THE VELLING–KIRILLOV METRIC ON THE UNIVERSAL TEICHMÜLLER CURVE

By

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Abstract. We extend Velling's approach and prove that the second variation of the spherical areas of a family of domains defines a Hermitian metric on the universal Teichmüller curve, whose pull-back to Diff $+(S^1)/S^1$ coincides with the Kirillov metric. We call this Hermitian metric the Velling–Kirillov metric. We show that the vertical integration of the square of the symplectic form of the Velling–Kirillov metric on the universal Teichmüller curve is the symplectic form that defines the Weil–Petersson metric on the universal Teichmüller space. Restricted to a finite dimensional Teichmüller space, the vertical integration of the corresponding form on the Teichmüller curve is also the symplectic form that defines the Weil–Petersson metric on the Teichmüller space.

1 Introduction

Let $T(1)$ be the universal Teichmüller space and $\mathcal{T}(1)$ be the corresponding universal Teichmüller curve. $T(1)$ and $\mathcal{T}(1)$ have the natural structure of infinite dimensional complex manifolds, and the natural projection $p : \mathcal{T}(1) \to T(1)$ is a holomorphic fibration. In [Vel], J. Velling introduced a metric on $T(1)$ by using spherical areas. Namely, consider the Bers embedding of $T(1)$ into the Banach space

$$A_\infty(\Delta) = \left\{ \phi \text{ holomorphic on } \Delta : \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2| < \infty \right\},$$

where Δ is the unit disc. For every $Q \in A_\infty(\Delta)$ and t small, the solution to the equation

$$\mathcal{S}(f^{tQ}) = tQ, \tag{1.1}$$

where $\mathcal{S}(f)$ is the Schwarzian derivative of the function f, defines a family of domains $\Omega_t = f^{tQ}(\Delta)$. Here f^{tQ} is normalized so that $f^{tQ}(0) = 0$, $f_z^{tQ}(0) = 1$

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and $f^{tQ}_{zz}(0) = 0$. Velling proved that the spherical area $A_S(\Omega_t)$ of the domain Ω_t satisfies

$$\frac{d^2}{dt^2}A_S(\Omega_t)\big|_{t=0} \geq 0.$$

This defines a Hermitian metric on the tangent space to $T(1)$ at the origin, identified with $A_\infty(\Delta)$. Our first result, Theorem 3.4, is the following explicit formula for this metric:

$$\| Q \|_S^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}A_S(\Omega_t)\big|_{t=0} = \sum_{n=2}^{\infty} n|a_n|^2,$$

where $Q(z) = \sum_{n=2}^{\infty}(n^3 - n)a_n z^{n-2}$. The series converges for all $Q \in A_\infty(\Delta)$. However, since the spherical area of the domain $f^{tQ}(\Delta)$ is not independent of the choice of the function f^{tQ} that satisfies (1.1), $\| \cdot \|_S$ does not naturally define a metric on $T(1)$ by right group translations.[1] Nevertheless, Velling's approach can be generalized to define a metric on the universal Teichmüller curve $\mathcal{T}(1)$. This is achieved by a natural identification of $\mathcal{T}(1)$ with the space $\mathrm{Homeo}_{qs}(S^1)/S^1$ — the subgroup of orientation preserving quasisymmetric homeomorphisms of the unit circle that fix the point 1, and with the space

$$\begin{aligned}\tilde{\mathcal{D}} = \{f : \Delta \longrightarrow \hat{\mathbb{C}} \text{ a univalent function } &: f(0) = 0, f'(0) = 1,\\ &f \text{ has a quasiconformal extension to } \hat{\mathbb{C}}\},\end{aligned}$$

which we prove in Section 2. This endows $\mathcal{T}(1)$ with a group structure.[2] Following Velling's approach to $T(1)$, given a one-parameter family of univalent functions $f^t : \Delta \to \hat{\mathbb{C}} \in \tilde{\mathcal{D}}$, $f^t|_{t=0} = \mathrm{id}$, which defines a tangent vector v corresponding to $\frac{d}{dt}f^t|_{t=0}$ at the origin, we define a metric on the tangent space to $\mathcal{T}(1)$ at the origin by

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}A_S(f^t(\Delta))\big|_{t=0}$$

and extend it to every point of $\mathcal{T}(1)$ by right translations. This metric is Hermitian and Kähler. More remarkably, its pull-back via the embedding $\mathrm{Diff}_+(S^1)/S^1 \hookrightarrow \mathrm{Homeo}_{qs}(S^1)/S^1 \simeq \mathcal{T}(1)$ is precisely the metric

$$\| v \|^2 = \sum_{n=1}^{\infty} n|c_n|^2$$

on $\mathrm{Diff}_+(S^1)/S^1$ introduced by Kirillov [Kir87, KY87] via the coadjoint orbit method. Here $v = \sum_n c_n e^{in\theta}\partial/\partial\theta$, $c_{-n} = \overline{c_n}$ is a vector field on S^1. We call this

[1] The metric on $T(1)$ defined as a pull-back of the Hermitian metric on $A_\infty(\Delta)$ given by $\| \cdot \|_S$ is not natural. It does not induce a metric on finite dimensional Teichmüller spaces embedded in $T(1)$ since these embeddings are base-point dependent.

[2] It is well-known (see, e.g., [Nag93, Leh87]) that $\mathcal{T}(1)$ is not a topological group.

Kähler metric on $\mathcal{T}(1)$ the Velling–Kirillov metric and prove that it is the unique right invariant Kähler metric on $\mathcal{T}(1)$.

Let κ be the symplectic form of the Velling–Kirillov metric on $\mathcal{T}(1)$. We consider the $(1,1)$ form ω on $T(1)$, which is the vertical integration of the $(2,2)$ form $\kappa\wedge\kappa$ on $\mathcal{T}(1)$, i.e., integration of $\kappa\wedge\kappa$ over the fibers of the fibration $p:\mathcal{T}(1)\to T(1)$. We show that this is equivalent to Velling's suggestion of averaging the Hermitian form $\|\cdot\|_S$ along the fibers of $\mathcal{T}(1)$ over $T(1)$. Our second result, which we prove in Theorems 4.2 and 4.3, is that ω is the symplectic form of the Weil–Petersson metric on $T(1)$, defined only on tangent vectors which correspond to $H^{3/2}$ vector fields on S^1.

When Γ is a cofinite Fuchsian group, the Teichmüller space $T(\Gamma)$ of Γ embeds holomorphically in $T(1)$. The Bers fiber space $\mathcal{BF}(\Gamma)$ is the inverse image of $T(\Gamma)$ under the projection map $\mathcal{T}(1)\to T(1)$, and the Teichmüller curve $\mathcal{F}(\Gamma)$ is a quotient space of $\mathcal{BF}(\Gamma)$. The symplectic form κ is well-defined when restricted to $\mathcal{F}(\Gamma)$. We prove in Theorems 4.9 and 4.10 that the vertical integration of $\kappa\wedge\kappa$ via the map $\mathcal{F}(\Gamma)\to T(\Gamma)$ is the symplectic form that defines the Weil–Petersson metric on Γ.

In the Appendix, we consider an analogue of the Bers embedding for $\mathcal{T}(1)$. We prove that $\mathcal{T}(1)$ embeds into the Banach space

$$\mathcal{A}_\infty(\Delta)=\left\{\psi \text{ holomorphic on } \Delta : \sup_{z\in\Delta}|\psi(z)(1-|z|^2)|<\infty\right\},$$

and its image contains an open ball about the origin of $\mathcal{A}_\infty(\Delta)$. We also verify that $\mathcal{A}_\infty(\Delta)$ and $A_\infty(\Delta)\oplus\mathbb{C}$ induce the same complex structure on $\mathcal{T}(1)$. These results are not used in the main text.

The content of this paper is the following. In Section 2, we review different models for the universal Teichmüller space and the universal Teichmüller curve and study their relations with the homogeneous spaces of $\mathrm{Homeo}_{qs}(S^1)$. In Section 3, we review Velling's approach and define a metric on the universal Teichmüller curve. We prove that its pull-back to $\mathrm{Diff}_+(S^1)/S^1$ coincides with the Kirillov metric. In Section 4, we prove that the vertical integration of the square of the symplectic form of the Velling–Kirillov metric is the symplectic form that defines the Weil–Petersson metric on Teichmüller spaces. In the Appendix, we consider an embedding of $\mathcal{T}(1)$.

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available, which has been a great stimulation for the present work. The author has quoted or reproduced some of his results for the convenience of the reader.

2 Universal Teichmüller space and the universal Teichmüller curve

2.1 Teichmüller theory. Here we collect basic facts from Teichmüller theory. For details, see [Nag88, Ahl87, Leh87].

Let $T(1)$ be the universal Teichmüller space. There are two classical models of this space.

Let Δ be the open unit disc and $\Delta^* = \hat{\mathbb{C}} \setminus \overline{\Delta} = \{z \in \mathbb{C} \cup \{\infty\} \mid |z| > 1\}$ the exterior of the unit disc. Let $L^\infty(\Delta^*)$ (resp., $L^\infty(\Delta)$) be the complex Banach space of bounded Beltrami differentials on Δ^* (resp., Δ) and let $L^\infty(\Delta^*)_1$ be the unit ball of $L^\infty(\Delta^*)$. For any $\mu \in L^\infty(\Delta^*)_1$, we consider the following two constructions.

(I) Model A: w_μ theory.

We extend μ by reflection to Δ, i.e.,

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \quad z \in \Delta. \tag{2.1}$$

There is a unique quasiconformal map w_μ, fixing $-1, -i$ and 1, which solves the Beltrami equation

$$(w_\mu)_{\bar{z}} = \mu (w_\mu)_z.$$

It satisfies

$$\frac{1}{\overline{w_\mu(z)}} = w_\mu\left(\frac{1}{\bar{z}}\right) \tag{2.2}$$

by the reflection symmetry (2.1). As a result, w_μ fixes the unit circle S^1, Δ and Δ^*.

(II) Model B: w^μ theory.

We extend μ to be zero outside Δ^*. There is a unique quasiconformal map w^μ, holomorphic on the unit disc, which solves the Beltrami equation

$$w^\mu_{\bar{z}} = \mu w^\mu_z,$$

and is normalized such that $f = w^\mu|_\Delta$ satisfies $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$.

The universal Teichmüller space $T(1)$ is defined as a set of equivalence classes of normalized quasiconformal maps

$$T(1) = L^\infty(\Delta^*)_1 / \sim,$$

where $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on the unit circle, or equivalently, $w^\mu = w^\nu$ on the unit disc.

Using model B, we can identify $T(1)$ with the space

$$\mathcal{D} = \{f : \Delta \to \mathbb{C} \text{ univalent } : f(0) = 0, f'(0) = 1, f''(0) = 0; \\ f \text{ has a quasiconformal extension to } \mathbb{C}\}.$$

Let $\mathcal{S}(f)$ be the Schwarzian derivative of the function f, which is given by

$$\mathcal{S}(f) = \left(\frac{f_{zz}}{f_z}\right)_z - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2 = \frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

Let $A_\infty(\Delta)$ be the Banach space

$$A_\infty(\Delta) = \left\{\phi \text{ holomorphic on } \Delta : \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2| < \infty\right\}.$$

The Bers embedding $T(1) \hookrightarrow A_\infty(\Delta)$, which maps $[\mu]$ — the equivalence class of μ — to $\mathcal{S}(w^\mu|_\Delta)$, endows $T(1)$ with a unique structure of a complex Banach manifold such that the projection map

$$\Phi : L^\infty(\Delta^*)_1 \to T(1)$$

is a holomorphic submersion. In particular, $L^\infty(\Delta^*)_1$ and $A_\infty(\Delta)$ induce the same complex structure on $T(1)$.

The derivative of the map Φ at the origin

$$D_0\Phi : L^\infty(\Delta^*) \longrightarrow T_0T(1)$$

is a complex linear surjection, with kernel $\mathcal{N}(\Delta^*)$ — the space of infinitesimally trivial Beltrami differentials. Explicitly,

$$\mathcal{N}(\Delta^*) = \left\{\mu \in L^\infty(\Delta^*) : \iint_{\Delta^*} \mu\phi = 0 \, , \; \forall\phi \in A_1(\Delta^*)\right\},$$

where $A_1(\Delta^*)$ is the Banach space of L^1 (with respect to Lebesgue measure on Δ^*) holomorphic functions on Δ^*.

Define

$$A_\infty(\Delta^*) = \left\{\phi \text{ holomorphic on } \Delta^* : \sup_{z\in\Delta^*} |\phi(z)(1-|z|^2)^2| < \infty\right\}$$

and its complex anti-linear isomorphic space

$$\Omega^{-1,1}(\Delta^*) = \left\{\mu(z) = (1-|z|^2)^2\overline{\phi(z)} : \phi \in A_\infty(\Delta^*)\right\},$$

the space of harmonic Beltrami differentials on Δ^*. There is a canonical splitting

$$L^\infty(\Delta^*) = \mathcal{N}(\Delta^*) \oplus \Omega^{-1,1}(\Delta^*),$$

which identifies the tangent space at the origin of $T(1)$ with $\Omega^{-1,1}(\Delta^*)$. Moreover, the Bers embedding induces the isomorphism $\Omega^{-1,1}(\Delta^*) \xrightarrow{\sim} A_\infty(\Delta)$ given by

$$\mu \mapsto \phi(z) = -\frac{6}{\pi} \iint\limits_{\Delta^*} \frac{\mu(\zeta)}{(\zeta - z)^4} \left| \frac{d\zeta \wedge d\bar{\zeta}}{2} \right|. \tag{2.3}$$

$L^\infty(\Delta^*)_1$ has a group structure induced by the composition of quasiconformal maps,

$$\lambda * \mu = \nu, \quad \text{where } w_\nu = w_\lambda \circ w_\mu.$$

Explicitly, it is given by

$$\nu = \frac{\mu + (\lambda \circ w_\mu)\frac{\overline{(w_\mu)_z}}{(w_\mu)_z}}{1 + \overline{\mu}(\lambda \circ w_\mu)\frac{\overline{(w_\mu)_z}}{(w_\mu)_z}}.$$

This group structure descends to $T(1)$. Moreover, the right group translation by $[\mu]$, $R_{[\mu]} : T(1) \to T(1)$, $[\lambda] \mapsto [\lambda * \mu]$ is biholomorphic. However, the left group translation is not even a continuous map on $T(1)$ (see, e.g., [Nag88, Leh87]).

Remark 2.1. Conventionally, the model of the universal Teichmüller space is the complex conjugate of the one we define above. Consider the natural complex anti-linear isomorphism

$$\begin{array}{ccc} L^\infty(\Delta^*)_1 & \to & L^\infty(\Delta)_1 \\ \mu & \mapsto & \tilde{\mu} = \overline{\mu\left(\frac{1}{\bar{z}}\right)\frac{z^2}{\bar{z}^2}}, \qquad z \in \Delta. \end{array}$$

Setting $\tilde{\mu}$ to be zero outside Δ, we obtain a unique solution of the Beltrami equation

$$w^{\tilde{\mu}}_{\bar{z}} = \tilde{\mu} w^{\tilde{\mu}}_z,$$

which is holomorphic on Δ^* and normalized such that $g = w^{\tilde{\mu}}|_{\Delta^*}$ has Laurent expansion at ∞ given by

$$g(z) = z\left(1 + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots\right). \tag{2.4}$$

Thus $T(1)$ is identified with the space

$$\mathcal{D}^* = \{g : \Delta^* \to \hat{\mathbb{C}} \text{ univalent } : g \text{ has Laurent expansion at } \infty \text{ given by (2.4)}$$
$$\text{and has quasiconformal extension to } \hat{\mathbb{C}}\}.$$

The universal Teichmüller curve $\mathcal{T}(1)$ is a fiber space over $T(1)$. The fiber over each point $[\mu]$ is the quasidisc $w^\mu(\Delta^*) \in \hat{\mathbb{C}}$ with the complex structure induced from $\hat{\mathbb{C}}$,

$$\mathcal{T}(1) = \Big\{([\mu], z) : [\mu] \in T(1),\ z \in w^\mu(\Delta^*)\Big\}. \tag{2.5}$$

It is a Banach manifold modeled on $A_\infty(\Delta) \oplus \mathbb{C}$. We have a real analytic isomorphism between $T(1) \times \Delta^*$ and $\mathcal{T}(1)$ given by

$$([\mu], z) \mapsto ([\mu], w^\mu(z)).$$

2.2 Homogeneous spaces of $\operatorname{Homeo}_{qs}(S^1)$**.** Let $\operatorname{Homeo}_{qs}(S^1)$ be the group of orientation preserving quasisymmetric homeomorphisms of the unit circle S^1. It contains the subgroup of orientation preserving diffeomorphisms — $\operatorname{Diff}_+(S^1)$. We denote by $\operatorname{Möb}(S^1)$ the subgroup of Möbius transformations and, abusing notation, denote by S^1 the subgroup of rotations.

Consider the model A of the universal Teichmüller space $T(1)$ given above. Clearly, the map $T(1) \ni [\mu] \mapsto w_\mu|_{S^1} \in \operatorname{Homeo}_{qs}(S^1)$ is well-defined and one-to-one. The Ahlfors–Beurling extension theorem implies that its image consists of all normalized orientation preserving quasisymmetric homeomorphisms of the unit circle (see, e.g., [Ber72, Nag88, Leh87]); in other words,

$$T(1) \cong \operatorname{Homeo}_{qs}(S^1)/\operatorname{Möb}(S^1).$$

Let $\mu \in \Omega^{-1,1}(\Delta^*)$ be a tangent vector at the origin of $T(1)$. It generates the one-parameter flow $w_{t\mu}$; and the corresponding vector field is given by $\dot{w}_\mu \partial/\partial z$, where

$$\dot{w}_\mu(z) = \frac{(z+1)(z+i)(z-1)}{2\pi i} \iint\limits_{\mathbb{C}} \frac{\hat{\mu}(\zeta)}{(\zeta - z)(\zeta+1)(\zeta+i)(\zeta-1)} d\zeta \wedge d\bar{\zeta},$$

and $\hat{\mu}$ is the extension of μ by reflection to $\mathbb{C}$. Restricted to S^1, we have $\dot{w}_\mu(z) = iz\mathrm{u}(z)$, where $\mathrm{u}(e^{i\theta})\partial/\partial\theta$ is the vector field on S^1.

It was proved by Reimann (see [Rei76, GS92, Nag93]) that the tangent space to $\operatorname{Homeo}_{qs}(S^1)$ at the origin is the Zygmund space

$$\Lambda(S^1) = \left\{ \mathrm{u}(e^{i\theta})\frac{\partial}{\partial\theta} : \text{(i) } \mathrm{u} : S^1 \to \mathbb{R} \text{ is continuous, and} \right.$$
$$\left. \text{(ii) } F_{\mathrm{u}}(x) = \frac{1}{2}(x^2+1)\mathrm{u}\left(\frac{x-i}{x+i}\right) \text{ is in } \Lambda(\mathbb{R}) \right\},$$

where

$$\Lambda(\mathbb{R}) = \Big\{ F : \mathbb{R} \to \mathbb{R} : \text{(i)} F \text{ is continuous, and}$$
$$\text{(ii)} \left| F(x+t) + F(x-t) - 2F(x) \right| \leq B\,|t| \text{ for some } B, \forall x, t \in \mathbb{R} \Big\}.$$

By imposing extra normalization conditions, we can characterize the tangent space at the origin of $\mathrm{Homeo}_{qs}(S^1)/S^1$ and $\mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ in a similar way.

Remark 2.2. It is not known how to characterize the Zygmund space $\Lambda(S^1)$ using Fourier coefficients on S^1.

In [Kir87], Kirillov considered the Lie group $\mathrm{Diff}_+(S^1)$ and proved that there is a natural bijection between the space $\mathcal{K}$ of smooth contours of conformal radius 1 which contain 0 in their interior and the space $\mathrm{Diff}_+(S^1)/S^1$. We generalize this bijection in the following theorem.

Theorem 2.3. *There is a natural bijection between the space* $\mathrm{Homeo}_{qs}(S^1)/S^1$ *and the space* $\mathcal{K}_{qc}$ *of all quasicircles, i.e. images of the unit circle under quasiconformal maps of conformal radius 1 which contain* 0 *in their interior. Moreover, for every* $\gamma \in \mathrm{Homeo}_{qs}(S^1)/S^1$, *there exists two univalent functions* $f : \Delta \to \mathbb{C}$ *and* $g : \Delta^* \to \hat{\mathbb{C}}$ *determined by the following properties:*

1. *f and g admit quasiconformal extensions to quasiconformal mappings of $\hat{\mathbb{C}}$;*
2. $\gamma = g^{-1} \circ f|_{S^1} \mod S^1$;
3. $f(0) = 0,\ f'(0) = 1$;
4. $g(\infty) = \infty,\ g'(\infty) > 0$.

Proof. By the Ahlfors–Beurling extension theorem, an orientation preserving quasisymmetric homeomorphism γ of the unit circle can be extended to a quasiconformal map w of $\hat{\mathbb{C}}$ satisfying the reflection property (2.2). Let μ be the Beltrami differential of the map $w|_{\Delta^*}$. Up to a linear fractional transformation, w agrees with w_μ as defined in Section 2.1, i.e., $w = \sigma_1 \circ w_\mu$ for some $\sigma_1 \in \mathrm{PSU}(1,1)$. The corresponding map w^μ (Section 2.1) is holomorphic inside the unit disc Δ. Define $g = \sigma_2 \circ w^\mu \circ w^{-1}$, where $\sigma_2 \in \mathrm{PSL}(2,\mathbb{C})$ is uniquely determined by the requirement that $f = \sigma_2 \circ w^\mu$ satisfy $f(0) = 0$, $f'(0) = 1$ and g satisfy $g(\infty) = \infty$. The maps $f|_\Delta$ and $g|_{\Delta^*}$ are holomorphic. They do not depend on the extension of γ, and we have $\gamma = g^{-1} \circ f|_{S^1}$. The image of S^1 under f, which is the same as the image of S^1 under g, is by definition a quasicircle $\mathcal{C}$ with conformal radius 1. By post-composing w with a rotation, we can arrange for the map g also to satisfy $g'(\infty) > 0$.

Conversely, by definition, a quasicircle $\mathcal{C}$ is the image of S^1 under a quasiconformal map $h : \mathbb{C} \to \mathbb{C}$. Let μ_1 be the Beltrami differential of $h|_\Delta$, extended to Δ^* by reflection. Let w_{μ_1} be a solution of the corresponding Beltrami equation. Then $f = h \circ w_{\mu_1}^{-1}$ is a quasiconformal map which is holomorphic inside Δ. When 0 is in the interior of $\mathcal{C}$, there is a unique way to normalize w_{μ_1} by post-composition with a $\mathrm{PSU}(1,1)$ transformation such that $f(0) = 0$ and $f'(0) > 0$. The image of S^1 under f is the quasicircle $\mathcal{C}$. In fact, by the Riemann mapping theorem, $f|_\Delta$ is uniquely determined by $\mathcal{C}$ and the normalization conditions $f(0) = 0$, $f'(0) > 0$. That $\mathcal{C}$ has conformal radius 1 implies that $f'(0) = 1$. Let μ be the Beltrami differential of $f|_{\Delta^*}$, extended to Δ by reflection. Let w_μ be a solution of the corresponding Beltrami equation. Define $g = f \circ w_\mu^{-1} \circ \sigma$, where $\sigma \in \mathrm{PSU}(1,1)$ is uniquely determined so that $g(\infty) = \infty$ and $g'(\infty) > 0$. The map $\gamma = g^{-1} \circ f|_{S^1}$ is then an orientation-preserving quasisymmetric homeomorphism of the unit circle. □

The decomposition $\gamma = g^{-1} \circ f$ is known as conformal welding. Using the fact that the correspondence between f and the quasicircle $\mathcal{C}$ is one-to-one, we can identify $\mathrm{Homeo}_{qs}(S^1)/S^1$ with the space of univalent functions

$$\tilde{\mathcal{D}} = \{f : \Delta \longrightarrow \hat{\mathbb{C}} \text{ a univalent function } : f(0) = 0, f'(0) = 1,$$
$$f \text{ has a quasiconformal extension to } \mathbb{C}\}.$$

$\tilde{\mathcal{D}}$ is a complex subspace of the complex space of sequences $\{a_n\}$ (Fourier coefficients of the holomorphic function f). This induces a complex structure on $\mathrm{Homeo}_{qs}(S^1)/S^1$.

Remark 2.4. Observe that if $\gamma = w_\mu|_{S^1}$ up to post-composition with a $\mathrm{PSU}(1,1)$ transformation, then the corresponding f is equal to w^μ up to post-composition with a $\mathrm{PSL}(2,\mathbb{C})$ transformation.

We identify $\mathrm{Homeo}_{qs}(S^1)/S^1$ as the subgroup of $\mathrm{Homeo}_{qs}(S^1)$ consisting of quasisymmetric homeomorphisms that fix the point 1. Consider $\mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ as the subspace of $\mathrm{Homeo}_{qs}(S^1)/S^1$ corresponding to the natural inclusion $T(1) \simeq \mathcal{D} \hookrightarrow \tilde{\mathcal{D}} \simeq \mathrm{Homeo}_{qs}(S^1)/S^1$. Analogous to the isomorphism $T(1) \simeq \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$, we have

Theorem 2.5. *There is an isomorphism between $\mathcal{T}(1)$ and $\mathrm{Homeo}_{qs}(S^1)/S^1 \simeq \tilde{\mathcal{D}}$. Moreover, the complex structure of $\mathcal{T}(1)$ induced from $A_\infty(\Delta) \oplus \mathbb{C}$ coincides with the complex structure induced from $\tilde{\mathcal{D}}$.*

Proof. The fiber of $\mathrm{Homeo}_{qs}(S^1)/S^1$ over $\gamma \in \mathrm{Homeo}_{qs}(S^1)/\mathrm{Möb}(S^1)$ consists of all quasisymmetric homeomorphisms of the form $\sigma \circ \gamma \bmod S^1$, where $\sigma \in \mathrm{PSU}(1,1) \bmod S^1$ are parametrized by $w \in \Delta^* \simeq \mathrm{PSU}(1,1)/S^1$, i.e.,

$$\sigma_w(z) = \frac{1 - z\bar{w}}{z - w}. \tag{2.6}$$

Let f, g (resp. f_w, g_w) be the univalent functions corresponding to γ (resp., $\gamma_w = \sigma_w \circ \gamma$), i.e.,

$$\gamma = g^{-1} \circ f, \quad \sigma_w \circ \gamma = g_w^{-1} \circ f_w.$$

Using Remark 2.4, we have

$$f_w = \lambda_w \circ f, \qquad \text{and hence} \quad g_w = \lambda_w \circ g \circ \sigma_w^{-1}$$

for some $\lambda_w \in \mathrm{PSL}(2,\mathbb{C})$. The normalization conditions on $f_w \in \tilde{\mathcal{D}}$ and $f \in \mathcal{D}$ imply that

$$\lambda_w(z) = \frac{z}{c_w z + 1}, \quad \text{where } c_w = -\frac{1}{2}\frac{f_w''(0)}{f_w'(0)}. \tag{2.7}$$

The condition $g_w(\infty) = \infty$ implies that

$$c_w = -\frac{1}{g(w)}. \tag{2.8}$$

Let $[\mu]$ be the equivalence class which corresponds to γ under the isomorphism $T(1) \simeq \mathrm{Homeo}_{qs}(S^1)/\mathrm{Möb}(S^1)$. For $w \in \Delta^*$, the point $g(w)$ lies in $f(\Delta^*) = w^\mu(\Delta^*)$, since $f(\Delta^*) = g(\Delta^*)$. Hence the natural correspondence between $\mathrm{Homeo}_{qs}(S^1)/S^1$ ($\simeq \tilde{\mathcal{D}}$) and $\mathcal{T}(1)$, given by

$$\begin{aligned} &\sigma_w \circ \gamma \in \mathrm{Homeo}_{qs}(S^1)/S^1 \quad (f_w = \lambda_w \circ f \in \tilde{\mathcal{D}}), \\ &\sigma_w \circ \gamma \ \ (f_w = \lambda_w \circ f) \mapsto ([\mu], g(w)), \end{aligned} \tag{2.9}$$

is an isomorphism.

In the identification above, $T(1)$ is the natural subspace $\{([\mu],\infty) : [\mu] \in T(1)\}$ of $\mathcal{T}(1)$. The embedding $([\mu],\infty) \mapsto f$ of $T(1)$ into $\tilde{\mathcal{D}}$ is the pre-Bers embedding. Hence the complex structure of $T(1) \simeq A_\infty(\Delta)$ agrees with the complex structure induced from $\tilde{\mathcal{D}}$. From (2.9), (2.7), (2.8), we see that if we fix $[\mu]$ in $([\mu], z) \in \mathcal{T}(1)$, and change z holomorphically, the corresponding $f \in \tilde{\mathcal{D}}$ associated to $([\mu], z)$ changes by post-composition with $\lambda = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$, where the coefficient c depends holomorphically on z. This implies that the complex structure of $\mathcal{T}(1)$ induced from the embedding $\mathcal{T}(1) \hookrightarrow A_\infty(\Delta) \oplus \mathbb{C}$ agrees with the complex structure of $\tilde{\mathcal{D}}$ induced from the isomorphism (2.9). ☐

We can identify each point in $\mathcal{T}(1)$ as an equivalence class of quasiconformal mappings as in the proof of the theorem above. This immediately implies that $\mathcal{T}(1)$ also has a group structure coming from composition of quasiconformal maps, which is an extension of the group structure on $T(1)$. According to the definition and the identification given in the proof of Theorem 2.5, the group multiplication in terms of coordinates (2.5) is given by

$$([\lambda], z) * ([\mu], z_0) = ([\nu], z'), \tag{2.10}$$

where

$$\nu = \frac{\mu + (\lambda \circ w)\frac{\overline{w_z}}{w_z}}{1 + \overline{\mu}(\lambda \circ w)\frac{\overline{w_z}}{w_z}} \qquad \text{and} \qquad z' = w^{\nu} \circ w^{-1} \circ (w^{\lambda})^{-1}(z). \tag{2.11}$$

Here w is the quasiconformal map corresponding to the point $([\mu], z_0) \in \mathcal{T}(1) \simeq \mathrm{Homeo}_{qs}(S^1)/S^1$. The right group translation by $([\mu], z_0)$, $R_{([\mu],z_0)} : \mathcal{T}(1) \to \mathcal{T}(1)$ is biholomorphic (see [Ber73]). Thus we can identify the tangent space at $([\mu], z_0)$ with the tangent space at $(0, \infty)$ — the origin of $\mathcal{T}(1)$ — via the inverse of the derivative of the map $R_{([\mu],z_0)}$ at the origin, i.e., via the map $\left(D_{(0,\infty)}R_{([\mu],z_0)}\right)^{-1}$. Moreover, this identification and the group structure give rise to a splitting of the tangent space at each point of $\mathcal{T}(1)$ into horizontal and vertical directions. At the origin $(0, \infty)$, the vertical direction is spanned by $\{0\} \oplus \mathbb{C}$ and the horizontal direction is spanned by $\Omega^{-1,1}(\Delta^*) \oplus \{0\}$. A horizontal vector $(\nu, 0)$, $\nu \in \Omega^{-1,1}(\Delta^*)$, at the origin $(0, \infty)$ has a unique horizontal lift to each point $(0, z)$ on the fiber at $(0, \infty)$. Namely, let $([t\nu], z_t')$, $z_0' = z$ be a curve that defines the horizontal lift of $(\nu, 0)$ at the point $(0, z)$. For t small, z_t' is determined by the equation

$$([\lambda(t)], \infty) * (0, z) = ([t\nu], z_t'), \qquad \lambda(t) \in L^{\infty}(\Delta^*)_1.$$

The point $(0, z)$ corresponds to the map σ_z defined by (2.6) (the subscript z does not indicate a derivative). Using the formulas (2.10), (2.11), taking the derivative with respect to t and setting $t = 0$ (which we denote by $\cdot$), we have

$$\dot{\lambda} = \left(\nu \; \frac{\overline{\sigma_z'}}{\sigma_z'}\right) \circ \sigma_z^{-1} \qquad \text{and} \qquad \dot{z}' = \dot{w}^{\nu}(z). \tag{2.12}$$

Hence the horizontal tangent vector $(\nu, 0)$ at $(0, \infty)$ is lifted to the vector $(\nu, \dot{w}^{\nu}(z))$ at $(0, z)$, and the latter is identified with the horizontal tangent vector $\left(\dot{\lambda}, 0\right)$ at the origin $(0, \infty)$ of $\mathcal{T}(1)$.

2.3 Identification of tangent spaces. Here we want to identify the tangent spaces of the different models of the universal Teichmüller curve and universal Teichmüller space. We need the following two lemmas.

Lemma 2.6. *Let $Q(z) = \sum_{n=2}^{\infty}(n^3 - n)a_n z^{n-2} \in A_\infty(\Delta)$. Then the series $\sum_{n=2}^{\infty} n^{2s}|a_n|^2$ is convergent for all real $s < 1$.*

Proof. Since

$$Q \in A_\infty(\Delta) = \{\phi \text{ holomorphic on } \Delta : \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2| < \infty\},$$

we have for any $\alpha < 1$,

$$\iint_{\Delta} |Q(z)(1-|z|^2)^2|^2 \frac{dxdy}{(1-|z|^2)^\alpha} < \infty,$$

where $z = x + iy$. This integral is equal to

$$\pi \sum_{n=2}^{\infty} (n^3 - n)^2 \frac{\Gamma(5-\alpha)\Gamma(n-1)}{\Gamma(4+n-\alpha)} |a_n|^2.$$

Stirling's formula for the gamma function Γ implies that

$$\lim_{n\to\infty} \frac{\Gamma(n-1)(n^3-n)^2}{\Gamma(4+n-\alpha)n^{1+\alpha}} = 1.$$

By the comparison test, the series

$$\sum_{n=2}^{\infty} n^{1+\alpha}|a_n|^2$$

is convergent for all $\alpha < 1$, which implies the assertion. □

Remark 2.7. We have used an idea of Velling [Vel] in the proof of this theorem.

Lemma 2.8 (*[Zyg88]*). *If the function $f(z) = \mathfrak{a}_0 + \mathfrak{a}_1 z + \cdots + \mathfrak{a}_n z^n + \cdots$ is holomorphic on Δ and continuous on $\Delta \cup S^1$, and the series $\sum_n n|\mathfrak{a}_n|^2$ is convergent, then the series*

$$\mathfrak{a}_0 + \mathfrak{a}_1 e^{i\theta} + \cdots + \mathfrak{a}_n e^{in\theta} + \cdots$$

converges uniformly to $f(e^{i\theta})$ on $0 \le \theta \le 2\pi$.

First, we look at the isomorphism between the universal Teichmüller curve

$$\begin{aligned} \mathcal{W} : \mathrm{Homeo}_{qs}(S^1)/S^1 &\longrightarrow \tilde{\mathcal{D}}, \\ \gamma &\mapsto f. \end{aligned}$$

It establishes the relation between the real analytic (through $\mathrm{Homeo}_{qs}(S^1)/S^1$)) and complex analytic (through $\tilde{\mathcal{D}}$) descriptions of $\mathcal{T}(1)$. Infinitesimally, it takes the following explicit form.

Theorem 2.9. *The derivative of $\mathcal{W}$ at the origin is the linear mapping $D_0\mathcal{W} : T_0 \operatorname{Homeo}_{qs}(S^1)/S^1 \to T_0\tilde{\mathcal{D}}$ given by*

$$\sum_{n\neq 0} c_n e^{in\theta} \mapsto i \sum_{n=1}^{\infty} c_n z^{n+1}.$$

Proof. Consider the smooth one parameter flow $\gamma^t = (g^t)^{-1} \circ f^t|_{S^1}$, $\gamma^t|_{t=0} = \mathrm{id}$. It is known (see, e.g., [Leh87]) that γ^t, f^t and g^t can be extended to quasi-conformal mappings of $\hat{\mathbb{C}}$, real analytic on $\hat{\mathbb{C}} \setminus S^1$. The corresponding vector fields

$$\frac{d}{dt}\gamma^t, \quad \frac{d}{dt}f^t \quad \text{and} \quad \frac{d}{dt}g^t$$

are continuous on $\hat{\mathbb{C}}$, real analytic on $\hat{\mathbb{C}} \setminus S^1$.

We write the perturbative expansion

$$f^t(z) = z + tu + O(t^2) = z + tz(a_1 z + a_2 z^2 + \cdots) + O(t^2),$$

for $z \in \Delta$, and

$$g^t(z) = z + tv + O(t^2) = z + tz(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots) + O(t^2),$$

for $z \in \Delta^*$.

We denote

$$\dot{\gamma} = \frac{d}{dt}\gamma^t\Big|_{t=0}, \quad \dot{f} = \frac{d}{dt}f^t\Big|_{t=0} \quad \text{and} \quad \dot{g} = \frac{d}{dt}g^t\Big|_{t=0},$$

so that $\dot{f}|_\Delta = u$ and $\dot{g}|_{\Delta^*} = v$.

Under the Bers embedding, $\mathcal{S}(f^t|_\Delta)$ belongs to a bounded subspace of $A_\infty(\Delta)$; and the corresponding tangent vector to $T(1)$ at the origin is

$$u_{zzz} = \frac{d}{dt}\mathcal{S}(f^t|_\Delta)\Big|_{t=0} \in A_\infty(\Delta).$$

Since $u = \sum_{n=1}^{\infty} a_n z^{n+1}$ is holomorphic on Δ and continuous on $\mathbb{C}$, Lemma 2.6 (with $s = \frac{1}{2}$) and Lemma 2.8 imply that the series

$$\sum_{n=1}^{\infty} a_n e^{i(n+1)\theta}$$

converges uniformly to the continuous function $u|_{S^1}(e^{i\theta})$ on the unit circle S^1.

Similar arguments imply that the series

$$\sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}$$

converges uniformly to the continuous function $v|_{S^1}(e^{i\theta})$ on S^1.

Taking the derivative with respect to t of the relation $\gamma^t = (g^t)^{-1} \circ f^t$ and setting $t = 0$, we have

$$\dot{\gamma} = -\dot{g} + \dot{f}. \tag{2.13}$$

This shows that the series

$$\sum_{n=1}^{\infty} a_n e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}$$

converges uniformly to the function $\dot{\gamma}|_{S^1}$. In particular, it is the Fourier series of $\dot{\gamma}|_{S^1}$. Let $\mathrm{u}(e^{i\theta})\partial/\partial\theta$ be the corresponding vector field, so that $\dot{\gamma} = iz\mathrm{u}(z)$ on S^1. We have proved that the Fourier series of $\mathrm{u}(e^{i\theta})$

$$\sum_{n\in\mathbb{Z}} c_n e^{in\theta}, \quad c_{-n} = \overline{c_n}$$

converges uniformly to $\mathrm{u}(e^{i\theta})$. Moreover,

$$i\sum_{n\in Z} c_n e^{i(n+1)\theta} = \sum_{n=1}^{\infty} a_n e^{i(n+1)\theta} - \sum_{n=0}^{\infty} b_n e^{i(1-n)\theta}.$$

Comparing coefficients, we have

$$a_n = ic_n, \quad b_n = -ic_{-n}, \quad n \geq 1.$$

Moreover, we have the relation

$$a_n = \overline{b_n}.$$

□

By imposing extra normalization conditions, we can pass from the models for $\mathcal{T}(1)$ to the models for $T(1)$.

Remark 2.10. In [Nag93], Nag proved a result similar to Theorem 2.9 for $T(1)$ by using explicit formulas for $\dot{\gamma}$ and $\dot{f}$ from the theory of quasiconformal mappings. Here we use a slightly different approach.

For the second isomorphism between the universal Teichmüller space, we combine the Ahlfors–Beurling extension theorem and the Bers embedding and get the map

$$\begin{aligned} \mathcal{B} : \mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1) &\to (L^\infty(\Delta^*)_1/\sim) \to A_\infty(\Delta), \\ \gamma &\mapsto \qquad [\mu] \qquad \mapsto \mathcal{S}(w^\mu|_\Delta), \end{aligned}$$

where $\gamma = w_\mu|_{S^1}$. Our argument in the proof of Theorem 2.9 gives immediately

Theorem 2.11. *The derivative of the map $\mathcal{B}$ at the origin is the linear mapping $D_0\mathcal{B} : T_0\left(\mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)\right) \to A_\infty(\Delta)$ given by*

$$\sum_{n\neq -1,0,1} c_n e^{in\theta} \mapsto i\sum_{n=2}^{\infty}(n^3-n)c_n z^{n-2}.$$

Remark 2.12. Lemmas 2.6, 2.8 and Theorem 2.9 imply that the tangent vectors at the origin of $\mathrm{Homeo}_{qs}(S^1)/S^1$ have Fourier series $\sum_n c_n e^{in\theta}$ which converge absolutely and uniformly and belong to the Sobolev class H^s for all $s<1$. Here the Sobolev space $H^s(S^1)$ is defined as

$$H^s(S^1) = \left\{ \mathfrak{u}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \mathfrak{a}_n e^{in\theta} : \sum_{n\in\mathbb{Z}} |n|^{2s}|\mathfrak{a}_n|^2 < \infty \right\}.$$

In light of Theorem 2.9, we say that a tangent vector $u = \sum_{n=1}^{\infty} a_n z^{n+1} \in T_0\tilde{\mathcal{D}}$ is in H^s if it is the image of a H^s vector $\sum_n c_n e^{in\theta}$ under the map $D_0\mathcal{W}$.

2.3.1 More on complex structures.

The almost complex structure J at the origins of $\mathrm{Diff}_+(S_1)/S^1$ and $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ is defined by the linear map $J : T_0 \to T_0$ given by

$$Jv = i\sum_n sgn(n) c_n e^{in\theta}\frac{\partial}{\partial\theta}, \quad \text{where } v = \sum_n c_n e^{in\theta}\frac{\partial}{\partial\theta}. \tag{2.14}$$

See references in [NV90]. (Note that we differ from the definition in [NV90] by a negative sign.) By Remark 2.12, J extends to almost complex structures on $\mathrm{Homeo}_{qs}(S^1)/S^1$ and $\mathrm{Homeo}_{qs}(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$.

In [NV90], Nag and Verjoysky proved that the almost complex structure J on $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$ is integrable and the corresponding complex structure is the pull-back of the complex structure on $T(1)$, induced by the complex structure of $L^\infty(\Delta)_1$. Adapting their proof to our convention, we immediately see that the complex structure J on $\mathrm{Homeo}_{qs}(S^1)/S^1$ coincides with the complex structure induced from $\mathcal{T}(1)$.

Under this convention, the holomorphic tangent vectors are of the form

$$w = \frac{v - iJv}{2} = \sum_{n>0} c_n e^{in\theta},$$

and the antiholomorphic tangent vectors are of the form

$$\bar{w} = \frac{v + iJv}{2} = \sum_{n<0} c_n e^{in\theta}.$$

2.4 Metrics. We are interested in homogeneous Hermitian metrics, i.e., Hermitian metrics which are invariant under the right group action on the homogeneous spaces of $\mathrm{Homeo}_{qs}(S^1)$. In [Kir87] and [KY87], Kirillov and Yuriev studied Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$. It is known that the homogeneous Kähler metrics on $\mathrm{Diff}_+(S^1)/S^1$ must be of the form

$$\| v \|_{a,b}^2 = \sum_{n>0} (an^3 + bn)|c_n|^2, \tag{2.15}$$

where $v = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}\partial/\partial\theta \in T_0 \mathrm{Diff}_+(S^1)/S^1$. The metric $\| \cdot \|_{0,1}$ is called the Kirillov metric.

On the other hand, since the vector fields $e^{-i\theta}\partial/\partial\theta, \partial/\partial\theta, e^{i\theta}\partial/\partial\theta$ generate the $\mathrm{PSU}(1,1)$ action on S^1, (2.15) defines a metric on $\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ if and only if $an^3 + bn = 0$ for $n = -1, 0, 1$. This implies that, up to a constant, there is a unique homogeneous Kähler metric on $\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ given by

$$\| v \|^2 = \frac{\pi}{2} \sum_{n>0} (n^3 - n)|c_n|^2. \tag{2.16}$$

Let Γ be a Fuchsian group realized as a subgroup of $\mathrm{PSU}(1,1)$ acting on Δ^*. Let $L^\infty(\Delta^*, \Gamma)$ be the space of Beltrami differentials for Γ, i.e.,

$$L^\infty(\Delta^*, \Gamma) = \left\{ \mu \in L^\infty(\Delta^*) : \mu \circ \gamma \frac{\overline{\gamma'}}{\gamma'} = \mu, \forall \gamma \in \Gamma \right\}.$$

The Teichmüller space $T(\Gamma)$ of Γ is the subspace of the universal Teichmüller space

$$T(\Gamma) = L^\infty(\Delta^*, \Gamma)_1 / \sim,$$

where

$$L^\infty(\Delta^*, \Gamma)_1 = L^\infty(\Delta^*)_1 \cap L^\infty(\Delta^*, \Gamma),$$

and $\sim$ is the same equivalence relation we use to define $T(1)$. The tangent space at the origin of $T(\Gamma)$ is identified with the space of harmonic Beltrami differentials of Γ

$$\Omega^{-1,1}(\Delta^*, \Gamma) = \Omega^{-1,1}(\Delta^*) \cap L^\infty(\Delta^*, \Gamma).$$

When Γ is a cofinite Fuchsian group, i.e., when the quotient Riemann surface $\Gamma\backslash\Delta^*$ has finite hyperbolic area, there is a canonical Hermitian metric on $T(\Gamma)$ given by

$$\langle \mu, \nu \rangle = \iint_{\Gamma\backslash\Delta^*} \mu \overline{\nu} \rho, \quad \mu, \nu \in \Omega^{-1,1}(\Delta^*, \Gamma),$$

where ρ is the area form of the hyperbolic metric on Δ^*. This metric is called the Weil–Petersson metric. The notation $T(1)$ for the universal Teichmüller space indicates that it corresponds to the case $\Gamma = \{\mathrm{id}\}$. This suggests defining the Weil–Petersson metric on $T(1)$ by

$$\langle \mu, \nu \rangle = \iint_{\Delta^*} \mu \bar{\nu} \rho, \quad \mu, \nu \in \Omega^{-1,1}(\Delta^*).$$

However, this integral does not converge for all $\mu, \nu \in \Omega^{-1,1}(\Delta^*)$. In particular, it diverges when both μ, ν are Beltrami differentials of a Fuchsian group which contains infinitely many elements. However, it was proved by Nag and Verjoysky in [NV90] that the integral is convergent on Sobolev class $H^{3/2}$ vector fields, which contains the C^2 class vector fields. More precisely, they proved that the pull-back of the Weil–Petersson metric on $T(1)$ to $\mathrm{Diff}_+(S^1)/\,\mathrm{Möb}(S^1)$ coincides with the unique homogeneous Kähler metric (2.16) on $\mathrm{Diff}_+ S^1/\,\mathrm{Möb}(S^1)$ (up to a factor 4). Henceforth, when we say the Weil–Petersson metric on $T(1)$, we understand that it is only defined on tangent vectors in the Sobolev class $H^{3/2}$.

Under the Bers embedding, the Weil–Petersson metric on $T(1)$ induces a metric on $A_\infty(\Delta)$. It is given by

Theorem 2.13. *For $Q = u_{zzz} \in A_\infty(\Delta)$, identified as a tangent vector to $T(1)$ at the origin such that $u = \sum_{n=1}^{\infty} a_n z^{n+1} \in H^{3/2}$, the Weil–Petersson metric has the form*

$$\| Q \|_{WP}^2 = \frac{\pi}{2} \sum_{n=2}^{\infty} (n^3 - n)|a_n|^2 = \frac{1}{4} \iint_{\Delta} |Q(z)|^2 (1 - |z|^2)^2 dxdy.$$

Proof. The first equality follows immediately from the identification of tangent spaces given by Theorem 2.11. The second equality is an explicit computation of the integral. □

Remark 2.14. The derivative of the map $\tilde{\mathcal{D}} \hookrightarrow A_\infty(\Delta)$ at the origin, $\dot{f} \mapsto \dot{f}_{zzz}$ can be viewed as a linear mapping sending vector fields to quadratic differentials. The theorem states that the Weil–Petersson metric on $A_\infty(\Delta)$ given by the Bers embedding $T(1) \hookrightarrow A_\infty(\Delta)$ is the usual Weil–Petersson metric defined on the space of quadratic differentials. This can also be proved directly by using the isomorphism (2.3). In particular, we have

$$\| Q \circ \gamma (\gamma')^2 \|_{WP}^2 = \| Q \|_{WP}^2, \quad \text{for all } \gamma \in \mathrm{PSU}(1,1).$$

Remark 2.15. Analogues of Theorems 2.9, 2.11 and 2.13 hold for finite dimensional Teichmüller spaces $T(\Gamma)$ embedded in the universal Teichmüller space $T(1)$.

According to Remark 2.12, the Kirillov metric on $\mathrm{Diff}_+(S^1)/S^1$ extends to $\mathcal{T}(1)$. Namely, at the origin, it is of the form

$$\| v \|^2 = \sum_{n>0} n|c_n|^2, \tag{2.17}$$

where $v = \sum_n c_n e^{in\theta}\partial/\partial\theta$ is the corresponding tangent vector. The series (2.17) is convergent. Using the right translations, we define a homogeneous Kähler metric on $\mathcal{T}(1)$.

Since every homogeneous Kähler metric on $\mathrm{Diff}_+(S^1)/S^1$ can be written as a linear combination of the metric (2.17) and the Weil–Petersson metric, and only the former is convergent for all the tangent vectors of $\mathcal{T}(1)$, we have

Theorem 2.16. *Every homogeneous Kähler metric on $\mathcal{T}(1)$ is a multiple of the metric* (2.17).

3 Velling's Hermitian form and the Velling–Kirillov metric

3.1 Spherical area theorem. The spherical area of a domain Ω in $\hat{\mathbb{C}}$ is

$$A_S(\Omega) = \iint\limits_{\Omega} \frac{4dxdy}{(1+|z|^2)^2}.$$

It is invariant under rotation, i.e., $A_S(\Omega) = A_S(e^{i\theta}(\Omega))$.

Following Velling [Vel], for $Q \in A_\infty(\Delta)$ and t small, we consider the one-parameter family of functions $f^{tQ} \in \mathcal{D}$ satisfying $\mathcal{S}(f^{tQ}) = tQ$ and the spherical areas of the domains $\Omega_t = f^{tQ}(\Delta)$,

$$\begin{aligned} A_S(\Omega_t) &= \iint\limits_{\Omega_t} \frac{4dxdy}{(1+|z|^2)^2} \\ &= 4\iint\limits_{\Delta} \frac{|df^{tQ}|^2}{(1+|f^{tQ}|^2)^2}. \end{aligned}$$

Velling's spherical area theorem is the following.

Theorem 3.1 (*Velling [Vel]*). *For $Q \in A_\infty(\Delta)$, we have*

$$\begin{aligned} &\frac{d}{dt}A_S(f^{tQ}(\Delta))|_{t=0} = 0, \\ &\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))|_{t=0} \geq 0, \end{aligned}$$

with equality if and only if $Q = 0$.

This follows from another result, proved by applying the classical area theorem.

Theorem 3.2 ***(Velling [Vel]).*** *Let $f : \Delta \longrightarrow \hat{\mathbb{C}}$ be a univalent function (perhaps meromorphic) with Taylor expansion $f(z) = z(1 + a_2 z^2 + a_3 z^3 + \cdots)$ at the origin. Then the spherical area $A_S(f(\Delta))$ satisfies*

$$A_S(f(\Delta)) \geq 2\pi,$$

with equality if and only if $f = \mathrm{id}$.

The second inequality in Velling's spherical area theorem implies that $\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))|_{t=0}$ is a Hermitian form on $A_\infty(\Delta)$. Our goal is to compute this form explicitly.

The following lemma is very useful for computations.

Lemma 3.3 ***([Zyg88]).*** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function on Δ and $\phi(r)$ an integrable function on $[0, 1)$. Then*

$$\iint_{\Delta} \phi(|z|) \operatorname{Re}(f(z)) dx dy = 2\pi \operatorname{Re}(a_0) \int_0^1 \phi(r) dr,$$

$$\iint_{\Delta} \phi(|z|) |f(z)|^2 dx dy = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 \phi(r) r^{2n+1} dr.$$

3.2 Velling's Hermitian form. Now we compute Velling's Hermitian form $\frac{d^2}{dt^2} A_S(f^{tQ}(\Delta))|_{t=0}$. For t small, we write the perturbative expansions

$$\begin{aligned} f^{tQ}(z) &= z + tu(z) + t^2 v(z) + O(t^3), \\ u(z) &= z(a_2 z^2 + a_3 z^3 + \cdots) = \sum_{n=2}^{\infty} a_n z^{n+1}, \\ v(z) &= z(b_2 z^2 + b_3 z^3 + \cdots) = \sum_{n=2}^{\infty} b_n z^{n+1}. \end{aligned} \tag{3.1}$$

Taking the t derivative of the equation $\mathcal{S}(f^{tQ}) = tQ$ and setting $t = 0$, we get the relation

$$\frac{\partial^3}{\partial z^3} u(z) = Q(z), \quad \text{i.e., } Q(z) = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2}.$$

Using the expansion

$$\frac{|f_z^{tQ}|^2}{(1 + |f^{tQ}|^2)^2} = \frac{|1 + tu_z + t^2 v_z|^2}{(1 + |z + tu + t^2 v|^2)^2} + O(t^3),$$

we obtain

$$\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))\big|_{t=0} = 8\iint\limits_{\Delta} \frac{\chi(z)dxdy}{(1+|z|^2)^2},$$

$$\chi(z) = (v_z + \overline{v_z} + |u_z|^2) - 2\frac{z\overline{v} + \bar{z}v + |u|^2 + (z\overline{u} + \bar{z}u)(u_z + \overline{u_z})}{1+|z|^2} + 3\frac{(z\overline{u}+\bar{z}u)^2}{(1+|z|^2)^2}.$$

Using the series expansion (3.1) and $v(0) = v'(0) = 0$, we see that v drops out from the integration. Applying Lemma 3.3, we get

$$\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))\big|_{t=0} = 16\pi\sum_{n=2}^{\infty} \mathfrak{c}_n|a_n|^2, \tag{3.2}$$

where

$$\mathfrak{c}_n = \int_0^1 \left(\frac{6r^{2n+4}}{(1+r^2)^4} - \frac{(4n+6)r^{2n+2}}{(1+r^2)^3} + \frac{(n+1)^2r^{2n}}{(1+r^2)^2}\right) rdr.$$

We compute $\mathfrak{c}_n$ by repeatedly using integration by parts:

$$\mathfrak{c}_n = \frac{1}{2}\int_0^1 \left(\frac{6r^{n+2}}{(1+r)^4} - \frac{(4n+6)r^{n+1}}{(1+r)^3} + \frac{(n+1)^2r^{n}}{(1+r)^2}\right) dr,$$

$$\int_0^1 \frac{r^{n+2}}{(1+r)^4}dr = -\frac{2n^2+7n+7}{24} + \frac{n(n+1)(n+2)}{6}\int_0^1 \frac{r^{n-1}}{1+r}dr,$$

$$\int_0^1 \frac{r^{n+1}}{(1+r)^3}dr = -\frac{2n+3}{8} + \frac{n(n+1)}{2}\int_0^1 \frac{r^{n-1}}{1+r}dr,$$

$$\int_0^1 \frac{r^{n}}{(1+r)^2}dr = -\frac{1}{2} + n\int_0^1 \frac{r^{n-1}}{1+r}dr.$$

When we substitute into $\mathfrak{c}_n$, all the terms with integrals cancel; and we are left with

$$\mathfrak{c}_n = n/8.$$

Therefore, we have

Theorem 3.4. *Let $Q \in A_\infty(\Delta)$. Then*

$$\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))\Big|_{t=0} = 2\pi\sum_{n=2}^{\infty} n|a_n|^2.$$

Remark 2.12 implies that the series is convergent for all $Q \in A_\infty(\Delta)$. Hence, we can define a Hermitian form on $A_\infty(\Delta)$ by

$$\| Q \|_S^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}A_S(f^{tQ}(\Delta))|_{t=0} = \sum_{n=2}^{\infty} n|a_n|^2,$$

where

$$Q(z) = \sum_{n=2}^{\infty}(n^3 - n)a_n z^{n-2};$$

we call this Velling's Hermitian form.

Remark 3.5. The first half of the computation above is reproduced from Velling's unpublished manuscript [Vel]. Velling gave the result in terms of (3.2). Our observation is that c_n can be computed explicitly.

Note that in evaluating the Hermitian form, we have chosen a particular normalized solution f^{tQ} to the equation $\mathcal{S}(f^{tQ}) = tQ$. Any other choice will differ from this one by post-composition with a PSL(2, $\mathbb{C}$) transformation. However, the spherical area of a domain $A_S(f(\Delta))$ is not invariant if f is post-composed with a PSL(2, $\mathbb{C}$) transformation. If we choose different normalization conditions to identify $T(1)$ as a subgroup of $\mathcal{T}(1)$, we get a different right invariant metric on $T(1)$. Hence the Hermitian form $\| \cdot \|_S$ does not naturally define a right invariant metric on $T(1)$.

On the other hand, since the correspondence between $\gamma \in \text{Homeo}_{qs}(S^1)/S^1$ and $f \in \tilde{\mathcal{D}}$ is canonical, we can use the same approach to define a metric on $\mathcal{T}(1) \simeq \text{Homeo}_{qs}(S^1)/S^1$. Namely, given the tangent vector $v = \sum_{n\neq 0} c_n e^{in\theta}\partial/\partial\theta$ at the origin with the associated one-parameter flow $\gamma^t = (g^t)^{-1} \circ f^t|_{S^1}$, we define a Hermitian form by

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}\Big|_{t=0} A_S(f^t(\Delta)).$$

The proof above holds with an extra term $n = 1$ (notice that we only need the fact there are no constant terms and terms linear in z in the first and second order perturbations), and we get

$$\| v \|^2 = \frac{1}{2\pi}\frac{d^2}{dt^2}\Big|_{t=0} A_S(f^t(\Delta)) = \sum_{n=1}^{\infty} n|a_n|^2 = \sum_{n=1}^{\infty} n|c_n|^2,$$

which coincides with the metric (2.17) at the origin. It is quite remarkable that this metric, introduced by Velling using classical function theory, coincides with the metric introduced by Kirillov using the orbit method. Henceforth, we call this metric on $\mathcal{T}(1)$ the Velling–Kirillov metric.

4 Metrics on Teichmüller spaces

4.1 Universal Teichmüller space. Let κ be the symplectic form of the Velling–Kirillov metric on $\mathcal{T}(1) \simeq \tilde{\mathcal{D}}$. We want to define a metric on $T(1)$ by vertical integration of the $(2,2)$ form $\kappa \wedge \kappa$. Namely, let

$$\omega = \iint_{\text{fiber}} \kappa \wedge \kappa$$

and define a Hermitian metric on $T(1)$ such that ω is the corresponding symplectic form.[3] Since κ defines a right invariant metric, ω also defines a right invariant metric. Hence we only have to compute the form ω at the origin of $T(1)$. We identify the tangent space of $\mathcal{T}(1)$ at the origin with $A_\infty(\Delta) \oplus \mathbb{C}$. The vertical tangent space is spanned by $\partial/\partial w$ and $\partial/\partial \overline{w}$, where w is the coordinate on $\mathbb{C}$. Observe that the horizontal and vertical tangent spaces are orthogonal with respect to the Velling–Kirillov metric. Hence, given a holomorphic tangent vector $Q \in A_\infty(\Delta)$, we have

$$\omega(Q, \bar{Q}) = 2 \iint_{\Delta^*} \kappa(\hat{Q}, \bar{\hat{Q}}) \kappa\Big(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\Big) dw \wedge d\overline{w},$$

where $\hat{Q}$ is the horizontal lift of $(Q, 0)$ to every point on the fiber. Using the right invariance of the Velling–Kirillov metric, we see at once that $\kappa(\partial/\partial w, \partial/\partial \overline{w}) dw \wedge d\overline{w}$ is the area form of a right invariant metric on Δ^*. Hence, up to a constant, it is the hyperbolic area form dA_H. Checking at the origin, we find that

$$\kappa\Big(\frac{\partial}{\partial w}, \frac{\partial}{\partial \overline{w}}\Big) dw \wedge d\overline{w} = \frac{dxdy}{(1 - |w|^2)^2} = \frac{1}{4} dA_H, \qquad w = x + iy.$$

Via the identification (2.12) and the isomorphism (2.3), $\hat{Q}$ at $(0, w)$ is identified with $Q \circ \sigma_w^{-1}((\sigma_w^{-1})')^2$ at the origin. Hence

$$\kappa(\hat{Q}, \bar{\hat{Q}})(w) = \frac{i}{2} \parallel Q \circ \sigma_w^{-1}((\sigma_w^{-1})')^2 \parallel_S^2 .$$

Under the change of variable $w \mapsto 1/w$, σ_w^{-1} is changed to γ_w, where modulo S^1,

$$\gamma_w(z) = \frac{z + w}{1 + z\overline{w}}.$$

Since pre-composing Q with a rotation does not change the Hermitian form $\parallel Q \parallel_S^2$, we finally get

$$\omega(Q, \bar{Q}) = \frac{i}{4} \iint_{\Delta} \parallel Q_w \parallel_S^2 dA_H, \qquad Q_w = Q \circ \gamma_w (\gamma_w')^2.$$

[3]Since the fiber is not compact, it is not a priori clear that we get a well-defined symplectic form on $T(1)$.

Thus our approach to defining a Hermitian metric on $T(1)$ coincides with Velling's suggestion [Vel] of averaging the Hermitian form $\|\cdot\|_S^2$ along the fiber to define Hermitian metric on $T(1)$, i.e.,

$$\| Q \|_V^2 = -2i\omega(Q,\bar{Q}) = \frac{1}{2}\iint_{\Delta} \| Q_w \|_S^2 \frac{4dxdy}{(1-|w|^2)^2}. \tag{4.1}$$

Remark 4.1. I am grateful to my advisor L. Takhtajan for his suggestion of using vertical integration to obtain a metric on $T(1)$.

Since the Hermitian form $\| Q \|_S^2$ is expressed in terms of the norm square of the corresponding coefficients $|a_n|^2$, to compute (4.1) it is sufficient to average $|a_n|^2$ for $n \geq 2$.

We set

$$Q_w(z) = Q \circ \gamma_w (\gamma_w')^2(z) = \sum_{n=2}^{\infty}(n^3-n)a_n^w z^{n-2}, \qquad \gamma_w(z) = \frac{z+w}{1+\overline{w}z}.$$

Then

$$a_n^w = \frac{1}{(n^3-n)}\frac{(Q\circ\gamma_w(\gamma_w')^2)^{(n-2)}}{(n-2)!}(0), \tag{4.2}$$

and

$$\| Q_w \|_S^2 = \sum_{n=2}^{\infty} n|a_n^w|^2.$$

Theorem 4.2. *Let $u(z) = \sum_{n=1}^{\infty} a_n z^{n+1} \in H^{3/2}$ and $Q = u_{zzz}$. Then*

$$\iint_{\Delta} |a_j^w|^2 \frac{4dxdy}{(1-|w|^2)^2} = \frac{2}{3(j^3-j)}\iint_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy$$
$$= \frac{4\pi}{3(j^3-j)}\sum_{n=2}^{\infty}(n^3-n)|a_n|^2.$$

Proof. Using (4.2), we set

$$a_j^w = \frac{1}{(j^3-j)}\frac{(Q\circ\gamma_w(\gamma_w')^2)^{(j-2)}}{(j-2)!}(0) = \frac{c_j(w)}{(j^3-j)},$$

and introduce the generating function for the $c_j(w)$'s,

$$\begin{aligned} f(u,w) &= \sum_{j=2}^{\infty} c_j(w)u^{j-2} \\ &= \sum_{j=2}^{\infty}\frac{(Q\circ\gamma_w(\gamma_w')^2)^{(j-2)}}{(j-2)!}(0)u^{j-2} \\ &= Q\circ\gamma_w(u)(\gamma_w'(u))^2. \end{aligned}$$

Writing $u = \rho e^{i\alpha}$, we have

$$\sum_{j=2}^{\infty} |c_j(w)|^2 \rho^{2j-4} = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\alpha}, w)|^2 d\alpha$$

and

$$\begin{aligned}\sum_{j=2}^{\infty} \iint\limits_{\Delta} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2}\rho^{2j-4} &= \frac{1}{2\pi}\int_0^{2\pi} \iint\limits_{\Delta} |f(\rho e^{i\alpha}, w)|^2 \frac{dxdy}{(1-|w|^2)^2} d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi} \iint\limits_{\Delta} |Q \circ \gamma_w(\rho e^{i\alpha})(\gamma_w'(\rho e^{i\alpha}))^2|^2 \frac{dxdy}{(1-|w|^2)^2} d\alpha.\end{aligned}$$

Denoting this integral by $\mathcal{I}$, substituting the series expansion of Q and using polar coordinates $w = re^{i\theta}$, we get

(4.3)
$$\begin{aligned}\mathcal{I} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1\int_0^{2\pi} d\theta r dr d\alpha \left|\frac{(1-r^2)}{(1+r\rho e^{i(\alpha-\theta)})^4}\right|^2 \\ \sum_{n=2}^{\infty}(n^3-n)a_n\left(\frac{\rho e^{i\alpha}+re^{i\theta}}{1+r\rho e^{i(\alpha-\theta)}}\right)^{n-2}\sum_{m=2}^{\infty}(m^3-m)\overline{a_m}\left(\frac{\rho e^{-i\alpha}+re^{-i\theta}}{1+r\rho e^{-i(\alpha-\theta)}}\right)^{m-2}.\end{aligned}$$

We do some "juggling",

$$\begin{aligned}&\left(\frac{\rho e^{i\alpha}+re^{i\theta}}{1+r\rho e^{i(\alpha-\theta)}}\right)^{n-2}\left(\frac{\rho e^{-i\alpha}+re^{-i\theta}}{1+r\rho e^{-i(\alpha-\theta)}}\right)^{m-2} \\ &= \left(\frac{\rho e^{i(\alpha-\theta)}+r}{1+r\rho e^{i(\alpha-\theta)}}\right)^{n-2}\left(\frac{\rho e^{-i(\alpha-\theta)}+r}{1+r\rho e^{-i(\alpha-\theta)}}\right)^{m-2} e^{i(n-m)\theta},\end{aligned}$$

and make a change of variable $\alpha \mapsto (\alpha + \theta)$ to get

$$\begin{aligned}
\mathcal{I} =& \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_0^{2\pi} d\theta r dr d\alpha \sum_{n,m\geq 2} (n^3 - n)(m^3 - m) a_n \overline{a_m} \\
& \left(\frac{\rho e^{i\alpha} + r}{1 + r\rho e^{i\alpha}}\right)^{n-2} \left(\frac{\rho e^{-i\alpha} + r}{1 + r\rho e^{-i\alpha}}\right)^{m-2} \left|\frac{(1 - r^2)}{(1 + r\rho e^{i\alpha})^4}\right|^2 e^{i(n-m)\theta} \\
=& \int_0^{2\pi} \int_0^1 r dr d\alpha \\
& \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho e^{i\alpha} + r}{1 + r\rho e^{i\alpha}}\right)^{n-2} \left(\frac{\rho e^{-i\alpha} + r}{1 + r\rho e^{-i\alpha}}\right)^{n-2} \left|\frac{(1 - r^2)}{(1 + r\rho e^{i\alpha})^4}\right|^2 \\
=& \int_0^{2\pi} \int_0^1 r dr d\alpha \\
& \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho + r e^{i\alpha}}{1 + \rho r e^{i\alpha}}\right)^{n-2} \left(\frac{\rho + r e^{-i\alpha}}{1 + \rho r e^{-i\alpha}}\right)^{n-2} \left|\frac{(1 - r^2)}{(1 + \rho r e^{i\alpha})^4}\right|^2 \\
=& \iint\limits_{\Delta} \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 \left(\frac{\rho + w}{1 + \rho w}\right)^{n-2} \left(\frac{\rho + \overline{w}}{1 + \rho \overline{w}}\right)^{n-2} \left|\frac{1 - |w|^2}{(1 + \rho w)^4}\right|^2 dx dy,
\end{aligned}$$

where we have done more juggling to get the second to last equality. Observe that

$$\begin{aligned}
\frac{\rho + w}{1 + \rho w} &= \gamma_\rho(w), \\
\frac{1}{(1 + \rho w)^4} &= \frac{\gamma_\rho'(w)^2}{(1 - \rho^2)^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \iint\limits_{\Delta} \left(\frac{\rho + w}{1 + \rho w}\right)^{n-2} \left(\frac{\rho + \overline{w}}{1 + \rho \overline{w}}\right)^{n-2} \left|\frac{1 - |w|^2}{(1 + \rho w)^4}\right|^2 dx dy \\
=& \iint\limits_{\Delta} \left((z^{n-2}) \circ \gamma_\rho (\gamma_\rho')^2\right)(w) \overline{\left((z^{n-2}) \circ \gamma_\rho (\gamma_\rho')^2\right)(w)} \frac{(1 - |w|^2)^2}{(1 - \rho^2)^4} dx dy \\
=& \iint\limits_{\Delta} w^{n-2} \overline{w^{n-2}} \frac{(1 - |w|^2)^2}{(1 - \rho^2)^4} dx dy,
\end{aligned}$$

using $\mathrm{PSU}(1,1)$-invariance of the Weil–Petersson metric. This gives

$$\begin{aligned}\mathcal{I} =& \iint\limits_{\Delta} \sum_{n=2}^{\infty}(n^3-n)^2|a_n|^2 w^{n-2}\overline{w^{n-2}}\frac{(1-|w|^2)^2}{(1-\rho^2)^4}dxdy \\ =& \frac{1}{(1-\rho^2)^4}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy \\ =& \sum_{j=2}^{\infty}\frac{j^3-j}{6}\rho^{2j-4}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy.\end{aligned}$$

Comparing coefficients, we get

$$\iint\limits_{\Delta} |c_j(w)|^2\frac{dxdy}{(1-|w|^2)^2} = \frac{j^3-j}{6}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy,$$

$$\iint\limits_{\Delta} |a_j^w|^2\frac{4dxdy}{(1-|w|^2)^2} = \frac{2}{3(j^3-j)}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy,$$

which finishes the proof. □

Theorem 4.3. *Let $Q = u_{zzz} \in A_\infty(\Delta)$ be a tangent vector to $T(1)$ at the origin such that $u \in H^{3/2}$. Then*

$$\| Q \|_V^2 = \iint\limits_{\Delta} \| Q_w \|_S^2 \frac{2dxdy}{(1-|w|^2)^2} = \frac{1}{4}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy,$$

which is the Weil–Petersson metric.

Proof. This is just a simple sum of the telescoping series:

$$\begin{aligned}\iint\limits_{\Delta} \| Q_w \|_S^2 \frac{2dxdy}{(1-|w|^2)^2} &= \sum_{j=2}^{\infty} j\iint\limits_{\Delta} |a_j^w|^2\frac{2dxdy}{(1-|w|^2)^2} \\ &= \sum_{j=2}^{\infty}\frac{1}{3(j-1)(j+1)}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy \\ &= \frac{1}{4}\iint\limits_{\Delta} |Q(w)|^2(1-|w|^2)^2 dxdy.\end{aligned}$$

□

4.2 Finite-dimensional Teichmüller spaces. Let Γ be a cofinite Fuchsian group. The tangent space to $T(\Gamma)$ at the origin is identified with

$$A_\infty(\Delta, \Gamma) = \{Q \in A_\infty(\Delta) : Q \circ \gamma(\gamma')^2 = Q, \forall \gamma \in \Gamma\},$$

and the Weil–Petersson metric is given by

$$\| Q \|_{WP}^2 = \frac{1}{4} \iint\limits_{\Gamma\backslash\Delta} |Q(w)|^2 (1-|w|^2)^2 dxdy. \tag{4.4}$$

The inverse image of $\mathcal{T}(\Gamma)$ under the projection map $\mathcal{T}(1) \to \mathcal{T}(\Gamma)$ is the Bers fiber space $\mathcal{BF}(\Gamma)$. The quasi-Fuchsian group $\Gamma^\mu = w^\mu \circ \Gamma \circ (w^\mu)^{-1}$ acts on the fiber $w^\mu(\Delta^*)$ at the point $[\mu] \in \mathcal{T}(\Gamma)$. The quotient space of each fiber is a corresponding Riemann surface. They glue together to form the fiber space $\mathcal{F}(\Gamma)$ over $\mathcal{T}(\Gamma)$, which is called the Teichmüller curve of Γ. First we have

Lemma 4.4. *Let Γ be a Fuchsian group. The symplectic form κ on $\mathcal{T}(1)$ restricted to $\mathcal{BF}(\Gamma)$ is equivariant with respect to the group action on each fiber.*

Proof. We only need to check this statement on the fiber at the origin. The form κ restricted to the vertical direction is clearly equivariant. We are left to verify that if $w \in \Delta^*$, $\gamma \in \Gamma$ and $Q \in A_\infty(\Delta, \Gamma)$, then

$$\kappa(\hat{Q}, \overline{\hat{Q}})(w) = \kappa(\hat{Q}, \overline{\hat{Q}})(w'),$$

where $w' = \gamma(w)$. Note that the PSU(1,1) transformation $\sigma_{w'} \circ \gamma \circ \sigma_w^{-1}$ fixes ∞, hence is a rotation. Using the fact that the Hermitian form $\| Q \|_S^2$ is invariant if Q is pre-composed with a rotation, we have

$$\begin{aligned} \| Q \circ \sigma_{w'}^{-1} \left((\sigma_{w'}^{-1})'\right)^2 \|_S^2 &= \| \left(Q \circ \gamma (\gamma')^2\right) \circ \sigma_w^{-1} \left((\sigma_w^{-1})'\right)^2 \|_S^2 \\ &= \| Q \circ \sigma_w^{-1} \left((\sigma_w^{-1})'\right)^2 \|_S^2 . \end{aligned}$$

□

The lemma implies that κ descends to a well-defined symplectic form on $\mathcal{F}(\Gamma)$. We vertically integrate the $(2,2)$-form $\kappa \wedge \kappa$ on $\mathcal{F}(\Gamma)$ to define the Hermitian metric on $\mathcal{T}(\Gamma)$. Using the same reasoning as in Section 4.1, we get

$$\| Q \|_V^2 = \frac{1}{2} \iint\limits_{\Gamma\backslash\Delta} \| Q_w \|_S^2 \, dA_H, \quad Q \in A_\infty(\Delta, \Gamma). \tag{4.5}$$

We want to compute this integral using a regularization technique suggested by J. Velling [Vel].

Theorem 4.5. *Let Γ be a cofinite Fuchsian group and $h \in L^\infty(\Delta)$ be Γ-automorphic. Then*

$$\iint\limits_{\Gamma\backslash\Delta} h(w) dA_H = \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} h(w) dA_H}{\iint_{\Delta_{r'}} dA_H},$$

where $\mathrm{Area}_H(\Gamma\backslash\Delta)$ *is the hyperbolic area of the quotient Riemann surface* $\Gamma\backslash\Delta$ *and* $\Delta_{r'} = \{z : |z| < r'\}$.

Proof. We use the fact that for any $z \in \Delta$, the number of elements $\gamma \in \Gamma$ such that $\gamma(z)$ is in the disc $\Delta_{r'}$, is given asymptotically in terms of r' by

$$\frac{1}{\mathrm{Area}_H(\Gamma\backslash\Delta)} \iint\limits_{\Delta_{r'}} dA_H\Big(1 + o(1)\Big), \quad \text{as } r' \to 1^-, \tag{4.6}$$

where the $o(1)$ term is uniform for all z in a compact set (see [Pat75]).

Let F be a fundamental domain of Γ. Given $E \subset F$, let $E' = \bigcup_{\gamma\in\Gamma} \gamma(E)$. Let χ_A be the characteristic function of the set A. Since Γ is cofinite, using (4.6), we have

$$\iint\limits_{F} \chi_E dA_H = \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} \chi_{E'} dA_H}{\iint_{\Delta_{r'}} dA_H} + o(1).$$

Here the $o(1)$ term is uniform for all the sets $E \subset F$. Since

$$\sup_{w\in\Delta} |h(w)| < \infty,$$

standard approximations of h by bounded step functions give our assertion. □

Corollary 4.6.

$$\| Q \|_{WP}^2 = \lim_{r'\to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |Q(w)|^2 \frac{(1-|w|^2)^2}{4} dxdy}{\iint_{\Delta_{r'}} dA_H}.$$

Proof. Take $h(w) = \big|Q(w)(1 - |w|^2)^2\big|^2$. Since $Q \in A_\infty(\Delta)$, h is in $L^\infty(\Delta)$. □

Lemma 4.7. *Let* $Q \in A_\infty(\Delta)$. *Then*

$$\sup_{w\in\Delta} \| Q_w \|_S^2 < \infty.$$

Proof. Let $\left(Q \circ \gamma_w (\gamma_w')^2\right)(z) = Q_w(z) = \sum_{n=2}^{\infty} a_n^w z^{n-2}$. The proof of Lemma 2.6 with $\alpha = 0$ implies that

$$\| Q_w \|_S^2 = \sum_{n=2}^{\infty} n|a_n^w|^2 < C \iint\limits_{\Delta} \big|Q_w(z)(1 - |z|^2)^2\big|^2 \, dxdy, \quad z = x + iy,$$

where C is a constant independent of $Q \in A_\infty(\Delta)$. After the change of variable $z \mapsto \gamma_w^{-1}(z)$, the integral on the right hand side becomes

$$\iint\limits_{\Delta} \left|Q(z)(1-|z|^2)^2\right|^2 \left|(\gamma_w^{-1})'(z)\right|^2 dxdy.$$

Since $Q \in A_\infty(\Delta)$, $\left|Q(z)(1-|z|^2)^2\right|^2$ is bounded on Δ; thus the formula (see [Kra72])

$$\iint\limits_{\Delta} \left|(\gamma_w^{-1})'(z)\right|^2 dxdy = \iint\limits_{\Delta} \frac{(1-|w|^2)^2}{|1-z\bar{w}|^4} dxdy = 2\pi$$

concludes the proof of the lemma. □

Theorem 4.5 and Lemma 4.7 imply that our approach to defining a Hermitian metric on $T(\Gamma)$ agrees with J. Velling's original suggestion of using regularized integrals. Namely, one has from Theorem 4.5 and Lemma 4.7

Corollary 4.8.

$$\| Q \|_V^2 = \frac{1}{2} \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} \| Q_w \|_S^2 \, dA_H}{\iint_{\Delta_{r'}} dA_H}.$$

Now we start to compute $\| Q \|_V^2$. First we have

Theorem 4.9. *Let Γ be a cofinite Fuchsian group, $Q \in A_\infty(\Delta, \Gamma)$. Then*

$$\lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |a_j^w|^2 dA_H}{\iint_{\Delta_{r'}} dA_H} = \frac{8}{3(j^3-j)} \| Q \|_{WP}^2 .$$

Proof. The proof is almost the same as that of Theorem 4.2. We have

$$\begin{aligned} \mathcal{I} &= \sum_{j=2}^{\infty} \iint\limits_{\Delta_{r'}} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2} \rho^{2j-4} \\ &= \iint\limits_{\Delta_{r'}} \sum_{n=2}^{\infty} (n^3-n)^2 |a_n|^2 \left(\frac{\rho+w}{1+\rho w}\right)^{n-2} \left(\frac{\rho+\bar{w}}{1+\rho\bar{w}}\right)^{n-2} \left|\frac{1-|w|^2}{(1+\rho w)^4}\right|^2 dxdy. \end{aligned}$$

Now observe that if $\gamma \in \mathrm{PSU}(1,1)$ and $Q \in A_\infty(\Delta, \Gamma)$, then $Q \circ \gamma(\gamma')^2 \in A_\infty(\Delta, \gamma^{-1}\Gamma\gamma)$, and

$$(\| Q \|_{WP}^2)_{T(\Gamma)} = (\| Q \circ \gamma(\gamma')^2 \|_{WP}^2)_{T(\gamma^{-1}\Gamma\gamma)}.$$

In particular, for any $u = \rho e^{i\alpha} \in \Delta$,

$$\| Q \|_{WP}^2 = \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |(Q \circ \gamma_u (\gamma_u')^2)(w)|^2 \frac{(1-|w|^2)^2}{4} dxdy}{\iint_{\Delta_{r'}} dA_H},$$

since $\mathrm{Area}_H(\Gamma\backslash\Delta) = \mathrm{Area}_H(\gamma_u^{-1}\Gamma\gamma_u\backslash\Delta)$. It follows that

$$\| Q \|_{WP}^2 = \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \frac{1}{2\pi}\int_0^{2\pi} \iint_{\Delta_{r'}} |(Q \circ \gamma_u (\gamma_u')^2)(w)|^2 \frac{(1-|w|^2)^2}{4} dxdyd\alpha}{\iint_{\Delta_{r'}} dA_H}.$$

But

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \iint\limits_{\Delta_{r'}} |(Q \circ \gamma_u (\gamma_u')^2)(w)|^2 \frac{(1-|w|^2)^2}{4} dxdyd\alpha \\
=&\frac{1}{2\pi}\int_0^{2\pi}\int_0^{r'}\int_0^{2\pi} d\theta r dr d\alpha \left|\frac{(1-r^2)(1-\rho^2)^2}{2(1+r\rho e^{i(\theta-\alpha)})^4}\right|^2 \\
&\sum_{n=2}^{\infty}(n^3-n)a_n\left(\frac{\rho e^{i\alpha}+re^{i\theta}}{1+r\rho e^{i(\theta-\alpha)}}\right)^{n-2} \sum_{m=2}^{\infty}(m^3-m)\overline{a_m}\left(\frac{\rho e^{-i\alpha}+re^{-i\theta}}{1+r\rho e^{-i(\theta-\alpha)}}\right)^{m-2}.
\end{aligned}$$

This is similar to the integral (4.3) with the role of θ and α interchanged, so it is equal to

$$\begin{aligned}
&\frac{(1-\rho^2)^4}{4}\iint\limits_{\Delta_{r'}} \sum_{n=2}^{\infty}(n^3-n)^2|a_n|^2\left(\frac{\rho+w}{1+\rho w}\right)^{n-2}\left(\frac{\rho+\overline{w}}{1+\rho\overline{w}}\right)^{n-2}\left|\frac{1-|w|^2}{(1+\rho w)^4}\right|^2 dxdy \\
=&\frac{(1-\rho^2)^4}{4}\mathcal{I}.
\end{aligned}$$

Hence

$$\sum_{j=2}^{\infty} \lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2}}{\iint_{\Delta_{r'}} dA_H}\rho^{2j-4} = \frac{4}{(1-\rho^2)^4} \| Q \|_{WP}^2 .$$

Comparing coefficients, we have

$$\lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |c_j(w)|^2 \frac{dxdy}{(1-|w|^2)^2}}{\iint_{\Delta_{r'}} dA_H} = \frac{2(j^3-j)}{3} \| Q \|_{WP}^2$$

and

$$\lim_{r' \to 1^-} \frac{\mathrm{Area}_H(\Gamma\backslash\Delta) \iint_{\Delta_{r'}} |a_j^w|^2 dA_H}{\iint_{\Delta_{r'}} dA_H} = \frac{8}{3(j^3-j)} \| Q \|_{WP}^2 . \qquad \square$$

As in the case of Theorem 4.3, this immediately implies

Theorem 4.10. *Let Γ be a cofinite Fuchsian group and $Q \in A_\infty(\Delta, \Gamma)$ a tangent vector to $T(\Gamma)$ at the origin. Then*

$$\| Q \|_V^2 = \| Q \|_{WP}^2 .$$

For a general Fuchsian group Γ and $Q \in A_\infty(\Delta, \Gamma)$, we can define

$$\| Q \|_V^2 = \lim_{r' \to 1^-} \frac{\iint_{\Delta_{r'} \cap F(\Gamma)} dA_H \iint_{\Delta_{r'}} \| Q_w \|_S^2 \, dA_H}{\iint_{\Delta_{r'}} dA_H},$$

whenever the limit is finite. Here $F(\Gamma)$ is a fundamental domain of Γ on Δ. When Γ is the trivial group, this reduces to integrating over the whole disc, which coincides with our original definition.

Appendix A Embedding of $\mathcal{T}(1)$

Consider the Banach space

$$\mathcal{A}_\infty(\Delta) = \left\{ \psi \text{ holomorphic on } \Delta : \sup_{z \in \Delta} |\psi(z)(1 - |z|^2)| < \infty \right\}.$$

Analogous to the Bers embedding $T(1) \simeq \mathcal{D} \hookrightarrow A_\infty(\Delta)$ (defined in Section 2.1), which is achieved by the mapping $f \in \mathcal{D} \mapsto \mathcal{S}(f) \in A_\infty(\Delta)$, we prove that there is an embedding $\mathcal{T}(1) \simeq \tilde{\mathcal{D}} \hookrightarrow \mathcal{A}_\infty(\Delta)$, achieved by the mapping $f \in \tilde{\mathcal{D}} \mapsto \theta(f)$, where

$$\theta(f) = \frac{d}{dz} \log f_z = \frac{f_{zz}}{f_z}.$$

By the classical distortion theorem (see, e.g., [Ahl73]), $f \in \tilde{\mathcal{D}}$ implies that

$$\left| \frac{f_{zz}}{f_z} - \frac{2\bar{z}}{(1 - |z|^2)} \right| \leq \frac{4}{1 - |z|^2}.$$

Hence $\theta(f) \in \mathcal{A}_\infty(\Delta)$, and the map $\theta : \tilde{\mathcal{D}} \to \mathcal{A}_\infty(\Delta)$ is well-defined. We claim that this map is an embedding, and the image contains an open ball.

Lemma A.1. *The map θ is injective.*

Proof. If $f, g \in \tilde{\mathcal{D}}$ are such that $\theta(f) = \theta(g)$, then

$$\frac{d}{dz} \log f_z = \frac{d}{dz} \log g_z.$$

This implies that $f = c_1 g + c_2$ for some constants c_1 and c_2. The normalization conditions $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$ (from the definition of $f, g \in \tilde{\mathcal{D}}$) imply that $c_1 = 1, c_2 = 0$. Hence $f = g$. □

We use the following notation for the sup-norms of $\mathcal{A}_\infty(\Delta)$ and $A_\infty(\Delta)$:

$$\| \psi \|_{\infty,1}= \sup_{z\in\Delta} |\psi(z)(1-|z|^2)|, \qquad \psi \in \mathcal{A}_\infty(\Delta);$$

$$\| \phi \|_{\infty,2}= \sup_{z\in\Delta} |\phi(z)(1-|z|^2)^2|, \qquad \phi \in A_\infty(\Delta).$$

Notice that $\mathcal{S}(f) = \theta(f)_z - \frac{1}{2}\theta(f)^2$. For $\psi \in \mathcal{A}_\infty(\Delta)$, we define

$$\psi \mapsto \Psi(\psi) = \psi_z - \frac{1}{2}\psi^2.$$

We claim that this is a map from $\mathcal{A}_\infty(\Delta)$ to $A_\infty(\Delta)$. First, we have the following continuity theorem.

Theorem A.2. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\psi \in \mathcal{A}_\infty(\Delta)$ satisfies $\| \psi \|_{\infty,1} < \delta$, then $\Psi(\psi) \in A_\infty(\Delta)$ and $\| \Psi(\psi) \|_{\infty,2} < \varepsilon$.*

Proof. Fix $\delta > 0$ and assume that $\psi \in \mathcal{A}_\infty(\Delta)$ satisfies $\| \psi \|_{\infty,1} < \delta$. We use the Cauchy formula

$$\psi_z(z) = \frac{1}{2\pi i}\oint_{|w|=r} \frac{\psi(w)}{(w-z)^2}dw, \qquad |z| < r < 1,$$

to estimate $\psi_z(z)$. Since $\sup_{w\in\Delta} |\psi(w)(1-|w|^2)| < \delta$,

$$|\psi_z(z)| \le \frac{\delta}{2\pi(1-r^2)}\oint_{|w|=r}\frac{|dw|}{|w-z|^2}.$$

Elementary computation gives

$$\frac{1}{2\pi}\oint_{|w|=r}\frac{|dw|}{|w-z|^2} = \frac{r}{r^2-|z|^2}.$$

Choosing $r = (1+|z|)/2$, after some elementary computations, we obtain[4]

$$|\psi_z(z)(1-|z|^2)^2| \le \frac{8\delta(1+|z|)^3}{(|z|+3)(1+3|z|)} \le \frac{64\delta}{3} \qquad \text{for } |z| \le 1.$$

Hence

$$\left|\left(\psi_z(z) - \frac{1}{2}(\psi(z))^2\right)(1-|z|^2)^2\right| \le \frac{64\delta}{3} + \frac{\delta^2}{2}.$$

Given $\varepsilon > 0$, we can always find $\delta > 0$ such that $64\delta/3 + \delta^2/2 < \varepsilon$. This proves our assertion. □

[4]This is not the sharpest estimate.

Corollary A.3. *$\psi \mapsto \Psi(\psi)$ is a holomorphic map from $\mathcal{A}_\infty(\Delta)$ to $A_\infty(\Delta)$.*

Proof. The map $\psi \mapsto \psi_z$ is linear. From the proof of the theorem above, we see that it is a continuous map from $\mathcal{A}_\infty(\Delta)$ to $A_\infty(\Delta)$. The map $\psi \mapsto -\frac{1}{2}\psi^2$ is clearly a continuous map from $\mathcal{A}_\infty(\Delta)$ to $A_\infty(\Delta)$. Hence Ψ is a continuous map from $\mathcal{A}_\infty(\Delta)$ to $A_\infty(\Delta)$.

To prove holomorphy, it is sufficient to note that for any $\psi, \varphi \in \mathcal{A}_\infty(\Delta)$ and $\epsilon \in \mathbb{C}$ in a neighbourhood of 0, the Frechet derivative

$$\lim_{\epsilon \to 0} \frac{\Psi(\psi + \epsilon\varphi) - \Psi(\psi)}{\epsilon} = \varphi_z - \psi\varphi$$

exists in the $\|\cdot\|_{\infty,2}$ norm, since

$$\left\| \frac{\Psi(\psi + \epsilon\varphi) - \Psi(\psi)}{\epsilon} - (\varphi_z - \psi\varphi) \right\|_{\infty,2} = \epsilon \| \frac{1}{2}\varphi^2 \|_{\infty,2} = \frac{1}{2}\epsilon \| \varphi \|_{\infty,1}^2$$

tends to 0 as $\epsilon \to 0$. □

Theorem A.4. *The image of θ contains an open ball about the origin of $\mathcal{A}_\infty(\Delta)$.*

Proof. By Theorem A.2, there exists α such that if $\| \psi \|_{\infty,1} < \alpha$, then $\phi = \Psi(\psi)$ satisfies $\| \phi \|_{\infty,2} < 2$. By the Ahlfors-Weill theorem, there exists a univalent function $f_1 : \Delta \to \mathbb{C}$ such that $\mathcal{S}(f_1) = \phi$ and f_1 has a quasiconformal extension to $\mathbb{C}$. On the other hand, there exists a unique holomorphic function $f : \Delta \to \mathbb{C}$ which solves the ordinary differential equation

$$\frac{d}{dz} \log f_z = \psi; \qquad f(0) = 0, f'(0) = 1.$$

Obviously, $\mathcal{S}(f) = \Psi(\psi) = \phi$. Hence f and f_1 agree up to post-composition with a PSL(2, $\mathbb{C}$) transformation. This implies that f also has a quasiconformal extension to $\mathbb{C}$ and $f \in \tilde{\mathcal{D}}$. Hence the image of θ contains the open ball of radius α. □

From Lemma A.1 and Theorem A.4, it follows that there is an embedding of $\tilde{\mathcal{D}}$ into $\mathcal{A}_\infty(\Delta)$ whose image contains an open ball about the origin. This implies that $\tilde{\mathcal{D}}$ has a Banach manifold structure modeled on $\mathcal{A}_\infty(\Delta)$. We want to compare this structure with the structure induced from the embedding $\tilde{\mathcal{D}} \simeq \mathcal{T}(1) \hookrightarrow A_\infty(\Delta) \oplus \mathbb{C}$. We define a map $\widehat{\Psi} : \mathcal{A}_\infty(\Delta) \to A_\infty(\Delta) \oplus \mathbb{C}$ by

$$\psi \mapsto \left(\Psi(\psi), \frac{1}{2}\psi(0) \right).$$

Theorem A.5. *The map $\widehat{\Psi}$ is holomorphic and one-to-one.*

Proof. Holomorphy follows directly from Corollary A.3. To prove injectivity, suppose $\widehat{\Psi}(\psi_1) = \widehat{\Psi}(\psi_2)$. For $j = 1, 2$, let

$$f_j(z) = \int_0^z e^{\int_0^w \psi_j(u)du} dw.$$

Then $\frac{d}{dz} \log f_j' = \psi_j$, $f_j(0) = 0$, $f_j'(0) = 1$. This implies that $\mathcal{S}(f_1) = \Psi(\psi_1) = \Psi(\psi_2) = \mathcal{S}(f_2)$. Hence $f_1 = \sigma \circ f_2$ for some $\sigma \in \mathrm{PSL}(2, \mathbb{C})$. Now $f_j(0) = 0, f_j'(0) = 1$, $j = 1, 2$ implies that $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ for some $c \in \mathbb{C}$. We also have

$$\frac{d}{dz} \log f_1' = \frac{d}{dz} (\log \sigma' \circ f_2 + \log f_2').$$

Setting $z = 0$ gives

$$\psi_1(0) = -2c + \psi_2(0).$$

Thus $\widehat{\Psi}(\psi_1) = \widehat{\Psi}(\psi_2)$ implies $c = 0$ and $f_1 = f_2$, $\psi_1 = \psi_2$. □

Theorem A.6. *The Banach spaces $\mathcal{A}_\infty(\Delta)$ and $A_\infty(\Delta) \oplus \mathbb{C}$ induce the same Banach manifold structure on $\tilde{\mathcal{D}}$.*

Proof. From our discussion in Section 2.2 (in particular (2.9), (2.8), (2.7)), we know that the embedding $\tilde{\mathcal{D}} \hookrightarrow A_\infty(\Delta) \oplus \mathbb{C}$ factors through the map $\widehat{\Psi}$, i.e., it is given by $f \mapsto \theta(f) \xrightarrow{\widehat{\Psi}} (\mathcal{S}(f), \frac{1}{2}\theta(f)(0))$. Let U (resp., V) be the image of $\tilde{\mathcal{D}}$ in $\mathcal{A}_\infty(\Delta)$ (resp., $A_\infty(\Delta) \oplus \mathbb{C}$). Bers proved that V is open in $A_\infty(\Delta) \oplus \mathbb{C}$ ([Ber73]) using a theorem of Ahlfors ([Ahl63]) which says that the image of $T(1)$ in $A_\infty(\Delta)$ is open. The continuity of the map $\widehat{\Psi}$ implies that U is open in $\mathcal{A}_\infty(\Delta)$. Hence we have a holomorphic bijection $\widehat{\Psi}|_U : U \to V$. In order to conclude that this is a biholomorphic map between open subsets of Banach manifolds, by the inverse mapping theorem (see, e.g., [Lan95]), we only have to show that for any $\psi \in U$, the derivative of $\widehat{\Psi}$ at ψ, $D_\psi \widehat{\Psi}$, is a topological linear isomorphism between $\mathcal{A}_\infty(\Delta)$ and $A_\infty(\Delta) \oplus \mathbb{C}$.

From the proof of Corollary A.3, the linear map $D_\psi \widehat{\Psi} : \mathcal{A}_\infty(\Delta) \to A_\infty(\Delta) \oplus \mathbb{C}$ is given by

$$D_\psi \widehat{\Psi}(\varphi) = \left(\varphi_z - \psi\varphi, \frac{1}{2}\varphi(0)\right).$$

From the theory of ordinary differential equations, it is easy to prove that this map is injective. To prove surjectivity, let $f \in \tilde{\mathcal{D}}$ be such that $\theta(f) = \psi$. Given $(\phi, c) \in A_\infty(\Delta) \oplus \mathbb{C}$, consider

$$\varphi(z) = f'(z) \left(\int_0^z \frac{\phi(u)}{f'(u)} du + 2c\right).$$

It is straightforward to check that φ is the unique holomorphic function on Δ that satisfies

$$\varphi_z - \psi\varphi = \phi \quad \text{and} \quad \frac{1}{2}\varphi(0) = c.$$

What remains to be proved is that $\varphi \in \mathcal{A}_\infty(\Delta)$.

Let $\Omega = f(\Delta)$ and $\Omega^* = \hat{\mathbb{C}} \setminus \overline{\Omega}$ be the exterior of the domain Ω. Let $\lambda(w)|dw|$ be the Poincaré metric on Ω, which is given by

$$\lambda \circ f(z)|f'(z)| = \frac{1}{(1-|z|^2)}.$$

For $w \in \Omega$, let $\delta(w)$ denote the Euclidean distance from w to the boundary of Ω. The Koebe one-quarter theorem (see, e.g., [Nag88, Leh87]) implies that

$$\frac{1}{4} \leq \lambda(w)\delta(w) \leq 1. \tag{A.1}$$

Let $\tilde{\phi}(w) = \phi \circ f^{-1}(w)\left(f_w^{-1}(w)\right)^2$. Then

$$\int_0^z \frac{\phi(u)}{f'(u)}du = \int_0^w \tilde{\phi}(v)dv = \Phi(w),$$

where $w = f(z)$. Since

$$\sup_{w\in\Omega} |\lambda^{-2}(w)\tilde{\phi}(w)| = \sup_{z\in\Delta} |(1-|z|^2)^2\phi(z)| < \infty,$$

by a theorem of Bers ([Ber66]), there is a bounded harmonic Beltrami differential on Ω^*, $\mu : \Omega^* \to \mathbb{C}$, $\sup_{w\in\Omega^*}|\mu(w)| = \beta < \infty$, such that

$$\tilde{\phi}(w) = -\frac{6}{\pi}\iint_{\Omega^*} \frac{\mu(v)}{(v-w)^4}\left|\frac{dv\wedge d\bar{v}}{2}\right|, \qquad w \in \Omega.$$

This implies that

$$\Phi(w) = -\frac{2}{\pi}\iint_{\Omega^*} \frac{\mu(v)}{(v-w)^3}\left|\frac{dv\wedge d\bar{v}}{2}\right| + C_1,$$

where C_1 is a constant such that $\Phi(0) = 0$. Since every point $v \in \Omega^*$ is of distance at least $\delta(w)$ away from w, we have the following estimate:

$$\begin{aligned}
|\Phi(w)| &\leq \frac{2}{\pi}\iint_{|v-w|\geq\delta(w)} \frac{|\mu(v)|}{|v-w|^3}\left|\frac{dv\wedge d\bar{v}}{2}\right| + C_1 \\
&\leq \frac{2\beta}{\pi}\int_0^{2\pi}\int_{\delta(w)}^{\infty} \frac{1}{\rho^3}\,\rho d\rho d\theta + C_1 \\
&= \frac{4\beta}{\delta(w)} + C_1 \leq 16\beta\lambda(w) + C_1.
\end{aligned}$$

Using (A.1) and $w = f(z)$, we also have

$$\frac{1}{4} \leq \frac{1}{(1-|z|^2)|f'(z)|}\delta(f(z)) \leq 1,$$

$$\frac{\delta(f(z))}{1-|z|^2} \leq |f'(z)| \leq \frac{4\delta(f(z))}{1-|z|^2}.$$

Hence

$$\begin{aligned} |\varphi(z)| &\leq |f'(z)|\,|\Phi(f(z)) + C_2| \\ &\leq 16\beta\,|\lambda(f(z))|\,|f'(z)| + C_2\frac{4\delta(f(z))}{1-|z|^2} \\ &\leq \frac{C}{1-|z|^2}. \end{aligned}$$

Here C_2 and C are constants. To get the last inequality, we have used the fact that $\delta(f(z))$ is bounded for $z \in \Delta$. This concludes the proof that $\varphi \in \mathcal{A}_\infty(\Delta)$. □

Remark A.7. When Γ is a Fuchsian group, the embedding $\theta : \tilde{\mathcal{D}} \simeq \mathcal{T}(1) \to \mathcal{A}_\infty(\Delta)$ restricts to an embedding $\mathcal{BF}(\Gamma) \to \mathcal{A}_\infty(\Delta, \Gamma)$, where

$$\mathcal{A}_\infty(\Delta,\Gamma) = \left\{\psi \in \mathcal{A}_\infty(\Delta) : \psi_z - \frac{1}{2}\psi^2 \in A_\infty(\Delta,\Gamma)\right\}.$$

In contrast to the description of $A_\infty(\Delta,\Gamma)$ as $\|\cdot\|_{\infty,2}$ bounded holomorphic quadratic differentials of the Riemann surface $\Gamma\backslash\Delta$, $\mathcal{A}_\infty(\Delta,\Gamma)$ does not have an intrinsic characterization as a space of differentials on $\Gamma\backslash\Delta$. Rather, it is extrinsically defined as the space of solutions to the Ricatti equation

$$\psi_z - \frac{1}{2}\psi^2 = \phi, \quad \phi \in A_\infty(\Delta,\Gamma)$$

on the Riemann surface $\Gamma\backslash\Delta$. However, it contains the subspace of affine connections on $\Gamma\backslash\Delta$, i.e.,

$$\left\{\lambda : \Delta \to \mathbb{C} \text{ holomorphic } : \|\lambda\|_{\infty,1} < \infty, \lambda \circ \gamma\gamma' - \lambda = \frac{\gamma''}{\gamma'} \;\forall \gamma \in \Gamma\right\},$$

which is an affine space modeled on the space of $\|\cdot\|_{\infty,1}$ bounded holomorphic 1-forms on $\Gamma\backslash\Delta$.

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