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EQUIVALENCE OF THE 1-RATE MODEL TO THE CLASSICAL MODEL ON STRICTLY NONBLOCKING SWITCHING NETWORKS*

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Abstract. In the 1-rate(f) network, each link can carry up to f messages for some integer f. The classical model is the special case when $f = 1$. We show that a network is strictly nonblocking under the 1-rate(f) model if and only if it is strictly nonblocking under the classical model.

Key words. switching network, 1-rate network, multirate network, graph coloring, flow, strictly nonblocking

AMS subject classifications. 68M10, 15C15, 90B18

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1. Introduction.

A switching network consists of a set of nodes and a set of (directed) links. Typically, an outlink of a node is the inlink of another node, and vice versa. There are two special types of nodes: the *inputs* and the *outputs*. Each input (output) is a node which has no inlink (outlink) and exactly one outlink (inlink).

We view a network as a directed graph $G = (V, E)$, where each vertex is a node and each edge is a link. The inputs and outputs are subsets I, O of V. To emphasize the special roles of the inputs and outputs, we denote a network as $G = (V, E, I, O)$. A network is called *acyclic* if the directed graph G is acyclic; i.e., G contains no directed cycles.

Let $G = (V, E, I, O)$ and f be a positive integer. The 1-rate(f) network, denoted by (G, f), is a network G together with the capacity constraint that each edge can carry up to f messages. If $f = 1$, then the 1-rate network $(G, 1)$ is the *classical model*. In other words, a classical model is a network in which each edge can carry at most one message. In this paper, we consider only 1-rate networks.

A *traffic* of (G, f) is a sequence of input-output pairs (i, j), where $i \in I$ and $j \in O$. There are two types of traffics: requests and cancellations. A *request* is a pair (i, j) such that neither of i, j has appeared in more than $f - 1$ previous uncancelled requests. Namely, the pair requests a connection in the network. A *cancellation* is a previous request whose connection in the network is to be removed. A request (i, j) is *routed* if a directed i-j-path is chosen, without exceeding the capacity of the edges. So a request (i, j) can be routed in the network (which has already routed many previous requests) if and only if there exists a directed i-j-path, each of whose edges has not been used more than $f - 1$ times.

A *state* S of (G, f) is a collection of (not necessarily distinct) directed paths of G joining vertices of I to vertices of O such that each edge e is contained in at most f directed paths. Given a state S, let $S(e)$ denote the number of directed paths

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containing e. Then $0 \leq S(e) \leq f$. A state is *blocking* if there exist a vertex $i \in I$ and $j \in O$ such that both i and j are contained in fewer than f directed paths in S, and every directed i-j-path of G contains an edge e with $S(e) = f$. We say that (G, f) is strictly nonblocking if there is no blocking state.

The classical model is, of course, the dominating model in the study of switching networks. Recently, the multirate network has received increasing attention due to the popular attempt to integrate multimedia service into one network. Since the theory of the classical model is well established, it is profitable to ask how much of it can be extended to the multirate model. The 1-rate model is the simplest multirate model but also has its own application. It is used in the *digital symmetrical matrices* in time-space switching [7, 10]. The principle of providing more links between two nodes, known as *statistical line grouping* in [8], was promoted as a major technique to cut down network blocking. On the other hand, strict nonblockingness is one of the most fundamental properties of a switching network. Therefore, asking whether one model implies the other on this property can serve as a natural start to explore the relation between the classical model and the multirate model. In this paper we prove that if $G = (V, E, I, O)$ is an acyclic network, then the strict nonblockingness of a 1-rate network (G, f) is equivalent to that of the classical model $(G, 1)$.

2. Strictly nonblocking for (G, f) implies the same for $(G, 1)$.

We first prove the implication in one direction.

THEOREM 1. *If (G, f) is strictly nonblocking for some positive integer f, then $(G, 1)$ is strictly nonblocking.*

Proof. It suffices to prove that if $(G, 1)$ has a blocking state, then (G, f) has a blocking state. Suppose S is a blocking state of $(G, 1)$. Let S' be the collection of directed paths of G which is obtained by duplicating f times each directed path of S. Then S' is a state of (G, f) and for each edge e of G, $S'(e) = f \times S(e)$. As S is a blocking state of $(G, 1)$, there is an input $i \in I$ and an output $j \in O$ such that none of i, j is contained in any directed path of S, and any directed i-j-path of G contains an edge e with $S(e) = 1$. Then both of i and j are contained in no directed paths of S', and every directed i-j-path of G contains an edge e with $S'(e) = f$. Therefore S' is a blocking state of (G, f). ∎

In the remainder of this paper, we shall prove the other direction; i.e., if for some integer $f \geq 1$, (G, f) has a blocking state, then $(G, 1)$ has a blocking state. Let S be a blocking state of (G, f). Then there exist $i \in I$ and $j \in O$ such that both i, j are contained in at most $f - 1$ directed path of S, and any directed i-j-path contains an edge e with $S(e) = f$. We need to construct a blocking state S' for $(G, 1)$. One may attempt to partition the directed paths in S into f classes such that

(i) directed paths which share an edge belong to different classes;

(ii) there exists a class C not containing any directed path with end vertex i or j.

If such a partition exists, then it is easy to verify that the class C is a blocking state of $(G, 1)$. However, such a partition may not exist. Consider the following network: Figure 1 shows an example of $(G, 2)$, where G is a simple digraph (a pair of double links indicates a link carrying two paths). The collection of directed paths $S = \{P_1, P_2, P_3, P_4\}$ in Figure 1 is a blocking state for $(G, 2)$, where input i and output j each has generated one path, and hence a new request (i, j) is legitimate. However, it is impossible to partition the paths into two classes in such a way that directed paths sharing an edge belong to different classes, because every two directed paths share an edge. Thus to construct the blocking state S' for $(G, 1)$, we need to use directed paths not contained in the collection S.

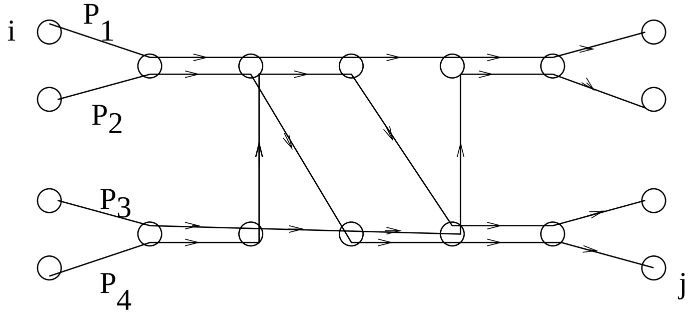

FIG. 1. *An example.*

3. Strictly nonblocking for $(G, 1)$ implies the same for $(G, 2)$. In this section, we consider the case $f = 2$.

THEOREM 2. *Suppose G is acyclic. If $(G, 1)$ is nonblocking, then $(G, 2)$ is nonblocking.*

Proof. Let S be a blocking state for $(G, 2)$. Thus there exist $i \in I$ and $j \in O$ such that both i, j are contained in at most one directed path of S, and any directed i-j-path contains an edge e with $S(e) = 2$.

We shall construct a blocking state for $(G, 1)$. For each vertex v of G, denote by $E^+(v)$ the outlinks of v and by $E^-(v)$ the inlinks of v. Let $E(v) = E^+(v) \cup E^-(v)$. Let

$$s^+(v) = \sum_{e \in E^+(v)} S(e) = \sum_{P \in S} |P \cap E^+(v)|,$$

$$s^-(v) = \sum_{e \in E^-(v)} S(e) = \sum_{P \in S} |P \cap E^-(v)|,$$

and

$$s(v) = s^+(v) + s^-(v) = \sum_{P \in S} |P \cap E(v)|.$$

Since each directed path $P \in S$ connects a vertex of I to a vertex of O, we conclude that for each vertex $v \notin I \cup O$, $|P \cap E^+(v)| = |P \cap E^-(v)|$. Hence $s^+(v) = s^-(v)$ and $s(v) = 2s^+(v)$. Let $E_1 = \{e \in E : S(e) = 1\}$, and let $E_2 = \{e \in E : S(e) = 2\}$. Then $s(v) = |E_1 \cap E(v)| + 2|E_2 \cap E(v)|$. If $v \notin (I \cup O)$, then $s(v)$ is even, and hence $|E_1 \cap E(v)|$ is even. Let $G_1 = (V, E_1)$ be the subgraph of G induced by the edge set E_1. As each vertex of $V - (I \cup O)$ has even degree in G_1, we can decompose G_1 into an edge-disjoint union of (not necessarily directed) cycles and paths, say

$$E_1 = (P_1 \cup P_2 \cup \cdots \cup P_l) \cup (C_1 \cup C_2 \cup \cdots \cup C_m),$$

where each path P_k connects two vertices of $I \cup O$. We color the edges of each P_k and C_l by two colors, a and b, as described below.

Given an undirected cycle (or a path), there are two choices for the *positive direction* of the cycle (or path). If the cycle is drawn on the plane, then either the clockwise direction or the counterclockwise direction can be chosen as the positive

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direction. For a path with end vertices i' and j', one can traverse the path from i' to j' or from j' to i'. Once a positive direction is chosen, then those directed edges that agree with the positive direction of the cycle (or path) are called *forward edges*, and those directed edges that oppose the positive direction are called *backward edges*. Arbitrarily choose a positive direction of C_l (or P_k) and color the forward edges of C_l (or P_k) by color a and backward edges by color b, except that if there exist an edge incident to i and an edge incident to j, then they should both be colored a. Observe that if these two edges are contained in a same path, then it is easy to see that they are in the same direction. Therefore whether the two edges are contained in a same path or contained in two distinct paths, by appropriately choosing the positive directions of the paths, they are both forward edges. So the required coloring exists.

Let $E_a \subset E_1$ be the edges of color a and $E_b \subset E_1$ be the edges of color b. Let $B_1 = E_a \cup E_2$ and $B_2 = E_b \cup E_2$. Suppose $v \notin (I \cup O)$. Let $i_a(v)$ (respectively, $o_a(v)$) be the number of inlinks (respectively, outlinks) of v of color a, and let $i_b(v)$ (respectively, $o_b(v)$) be the number of inlinks (respectively, outlinks) of v of color b.

If P_k (or C_l) contains v, then either P_k (or C_l) contains two inlinks or two outlinks of v which are colored by distinct colors or it contains one outlink and one inlink of v which are colored by the same color. Therefore

$$i_a(v) + o_b(v) = o_a(v) + i_b(v).$$

Let $i_2(v) = |E_2 \cap E^-(v)|$ and $o_2(v) = |E_2 \cap E^+(v)|$. Then

$$s^-(v) = i_a(v) + i_b(v) + 2i_2(v)$$

and

$$s^+(v) = o_a(v) + o_b(v) + 2o_2(v).$$

As $s^+(v) = s^-(v)$, we conclude that $i_a(v) + i_2(v) = o_a(v) + o_2(v)$ and $i_b(v) + i_2(v) = o_b(v) + o_2(v)$.

Let H_1 be the directed subgraph of G induced by the edge set $E_a \cup E_2$ and H_2 be the directed subgraph of G induced by the edge set $E_b \cup E_2$. Then for each vertex $v \notin (I \cup O)$, the number of inlinks of v in H_1 is $i_a(v) + i_2(v)$, and the number of outlinks of v in H_1 is $o_a(v) + o_2(v)$. So the number of inlinks of v is equal to the number of outlinks of v. As G is acyclic, H_1 is acyclic. Therefore H_1, and similarly H_2, can be decomposed into directed paths joining vertices of I to vertices of O. For $k = 1, 2$, denote by S_k the collection of directed paths which form a decomposition of H_k. For each edge e of G, $0 \le S_k(e) \le 1$ and $S(e) = S_1(e) + S_2(e)$. Moreover, both i and j are not contained in any directed paths of S_2. Any directed i-j-path of G contains an edge e with $S(e) = 2$, and hence $S_2(e) = 1$. Therefore S_2 is a blocking state of $(G, 1)$. ∎

4. Strictly nonblocking for $(G, 1)$ implies the same for (G, f). In this section, we prove that the strict nonblocking of the classical model implies the strict nonblocking of the 1-rate(f) model for any $f \ge 1$. Our proof needs a result concerning integer flows of graphs.

Let G be a directed graph. An *integer flow* of G is a mapping $\phi : E \to Z$ which assigns to each edge $e \in E$ an integer $\phi(e)$ such that for each vertex v of G,

$$\sum_{e \in E^+(v)} \phi(e) = \sum_{e \in E^-(v)} \phi(e).$$

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An integer flow ϕ is called a *nonnegative* k-flow if for each edge e, $0 \leq \phi(e) \leq k-1$. Lemma 1 is due to Little, Tutte, and Younger [9].

LEMMA 1. *For each nonnegative k-flow f of G, there exist $k-1$ nonnegative 2-flows ϕ_t $(t = 1, 2, \ldots, k-1)$ such that $\phi = \sum_{t=1}^{k-1} \phi_t$.*

LEMMA 2. *Suppose G is acyclic. If S is a state of (G, f), then there are f states $S_1, S_2, \ldots, S_f$ of $(G, 1)$ such that for each edge e of G, $S(e) = \sum_{i=1}^{f} S_i(e)$.*

Proof. Let S be a state of (G, f). Let G' be the directed graph obtained from G by identifying all the inputs and outputs, i.e., identifying all the vertices of $I \cup O$ into a single vertex v^*. We view S as a weight assignment to the edges of G'. It is easy to see that for each vertex v of G',

$$\sum_{e \in E^+(v)} S(e) = \sum_{e \in E^-(v)} S(e),$$

and for each edge of G',

$$0 \leq S(e) \leq f.$$

Therefore S is a nonnegative $(f+1)$-flow of G'. By Lemma 1, G' has f nonnegative 2-flows S_t $(t = 1, 2, \ldots, f)$ such that $S = \sum_{t=1}^{k-1} S_t$. Each nonnegative 2-flow S_t corresponds to a collection of edge disjoint directed cycles of G'. As G is acyclic, each directed cycle C contains the vertex v^*. In other words, each directed cycle C corresponds to a directed path of G joining a vertex of I to a vertex of O. Thus each S_t is indeed a state of $(G, 1)$. □

THEOREM 3. *If $(G, 1)$ is strictly nonblocking, then (G, f) is strictly nonblocking for any $f \geq 1$.*

Proof. Assume (G, f) is not strictly nonblocking and S is a blocking state of (G, f). Then there exist $i \in I$ and $j \in O$ such that both i and j are contained in fewer than f directed paths in S, and every directed i-j-path of G contains an edge e with $S(e) = f$. By Lemma 2, there exist f states, $S_1, S_2, \ldots, S_f$, of $(G, 1)$ such that for every edge e,

$$S(e) = \sum_{k=1}^{f} S_k(e).$$

As both i and j are contained in fewer than f directed paths in S, there exists $1 \leq a, b \leq f$ such that i is not contained in any path of S_a, and j is not contained in any path of S_b. If $a = b$, then S_a is a blocking state of $(S, 1)$. Assume $a \neq b$. Then $S_a \cup S_b$ is a blocking state of $(G, 2)$. By Theorem 2, $(G, 1)$ has a blocking state. □

COROLLARY 1. *Suppose $G = (V, E, O, I)$ is an acyclic network. Then for any positive integers f, f', (G, f) is strictly nonblocking if and only if (G, f') is strictly nonblocking.*

Proof. The strictly nonblocking of (G, f) is equivalent to the strictly nonblocking of $(G, 1)$ for any integer f. Hence strictly nonblocking of (G, f) is equivalent to strictly nonblocking of (G, f'). □

5. Some concluding remarks. Some other implications between the classical model and the multirate model are available from the literature. These involve some other notions of nonblockingness. A network is *wide-sense nonblocking* if every request can be routed, provided all routing follows a given algorithm. A network is *rearrangeably nonblocking* if all requests can be routed if they are given at once (instead of the usual "sequential" model).

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Let $C(n_1, r_1, m, n_2, r_2)$ denote the 3-stage Clos network whose nodes are partitioned into three stages (parts):

The first stage consists of r_1 nodes each with n_1 inlinks and m outlinks; the second stage consists of m nodes each with r_1 inlinks and r_2 outlinks; and the third stage consists of r_2 nodes each with m inlinks and n_2 outlinks such that there exists a link from each stage-i node to each stage-$(i+1)$ node but no other links between two nodes.

Clos [4] proved the following lemma.

LEMMA 3. *$C(n_1, r_1, m, n_2, r_2)$ is strictly nonblocking under the classical model if and only if*

$$m \geq \min\{n_1 + n_2 - 1, n_1 r_1, n_2 r_2\}.$$

Hwang and Yeh, as reported in [6], proved a similar result under a model slightly more general than the 1-rate(f) model; suppose each input has capacity f_0, each output has capacity f_0', each link between stage 1 and stage 2 has capacity f_1, and each link between stage 2 and stage 3 has capacity f_2.

LEMMA 4. *$C(n_1, r_1, m, n_2, r_2; f_0, f_0', f_1, f_2)$ is strictly nonblocking if and only if*

$$m \geq \left\lfloor \frac{\min\{n_1 f_1, n_2 r_2 f_2\} - 1}{f_0} \right\rfloor + \left\lfloor \frac{\min\{n_1 r_1 f_1, n_2 r_2\} - 1}{f_0} \right\rfloor + 1.$$

By setting $f_0 = f_0' = f_1 = f_2 = f$, we obtain the following corollary.

COROLLARY 2. *$C(n_1, r_1, m, n_2, r_2)$ is strictly nonblocking under the 1-rate(f) model if and only if*

$$m \geq \min\{n_1 + n_2 - 1, n_1 r_1, n_2 r_2\}.$$

Note that the conditions in Lemmas 3 and Corollary 2 are the same. Hence we obtain the following theorem.

THEOREM 4. *For $C(n_1, r_1, m, n_2, r_2)$, strictly nonblocking under the classical model implies the same for the 1-rate(f) model, and vice versa.*

Benes [1] proved the following lemma.

LEMMA 5. *$C(n, 2, m, n, 2)$ is wide-sense nonblocking under the classical model if and only if $m \geq \lfloor \frac{3n}{2} \rfloor$.*

On the other hand, Fishburn et al. [5] proved the following lemma.

LEMMA 6. *$C(n, 2, m, n, 2)$ is wide-sense nonblocking under the 1-rate(f) model if and only if $m \geq \lceil \frac{3n}{2} \rceil$.*

By comparing Lemmas 5 and 6, we obtain the following theorem.

THEOREM 5. *For $C(n, 2, m, n, 2)$, wide-sense nonblocking under the classical model does not imply the same for the 1-rate model.*

Finally, Chung and Ross [3] proved the following lemma.

LEMMA 7. *Rearrangeably nonblocking under the classical model implies the same for the 1-rate(f) model.*

For the other direction, only special cases have been proved. Slepian (see [1]) proved the following result (he ignored the terms $n_1 r_1$ and $n_2 r_2$, which reflect the boundary effects).

LEMMA 8. *$C(n_1, r_1, m, n_2, r_2)$ is rearrangeably nonblocking under the classical model if and only if $m \geq \max\{\min\{n_1, n_2 r_2\}, \min\{n_1 r_1, n_2\}\}$.*

On the other hand, Hwang and Yeh, as reported in [6], proved the following lemma.

LEMMA 9. *$C(n_1, r_1, m, n_2, r_2; f_0, f_0^{'}, f_1, f_2)$ is rearrangeably nonblocking if and only if*

$$m \geq \max\left\{\frac{\min\{n_1 f_1, n_2 r_2 f_2\}}{f_0}, \frac{\min\{n_1 r_1 f_1, n_2 f_2\}}{f_0^{'}}\right\}.$$

By setting $f_0 = f_0^{'} = f_1 = f_2$, we obtain the following corollary.

COROLLARY 3. *$C(n_1, r_1, m, n_2, r_2)$ is rearrangeably nonblocking under the 1-rate(f) model if and only if $m \geq \max\{\min\{n_1, n_2 r_2\}, \min\{n_1 r_1, n_2\}\}$.*

By comparing Lemma 8 and Corollary 3, we obtain the following theorem.

THEOREM 6. *For the 3-stage Clos network, rearrangeably nonblocking under the 1-rate(f) model implies the same for the classical model.*

Note that all these results deal with the very special 3-stage Clos networks. Chung and Ross, and the authors, are the only exceptions to attack the much harder general networks.

To summarize, we have

	classical		1-rate(f)	remark
strict		$\Longrightarrow$		proved
		$\Longleftarrow$		proved
wide-sense		$\Longrightarrow$		not true
		$\Longleftarrow$		possible
rearrangeable		$\Longrightarrow$		proved
		$\Longleftarrow$		possible

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