So, $\langle x, P_1, u, a, P_3, b, v, P_2, y\rangle$ forms a Hamiltonian path of $A_{n,2}-F$ from x to y. See Fig. 12a for an illustration.

Next, consider that $|F(A^{\langle n\rangle-\{i\}}_{n,2})| = 2n-7$. Then, $|EF(A^i_{n,2})| = 0$. By pigeonhole principle, there is an index i_1 such that $|EF(A^{i_1}_{n,2})| \ge 2$. Since $|F(A^{i_1}_{n,2})| \le n-5$ and $|EF(A^i_{n,2})| = 0$, $|GE^{i,i_1}(F)| \ge 3$ and then there is an edge $(u,u') \in GE^{i,i_1}(F)$ for $u \in V(A^i_{n,2}-\{x,y\})$ and $u' \in V(A^{i_1}_{n,2})$. Consider the two neighbors of u on P. There must be one of them, say v, such that $N^*_F(v) \ge 2$. (Otherwise, $|F| \ge 2(n-3) > 2n-7$.) Then, there is an edge $(v,b) \in GE^{i,i_2}(F)$ for $b \in V(A^{i_2}_{n,2})$ with $i_2 \ne i_1$. In $A^{i_1}_{n,2}$, since $|V(A^{i_1}_{n,2}-F)| \ge 4$, there is a vertex $v' \ne u' \in V(A^{i_1}_{n,2}-F)$ such that $|N^*_F(v')| \ge 3$. (Otherwise, $|F| \ge 3(n-4) > 2n-7$.) So, there is an edge $(v',a) \in GE^{i_1,i_3}(F)$ for $a \in V(A^{i_3}_{n,2})$ with $i_3 \ne i, i_2$. Let $P = \langle x, P_1, u, v, P_2, y\rangle$ and let P_3 be a Hamiltonian path of $A^{i_1}_{n,2}-F$ between u' and v'. Let $I = \langle n\rangle - \{i, i_1\}$. Since $|F(A^I_{n,2})| \le 2n-9 = n-7+|I|$, by Lemma 11, there is a Hamiltonian path P_4 of $A^I_{n,2}-F$ between a and b. Thus, $\langle x, P_1, u, u'P_3, v', a, P_4, b, v, P_2, y\rangle$ forms a Hamiltonian path of $A_{n,2}-F$. See Fig. 12b for an illustration.

Case 2: $|F(A^1_{n,2})| = n-4$ and $|F(A^2_{n,2})| \le n-5$. Then, $A^1_{n,2}-F$ is still Hamiltonian and all the other subgraphs are still Hamiltonian connected. Let C_1 be a Hamiltonian cycle of $A^1_{n,2}-F$. Consider the following four subcases:

Subcase 2.1: $i = j = 1$. If x and y are adjacent on C_1, the proof is similar to the first situation of subcase 1.2. So, consider that x and y are not adjacent on C_1. Then, $|V(A^1_{n,2}) - F| \ge 4$. (Otherwise, x and y must be adjacent.) Let $C_1 = \langle x, P_1, u, y, P_2, v, x\rangle$, where x, u, y, v are distinct. Since there are at most $(n-3)$ faults outside $A^1_{n,2}$, we can assume that $|N^*_F(u)| \ge 1$ and $|N^*_F(v)| \ge 2$. (Otherwise, $|F| \ge n-4+2(n-3) > 2n-7$.) So, there are two edges $(u,a) \in GE^{1,i'}(F)$ and $(v,b) \in GE^{1,j'}(F)$ for $a \in V(A^{i'}_{n,2})$ and $b \in V(A^{j'}_{n,2})$ with $i' \ne j'$. Let $I = \langle n\rangle - \{1\}$. Then, $|F(A^I_{n,2})| \le n-3 \le n+|I|-7$. By Lemma 11, there is a Hamiltonian path P_3 of $A^I_{n,2}-F$ from a to b. So, $\langle x, P_1, u, a, P_3, b, v, P_2, y\rangle$ forms a Hamiltonian path of $A_{n,2}-F$ from x to y. See Fig. 13 for an illustration.

Subcase 2.2: $i = j \ne 1$. Let P be a Hamiltonian path of $A^i_{n,2}-F$ from x to y. First, consider that there is an edge $(u',u) \in GE^{1,i}(F)$ for $u \in V(A^i_{n,2})$ and $u' \in V(A^1_{n,2})$. Without loss of generality, assume that $u \ne y$. (u may be x.) Let $P = \langle x, P_1, u, v, P_2, y\rangle$, where P_1 or P_2 may be of length 0. Suppose that there is an edge $(v,b) \in GE^{i,i'}$ for $b \in V(A^{i'}_{n,2})$ with $i' \ne 1$. Then, consider the two neighbors of u' on C_1, where C_1 is a fault-free Hamiltonian cycle of $A^1_{n,2}-F$. There must be one of them, say v', such that $N^*_F(v') \ge 3$. (Otherwise, $|F| \ge n-4+2(n-4) > 2n-7$.) Let $C_1 = \langle u', P_3, v', u'\rangle$. Then, there is an edge $(v',a) \in GE^{1,j'}(F)$ for $a \in V(A^{j'}_{n,2})$ with $j' \ne i, i'$. Let $I = \langle n\rangle - \{1, i\}$. Since $|F(A^I_{n,2})| \le n-3 \le n+|I|-7$, by Lemma 11, there is a Hamiltonian path $\langle a, P_4, b\rangle$ of $A^I_{n,2}-F$. Then, $\langle x, P_1, u, u', P_3, v', a, P_4, b, v, P_2, y\rangle$ forms a Hamiltonian path of $A_{n,2}-F$ from x to y. See Fig. 14a for an illustration.

Now, consider the case that there is no such vertex b as above. Then, there are $(n-3)$ edge (or vertex) faults connecting to v such that these faults are not in $A^1_{n,2}$ nor in $A^i_{n,2}$. Thus, besides these faults, all faults are in $A^1_{n,2}$. $F(A^i_{n,2}) = \emptyset$. Let $w \ne x, u, y, v$. Then, there is a Hamiltonian path P_1 of $A^i_{n,2} - \{x,u\}$ between w and y since $A^i_{n,2}$ is $(n-5)$-Hamiltonian connected. Clearly, $|N^*_F(w)| > 1$. There is

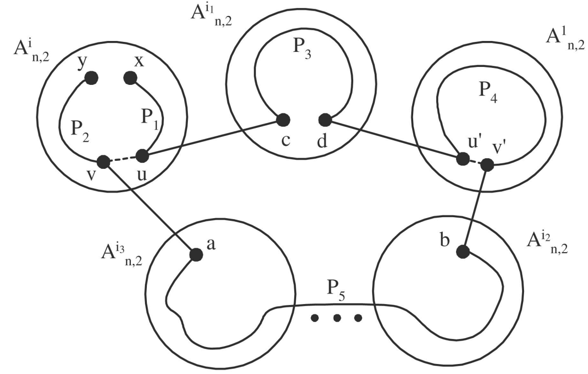


Fig. 15. Lemma 8, Part 2, situation 3 of subcase 2.2.

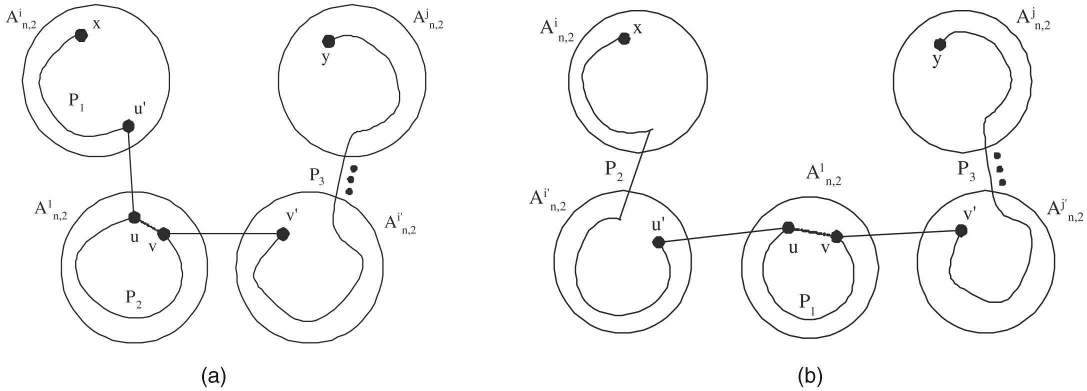


Fig. 16. Lemma 8, Part 2, subcase 2.4.

an edge $(w,c) \in GE^{i,i'}(F)$ for $c \in V(A^{i'}_{n,2})$ with $i' \neq 1$. Similarly to the previous situation, there is a neighbor v' of u' on C_1 such that $|N^*_F(v')| \geq 3$. Let $C_1 = \langle u', P_2, v', u' \rangle$. And, let $(v', a) \in GE^{1,j'}(F)$ for $a \in V(A^{j'}_{n,2})$ with $j' \neq i, i'$. Let $I = \langle n \rangle - \{1, i\}$. Since $|F(A^I_{n,2})| \leq n-3 \leq n+|I|-7$, there is a Hamiltonian path $\langle a, P_3, c \rangle$ of $A^I_{n,2} - F$. Then, $\langle x, (u,)u', P_2, v', a, P_3, c, w, P_1, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$. See Fig. 14b for an illustration.

Now, consider that $GE^{1,i}(F) = \emptyset$. Then, $|F(A^{\{1,i\}}_{n,2})| \geq n-2$ and at most $(n-5)$ faults are outside $A^{\{1,i\}}_{n,2}$. Thus, there are three indices i_1, i_2, i_3 such that, for each $l \in \{i_1, i_2, i_3\}$, $|F(A^l_{n,2})| + |E^{1,l} \cap F| + |E^{i,l} \cap F| = 0$. Since $|F(A^i_{n,2})| \leq n-5$, $|GE^{i,i_1}(F)| \geq 3$ and, so, there is an edge $(u,c) \in GE^{i,i_1}(F)$ for $u \in (V(A^i_{n,2}) - F \cup \{x,y\})$ and $c \in V(A^{i_1}_{n,2})$. Clearly, there is a neighbor v of u on P adjacent to $A^{i_2}_{n,2}$. Let b be the neighbor of v in $A^{i_2}_{n,2}$. Obviously, $(v,b) \in GE^{i,i_2}(F)$. Let $P = \langle x, P_1, u, v, P_2, y \rangle$, where P_2 may be of length 0. Then, consider $A^1_{n,2}$. Since $|F(A^1_{n,2})| = n-4$, $|GE^{1,i_1}(F)| \geq 2$ and, so, there is an edge $(u', d) \in GE^{1,i_1}(F)$ for $u' \in V(A^1_{n,2})$ and $d \neq c \in V(A^{i_1}_{n,2})$. Similarly, there is a neighbor v' of u' on C_1 adjacent to $A^{i_3}_{n,2}$. Let a be the neighbor of v' in $A^{i_3}_{n,2}$. Then, $(v', a) \in GE^{1,i_3}(F)$. Let $C_1 = \langle u', P_4, v', u' \rangle$. And, let $I = \langle n \rangle - \{1, i, i_1\}$. Then, $|F(A^I_{n,2})| \leq n-5 \leq n+|I|-7$. By Lemma 11, there is a Hamiltonian path $\langle a, P_5, b \rangle$ of $A^I_{n,2} - F$. So, $\langle x, P_1, u, c, P_3, d, u', P_4, v', a, P_5, b, v, P_2, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$ from x to y. See Fig. 15 for an illustration.

Subcase 2.3: $i = 1$ and $j \neq 1$. Since there are two neighbors of x on C_1 where C_1 is a Hamiltonian cycle of $A^1_{n,2} - F$, one of them, say u, has at least two neighbors in $A_{n,2} - F$. (Otherwise, $|F| \geq n-4+2(n-3) > 2n-7$.) So, there is an edge $(u,v) \in GE^{1,j'}(F)$ for $v \in V(A^{j'}_{n,2})$ with $j' \neq j$. By Lemma 11, there is a Hamiltonian path P_2 of $A^{\langle n \rangle - \{1\}}_{n,2} - F$ from v to y. Let $C_1 = \langle x, P_1, u, x \rangle$. Then, $\langle x, P_1, u, v, P_2, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$ from x to y.

Subcase 2.4: $i, j, 1$ are distinct. Since there are at most $(n-3)$ faults outside $A^1_{n,2}$, there is a subset S of $\langle n \rangle$ with $|S| \geq 2$ such that $|F(A^l_{n,2})| + |E^{1,l} \cap F| = 0$ for each $l \in S$. First, consider that i or $j \in S$. Without loss of generality, assume that $i \in S$. Since $|V(A^1_{n,2}) - F| \geq 3$, $|GE^{1,i}(F)| \geq 2$ and then there is a vertex u on C_1 such that $(u, u') \in GE^{1,i}(F)$ for $u' \neq x \in V(A^i_{n,2})$. Let P_1 be a Hamiltonian path of $A^i_{n,2}$ between x and u'. Since there are two neighbors of u on C_1, one of them, say v, satisfies that $|N^*_F(v)| \geq 3$. (Otherwise, $|F| \geq n-4+2(n-4) > 2n-7$.) Thus, there is an outer neighbor $v' \in V(A^{i'}_{n,2})$ of v in $A_{n,2} - F$ with $i' \notin \{1, i, j\}$. Let $C_1 = \langle u, P_2, v, u \rangle$. Let $I = \langle n \rangle - \{1, i\}$. Then, $F(A^I_{n,2})| \leq n-3 \leq n+|I|-7$. By Lemma 11, there is a Hamiltonian path P_3 of $A^I_{n,2} - F$ from v' to y. Thus, $\langle x, P_1, u', u, P_2, v, v', P_3, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$ from x to y. See Fig. 16a for an illustration.

Consider that $i, j \notin S$. Let i' and j' be two indices in S and $i' \neq j'$. Then, $|F(A^{i'}_{n,2})| + |E^{1,i'} \cap F| = 0$ and $|F(A^{i'}_{n,2})| + |E^{1,i'} \cap F| = 0$. Since $|V(A^1_{n,2}) - F| \geq 3$, there is a vertex u on C_1 adjacent to $A^{i'}_{n,2}$ and there is a neighbor v of u on C_1 adjacent to $A^{j'}_{n,2}$. Let u' be the neighbor of u in $A^{i'}_{n,2}$ and v' the neighbor of u in $A^{j'}_{n,2}$. Without loss of generality, we may assume that $|F(A^{\{i,i'\}}_{n,2})| \leq |F(A^{\{j,j'\}}_{n,2})|$. Then, $|F(A^{\{i,i'\}}_{n,2})| \leq n-5$. So, by Lemma 11, there is a Hamiltonian path P_2 of $A^{\{i,i'\}}_{n,2}$ between x and u'. Let $I = \langle n \rangle - \{1, i, i'\}$. Then, $|F(A^I_{n,2})| \leq n-4 \leq n+|I|-7$. By Lemma 11, there is a Hamiltonian path P_4 of $A^I_{n,2} - F$ from b to y. Then, $\langle x, P_3, w, w', P_2, a, u, P_1, v, b, P_4, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$ from x to y. See Fig. 16b for an illustration.

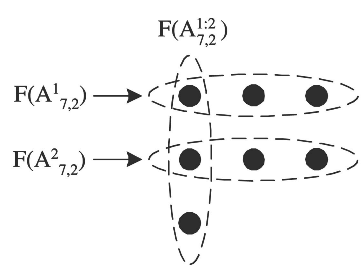
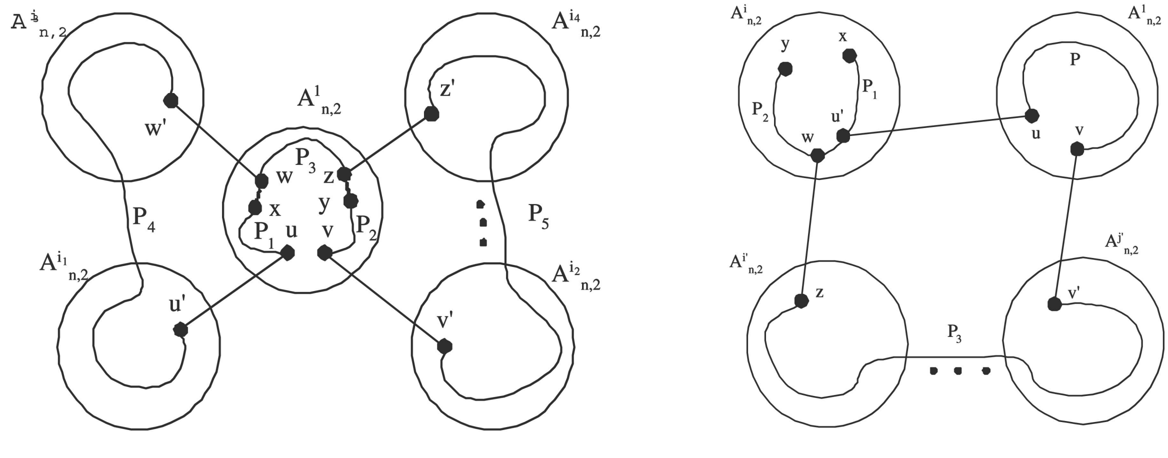
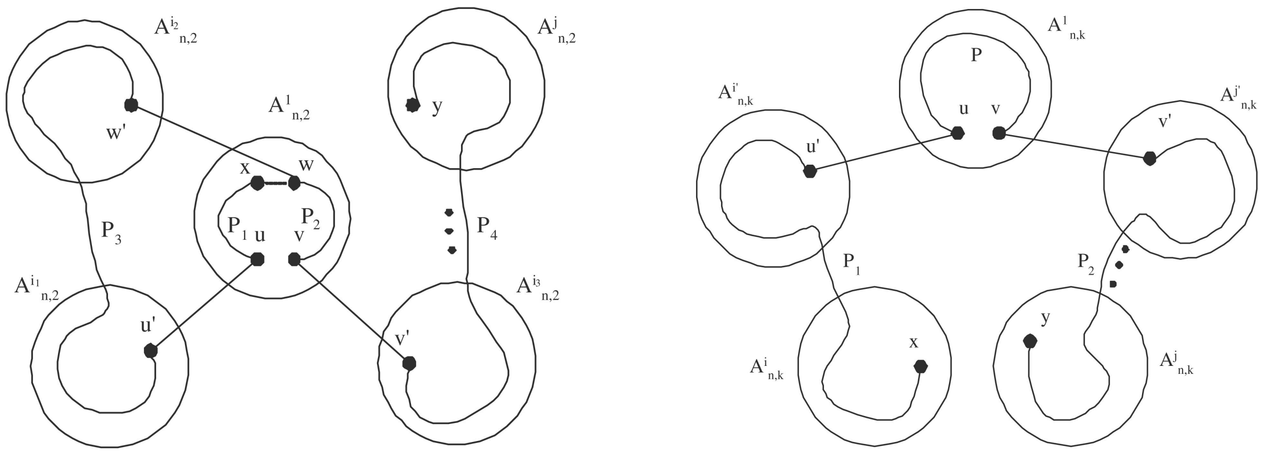


Fig. 17. Distribution of faults when $n = 7$ and $|F(A^2_{7,2})| = 3$.



(a) (b)

Fig. 18. Lemma 8, Part 2, subcase 4.1 and subcase 4.2.



(a) (b)

Fig. 19. Lemma 8, Part 2, subcase 4.3 and subcase 4.4.

Case 3: $|F(A^1_{n,2})| = n-4$ and $|F(A^2_{n,2})| = n-4$. Then, there is only one fault outside $A^1_{n,2}$ and $A^2_{n,2}$. By Proposition 4, $|F(A^2_{n,2})| = n-4 \leq |F| - 2(n-4) + 2$. Thus, $3n-14 \leq |F| \leq 2n-7$ and then $n=7$. Fig. 17 shows the distribution of faults when $n=7$. Clearly, there is no other index $i_1 \neq 2$ such that $|F(A^{1:i_1}_{7,2})| \geq 3$. Thus, this case is similar to the case that $|F(A^1_{n,2})| = n-4$ and $|F(A^2_{n,2})| \leq n-5$, which we have discussed in Case 2.

Case 4: $|F(A^1_{n,2})| = n-3$. Then,

$$|F| - 2(n-3) + 2 = 1 \geq |F(A^2_{n,2})| \geq \cdots \geq |F(A^n_{n,2})|.$$

Let $|F(A^{1:\hat{i}}_{n,2})| = \max_{l\in\langle n\rangle}\{|F(A^{1:l}_{n,2})|\}$. By Proposition 4, $|F(A^{1:\hat{i}}_{n,2})| = n-3$, $|F| = 2n-7$, and

$$F(A^{1:\hat{i}}_{n,2}) \cap F(A^1_{n,2}) = \{\hat{i}1\}.$$

Without loss of generality, assume that $\hat{i} = 2$. Then,

$$F \subseteq \{v_1v_2 \mid v_1 = 2 \text{ or } v_2 = 1\} \cup \{(u_1u_2, v_1v_2) \mid u_1 = v_1 = 2 \text{ or } u_2 = v_2 = 1\}.$$

Thus, for each vertex $u \in (A^1_{n,2} - F)$, u is not 21 and $N^*_F(u) = N^*(u)$. And, for any $l \neq 1$ and $l' \neq l$, $|F(A^l_{n,2})| \leq 1$ and $|E^{l,l'} \cap F| \leq 1$. Moreover, there are at least three subgraphs containing no fault since $|F - F(A^1_{n,2})| \leq n-4$. Similarly to Case 3 in Part 1, there is a Hamiltonian path of $A^1_{n,2} - F$. Let $\langle u, P, v\rangle$ be this path. Notice that such u and v are determined by $F(A^1_{n,2})$ and are not chosen freely. Now, consider the following four subcases:

Subcase 4.1: $i = j = 1$. First, consider that x and y are adjacent on P. Let $P = \langle u, P_1, x, y, P_2, v\rangle$. Notice that P_1 or P_2 may be of length 0. There are two edges $(u,u') \in GE^{1,i'}(F)$ and $(v,v') \in GE^{1,j'}(F)$ for $u' \in V(A^{i'}_{n,2})$ and $v' \in V(A^{j'}_{n,2})$ with $i' \neq j'$. Since $|F(A^{\langle n\rangle - \{1\}}_{n,2})| = n-4 \leq n + (n-1) - 7$, by Lemma 11, there is a Hamiltonian path $\langle u', P_3, v'\rangle$ of $A^{\langle n\rangle - \{1\}}_{n,2}$. Then, $\langle x, P_1, u, u', P_3, v', v, P_2, y\rangle$ forms a Hamiltonian path of $A_{n,2} - F$ from x to y.

Now, consider that x and y are not adjacent on P. Let $P = \langle u, P_1, x, w, P_2, z, y, P_3, v\rangle$. Notice that it is possible that w and z are the same. However, we take $|N^*(w) \cup N^*(z)|$ as $(n-2)$ in the following discussion. Since $n-2 \geq 4$, there are four edges, $(u,u') \in GE^{1,i_1}(F)$, $(v,v') \in GE^{1,i_2}(F)$, $(w,w') \in GE^{1,i_3}(F)$, and $(z,z') \in GE^{1,i_4}(F)$, for

$u' \in V(A^{i_1}_{n,2})$, $v' \in V(A^{i_2}_{n,2})$, $w' \in V(A^{i_3}_{n,2})$, and $z' \in V(A^{i_4}_{n,2})$ such that $i_1, i_2, i_3,$ and i_4 are four distinct indices. Since $d(u, w) = 1$ in $A^1_{n,2}$, by Lemma 2,

$$AS(u) \cup AS(v) = \langle n \rangle - \{1\}.$$

So, we may assume that $A^{i_1}_{n,2}$ and $A^{i_3}_{n,2}$ contain no fault. Let $I = \{i_1, i_3\}$ and $I' = \langle n \rangle - \{1, i_1, i_3\}$. Then, $|F(A^I_{n,2})| \leq 1$ and $|F(A^{I'}_{n,2})| \leq n - 4 \leq n + |I'| - 7$. By Lemma 11, there are a Hamiltonian path $\langle u', P_4, w' \rangle$ of $A^I_{n,2} - F$ and a Hamiltonian path $\langle z', P_5, v' \rangle$ of $A^{I'}_{n,2} - F$. Thus, $\langle x, P_1, u, u', P_4, w', w, P_2, z, z', P_5, v', v, P_3, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$. See Fig. 18a for an illustration.

Subcase 4.2: $i = j \neq 1$. Since $d(u, v) = 1$ in $A^1_{n,2}$, $i \in AS(u) \cup AS(v)$. We may assume that u is adjacent to $A^i_{n,2}$ and the neighbor of u in $A^i_{n,2}$ is u'. Without loss of generality, we may assume that $u' \neq y$. Since $|F(A^i_{n,2}| \leq 1$, there is a Hamiltonian path of $A^i_{n,2} - F$. Let $\langle x, P_1, u', w, P_2, y \rangle$ be the path, where $w \neq u'$. Since there are at most $(n-4)$ faults outside $A^1_{n,2}$, $|N^*_F(w)| \geq 2$. There is an edge $(w, z) \in GE^{i,i'}(F)$ for $z \in V(A^{i'}_{n,2})$ with $i \neq 1$. Since $|N^*_F(v)| = n - 2 \geq 4$, there is an edge $(v, v') \in GE^{1,j'}(F)$ for $v' \in V(A^{j'}_{n,2})$ with $j' \notin \{i, i'\}$. Let $I = \langle n \rangle - \{1, i\}$. Then, $|F(A^I_{n,2})| \leq n - 4 \leq n + |I| - 7$. By Lemma 11, there is a Hamiltonian path $\langle v', P_3, z \rangle$ of $A^I_{n,2} - F$. So, $\langle x, P_1, u', u, P, v, v', P_3, z, w, P_2, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$. See Fig. 18b for an illustration.

Subcase 4.3: $i = 1$ and $j \neq 1$. Without loss of generality, assume that x is not v. Let $P = \langle u, P_1, x, w, P_2, v \rangle$, where $x \neq w$. Since

$$|N^*_F(u)| = |N^*_F(v)| = |N^*_F(w)| = n - 2 \geq 4,$$

there are three edges,

$$(u, u') \in GE^{1,i_1}(F), (w, w') \in GE^{1,i_2}(F),$$

and

$$(v, v') \in GE^{1,i_3}(F),$$

for $u' \in V(A^{i_1}_{n,2})$, $w' \in V(A^{i_2}_{n,2})$, and $v' \in V(A^{i_3}_{n,2})$, where i_1, i_2, i_3, and j are distinct. Obviously, we may assume that $A^{i_1}_{n,2}$ and $A^{i_2}_{n,2}$ contain no fault. Let $I = \{i_1, i_2\}$ and $I' = \langle n \rangle - \{1, i_1, i_2\}$. Then, $|F(A^I_{n,2})| \leq 1$ and

$$|F(A^{I'}_{n,2})| \leq n - 4 \leq n + |I'| - 7.$$

By Lemma 11, there are a Hamiltonian path $\langle u', P_3, w' \rangle$ of $A^I_{n,2} - F$ and a Hamiltonian path $\langle v', P_4, y \rangle$ of $A^{I'}_{n,2} - F$. Then, $\langle x, P_1, u, u', P_3, w', w, P_2, v, v', P_4, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$. See Fig. 19a for an illustration.

Subcase 4.4: $i, j, 1$ are distinct. Since

$$|AS(u) \cup AS(v)| = n - 1,$$

there are two edges $(u, u') \in GE^{1,i'}(F)$ and $(v, v') \in GE^{1,j'}(F)$ for $u' \in V(A^{i'}_{n,2})$ and $v' \in V(A^{j'}_{n,2})$ such that i, j, i', j' are distinct and $A^{i'}_{n,2}$ contains no fault. Let $I = \{i, i'\}$ and $I' = \langle n \rangle - \{1, i, i'\}$. Then, $|F(A^I_{n,2})| \leq 2 \leq n - 5$ and $|F(A^{I'}_{n,2})| \leq n - 4 \leq n + |I'| - 7$. By Lemma 11, there are a Hamiltonian path $\langle x, P_1, u' \rangle$ of $A^I_{n,2} - F$ and a Hamiltonian path $\langle v', P_2, y \rangle$ of $A^{I'}_{n,2} - F$. Thus, $\langle x, P_1, u', u, P, v, v', P_2, y \rangle$ forms a Hamiltonian path of $A_{n,2} - F$. See Fig. 19b for an illustration.

Hence, the lemma follows. □

ACKNOWLEDGMENTS

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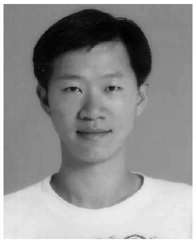

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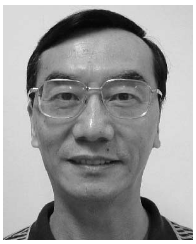

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