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On Integrals of Eisenstein Series and Derivatives of L-Series

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1 Introduction

In [1], inspired by a formula of Ramanujan (see [3, page 207])

$$q^{1/9}\frac{(1-q)(1-q^4)^4(1-q^7)^7\cdots}{(1-q^2)^2(1-q^5)^5(1-q^8)^8\cdots}=\exp\left\{-C-\frac{1}{9}\int_q^1\frac{f(-t)^9}{f(-t^3)^3}\frac{dt}{t}\right\},\qquad(1.1)$$

where $f(-q)=\prod_{n=1}^\infty(1-q^n)$ and C is a multiple of the value of certain Dirichlet L-series evaluated at 2, Ahlgren, Berndt, Yee, and Zaharescu established the following result that connects Eisenstein series, special values of Dirichlet L-series, and infinite products of certain form. (See [1] for a historical background of the above formula.)

Theorem 1.1 (Ahlgren, Berndt, Yee, and Zaharescu)**.** Suppose that α is real, $k\ge 2$ is an integer, and χ is a nontrivial Dirichlet character that satisfies $\chi(-1)=(-1)^k$. Then, for $0<q<1$,

$$q^\alpha\prod_{n=1}^\infty(1-q^n)^{\chi(n)n^{k-2}}=\exp\left\{-C-\int_q^1\left(\alpha-\sum_{n=1}^\infty\sum_{d|n}\chi(d)d^{k-1}t^n\right)\frac{dt}{t}\right\},\qquad(1.2)$$

where

$$C=L'(2-k,\chi).\qquad(1.3)$$

□

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This result is surprising in the sense that it connects three apparently disparate mathematical objects, and it is natural to ask whether there are similar identities for functions other than the Dirichlet characters. In this paper, we will show that this is indeed the case and that Theorem 1.1 is in fact a simple corollary of a general result.

Theorem 1.2. Let $a(n)$ be an arithmetic function such that

(i) $a(n) \ll n^{\lambda}$ for some real number λ,

(ii) the Dirichlet series $A(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ can be analytically continued to the half-plane $\{s : \operatorname{Re} s \geq -\epsilon\}$ and satisfies $|A(s)| \ll e^{(\pi/2-\epsilon')|\operatorname{Im} s|}$ in the region for some positive numbers ϵ and ϵ',

(iii) $A(0) = 0$.

Then, for $0 < q < 1$,

$$q^{\alpha} \prod_{n=1}^{\infty} (1-q^n)^{a(n)} = \exp\left\{ -C - \int_q^1 \left(\alpha - \sum_{n=1}^{\infty} \sum_{d|n} a(d) dt^n \right) \frac{dt}{t} \right\}, \tag{1.4}$$

where

$$C = A'(0). \tag{1.5}$$

□

If we set $a(n) = \chi(n)n^{k-2}$, where $\chi(n)$ is a nonprincipal character modulo N with $\chi(-1) = (-1)^k$, then the function $a(n)$ clearly satisfies the conditions in Theorem 1.2 (see [2]). Therefore, identity (1.2) follows.

Our line of approach is different from that in [1], in which the main ingredient is the representation

$$L'(2-k,\chi) = \lim_{q \to 1^-} \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} \frac{q^n}{n} \tag{1.6}$$

for integers $k \geq 2$ and nonprincipal Dirichlet character χ with $\chi(-1) = (-1)^k$, and Theorem 1.1 follows immediately from this assertion. To prove (1.6), they started out by writing the sum on the right-hand side as a Riemann sum for some integral in a clever way. Thus, evaluating the limit of the sum is equivalent to evaluating a certain integral, which is done by contour integration and the residue theorem.

Here, we provide a simpler and more natural approach to this problem. Assume that χ is a character modulo N. We first observe that the integrand on the right-hand side of (1.2), with a suitable choice of α, is the Fourier expansion of the Eisenstein series with the character χ of weight k associated with the cusp ∞ on $\Gamma_0(N)$. Thus, it is very natural to express the integrand as a Mellin integral. In particular, if we write t as e^{-u}, then the Mellin transform of the integrand with respect to the variable u contains the L-series $L(s,\chi)$. This explains how special values of L-series come into the identity. To prove Theorem 1.1 (and, implicitly, (1.6)), we only need to evaluate the integral of the Mellin integral in a straightforward manner. In fact, upon closer scrutiny, we can see that our argument actually works for any functions $a(n)$ satisfying the conditions in Theorem 1.2.

Finally, we remark that one may wonder what will happen if we replace the integrand on the right-hand side of (1.2) by Eisenstein series associated with cusps other than ∞. In that case, we can still obtain an expression for the right-hand side, but it is not as elegant as that on the left-hand side of (1.2).

2 Proof of Theorem 1.2

Let $a(n)$ be an arithmetic function satisfying the assumptions in Theorem 1.2 and let $F(q)$ be defined by

$$F(q) = \alpha - \sum_{n=1}^{\infty} \sum_{d|n} a(d) d q^n. \tag{2.1}$$

By the well-known formula

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) z^{-s} \, ds, \tag{2.2}$$

which holds for all complex numbers z with $\operatorname{Re} z > 0$ and all real numbers $c > 0$, we have, for all $u > 0$,

$$\begin{aligned} F(e^{-u}) &= \alpha - \sum_{d=1}^{\infty} a(d) d \sum_{n=1}^{\infty} e^{-ndu} \\ &= \alpha - \frac{1}{2\pi i} \sum_{d=1}^{\infty} a(d) d \sum_{n=1}^{\infty} \int_{\lambda+3-i\infty}^{\lambda+3+i\infty} \Gamma(s)(ndu)^{-s} \, ds \\ &= \alpha - \frac{1}{2\pi i} \int_{\lambda+3-i\infty}^{\lambda+3+i\infty} \Gamma(s)\zeta(s) A(s-1) u^{-s} \, ds, \end{aligned} \tag{2.3}$$

where $A(s)$ denotes the Dirichlet series $\sum_{n=1}^{\infty} a(n) n^{-s}$. Let c be a positive number less than 1, and denote $\log(q^{-1})$ and $\log(c^{-1})$ by x and δ, respectively. The last expression yields

$$\int_q^c F(t)\frac{dt}{t} = \int_\delta^x F\left(e^{-u}\right)du$$
$$= \alpha(x-\delta) - \frac{1}{2\pi i}\int_{\lambda+3-i\infty}^{\lambda+3+i\infty}\Gamma(s)\zeta(s)A(s-1)\frac{x^{1-s}-\delta^{1-s}}{1-s}ds. \tag{2.4}$$

Noting that $\Gamma(s) = (s-1)\Gamma(s-1)$ and changing the variable from s to $s+1$, we see that

$$\int_q^c F(t)\frac{dt}{t} = \alpha(x-\delta) + \frac{1}{2\pi i}\int_{\lambda+2-i\infty}^{\lambda+2+i\infty}\Gamma(s)\zeta(s+1)A(s)\left(x^{-s}-\delta^{-s}\right)ds. \tag{2.5}$$

We now consider the integrals involving x and δ separately.

For the integral involving x, we observe that

$$\zeta(s+1)A(s) = \sum_{d=1}^\infty\sum_{n=1}^\infty \frac{a(d)}{n}(nd)^{-s}. \tag{2.6}$$

Therefore, by (2.2) again, we have

$$\frac{1}{2\pi i}\int_{\lambda+2-i\infty}^{\lambda+2+i\infty}\Gamma(s)\zeta(s+1)A(s)x^{-s}ds = \sum_{d=1}^\infty a(d)\sum_{n=1}^\infty\frac{1}{n}e^{-n\,dx}$$
$$= -\sum_{d=1}^\infty a(d)\log\left(1-q^d\right). \tag{2.7}$$

For the integral involving δ we use the residue theorem. We move the line of integration to the vertical line $\mathrm{Re}\, s = -\epsilon$. This is justified by assumption (ii) and the upper-bound $|\Gamma(\sigma+it)| \ll |t|^{\sigma-1/2}e^{-\pi|t|/2}$. Each of the functions $\Gamma(s)$ and $\zeta(s)$ has a simple pole at $s=0$. By assumption (iii), the function $A(s)$ has a zero at $s=0$. The Taylor expansions at 0 of these three functions are given by

$$\Gamma(s) = \frac{1}{s}+\cdots, \qquad \zeta(s+1) = \frac{1}{s}+\cdots, \qquad A(s) = sA'(0)+\cdots. \tag{2.8}$$

Denoting $A'(0)$ by C, we thus have

$$\frac{1}{2\pi i}\int_{\lambda+2-i\infty}^{\lambda+2+i\infty}\Gamma(s)\zeta(s+1)A(s)\delta^{-s}ds = C + \frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty}\Gamma(s)\zeta(s+1)A(s)\delta^{-s}ds$$
$$= C + O\left(\delta^\epsilon\right). \tag{2.9}$$

Combining (2.4), (2.7), and (2.9), and letting $\delta \to 0$, we hence obtain

$$\int_q^1 F(t)\frac{dt}{t} = -\alpha \log q - C - \sum_{d=1}^{\infty} a(d) \log\left(1 - q^d\right), \tag{2.10}$$

which is equivalent to (1.4). This completes the proof of Theorem 1.2.

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References

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