



Partial wave analysis of scattering with nonlocal Aharonov–Bohm effect

De-Hone Lin

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30043, Taiwan

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Abstract

Partial wave theory of a two-dimensional scattering problem for an arbitrary short range potential and a nonlocal Aharonov–Bohm magnetic flux is established. The scattering process of a “hard disk” like potential and the magnetic flux is examined. Since the nonlocal influence of magnetic flux on the charged particles is universal, the nonlocal effect in hard disk case is expected to appear in quite general potential system and will be useful in understanding some phenomena in mesoscopic physics.

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In this Letter we study the scattering amplitude and the cross section of a charged particle moving in a short range potential with scattering center located at the origin, and the Aharonov–Bohm (AB) magnetic flux along z -axis [1]. The nonintegrable phase factor (NPF) [2–4] is used to couple the magnetic flux with the particle angular momentum such that the partial wave method can be conveniently developed [5–9]. As a realization of the method, a charged particle scattered by a “hard disk” like potential plus the magnetic flux is discussed in detail. Several interesting nonlocal effects of the magnetic flux in the hard disk model are concluded as follows: (1) in the long wave length limit (equivalently, short range potential) the total cross section is drastically suppressed at quantized magnetic flux $\Phi = (2n + 1)\Phi_0/2$, where $n = 0, 1, 2, \dots$, and Φ_0 is the fundamental magnetic flux quantum hc/e . The global influence of the magnetic flux on the cross section is manifested with Φ_0 periodicity. On the other hand, the cross section approaches the flux-free case in the short wave length limit, i.e., the quantum interference feature of the nonlocal effect gradually disappears, and the cross section approaches the classical limit. (2) If the hard disk is used to simulate the boson (fermion) moving in two-dimensional space, the scattering process of identical particles carrying the magnetic flux shows that the total cross section is suppressed at quantized magnetic flux $\Phi = (2n + 1)\Phi_0$ for bosons ($\Phi = 2n\Phi_0$ for fermions) and exhibits the global structure with $2\Phi_0$ periodicity. Since the nonlocal influence of the magnetic flux on the charged particle are universal, the influences should be general in

E-mail address: dhlin@mail.nctu.edu.tw (D.-H. Lin).

similar systems, and may be useful in understanding some transport phenomena in mesoscopic physics and account for the quantum Hall effect [6,10].

We consider a two-dimensional model. The fixed-energy Green's function $G^0(\mathbf{x}, \mathbf{x}'; E)$ for a charged particle with mass μ propagating from \mathbf{x}' to \mathbf{x} satisfies the Schrödinger equation

$$\left\{ E - \left[-\frac{\hbar^2 \nabla^2}{2\mu} + V(\mathbf{x}) \right] \right\} G^0(\mathbf{x}, \mathbf{x}'; E) = \delta(\mathbf{x} - \mathbf{x}'), \quad (1)$$

where $V(\mathbf{x})$ is the scalar potential and \mathbf{x} is the two-dimensional coordinate vector. In the cylindrically symmetric system, the Green's function can be decomposed as

$$G^0(\mathbf{x}, \mathbf{x}'; E) = \sum_{m=-\infty}^{\infty} G_m^0(\rho, \rho'; E) \frac{e^{im(\varphi-\varphi')}}{2\pi} \quad (2)$$

with (ρ, φ) being the polar coordinates in two-dimensional space and $G_m^0(\rho, \rho'; E)$ the radial Green's function. The left-hand side of Eq. (1) can then be cast into

$$\sum_{m=-\infty}^{\infty} \left\{ E + \left[\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{m^2}{\rho^2} \right) \right] - V(\rho) \right\} G_m^0(\rho, \rho'; E) \frac{e^{im(\varphi-\varphi')}}{2\pi}. \quad (3)$$

For a charged particle affected by a magnetic field, the Green's function $G(\mathbf{x}, \mathbf{x}'; E)$ is different from $G^0(\mathbf{x}, \mathbf{x}'; E)$ by a global NPF [2,7,8]

$$G(\mathbf{x}, \mathbf{x}'; E) = G^0(\mathbf{x}, \mathbf{x}'; E) \exp \left\{ \frac{ie}{\hbar c} \int_{\mathbf{x}'}^{\mathbf{x}} \mathbf{A}(\tilde{\mathbf{x}}) \cdot d\tilde{\mathbf{x}} \right\}. \quad (4)$$

Here the vector potential $\mathbf{A}(\mathbf{x})$ is used to describe the magnetic field. For an infinitely thin tube of finite magnetic flux along the z -direction, the vector potential can be expressed as

$$\mathbf{A}(\mathbf{x}) = 2g \frac{-y\hat{e}_x + x\hat{e}_y}{x^2 + y^2}, \quad (5)$$

where \hat{e}_x, \hat{e}_y stand for the unit vector along the x, y axis, respectively. Introducing the azimuthal angle $\varphi(\mathbf{x}) = \tan^{-1}(y/x)$ around the AB tube, the components of the vector potential can be expressed as $A_i = 2g\partial_i\varphi(\mathbf{x})$. The associated magnetic field lines are confined to an infinitely thin tube along the z -axis,

$$B_3 = 2g\epsilon_{3ij}\partial_i\partial_j\varphi(\mathbf{x}) = 4\pi g\delta(\mathbf{x}_\perp), \quad (6)$$

where \mathbf{x}_\perp stands for the transverse vector $\mathbf{x}_\perp \equiv (x, y)$. Since the magnetic flux through the tube is defined by the integral $\Phi = \int d^2x B_3$, the coupling constant g is related to the magnetic flux by $g = \Phi/4\pi$. By using the expression of $A_i = 2g\partial_i\varphi$, the angular difference between the initial point \mathbf{x}' and the final point \mathbf{x} in the exponent of the NPF is given by

$$\varphi - \varphi' = \int_{t'}^t d\tau \dot{\varphi}(\tau) = \int_{t'}^t d\tau \frac{-y\dot{x} + x\dot{y}}{x^2 + y^2} = \int_{\mathbf{x}'}^{\mathbf{x}} \frac{\tilde{\mathbf{x}} \times d\tilde{\mathbf{x}}}{\tilde{\mathbf{x}}^2}, \quad (7)$$

where $\dot{\varphi} = d\varphi/d\tau$. Given two paths C_1 and C_2 connecting \mathbf{x}' and \mathbf{x} , the integral differs by an integer multiple of 2π . The winding number is thus given by the contour integral over the closed difference path C :

$$n = \frac{1}{2\pi} \oint_C \frac{\tilde{\mathbf{x}} \times d\tilde{\mathbf{x}}}{\tilde{\mathbf{x}}^2}. \quad (8)$$

The magnetic interaction is therefore purely nonlocal and topological [5–8,11]. Its action takes the form $\mathcal{A}_{\text{mag}} = -\hbar\mu_0 2\pi n$, where $\mu_0 \equiv -2eg/\hbar c = -\Phi/\Phi_0$ is a dimensionless number with the customarily minus sign. The NPF now becomes $\exp\{-i\mu_0(2\pi n + \varphi - \varphi')\}$. The Green's function $G_n(\rho, \rho'; E)$ for a specific winding number n can be obtained by converting the summation over m in Eq. (3) into an integral over z and another summation over n by the Poisson's summation formula (e.g., Ref. [13, p. 469])

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} dz \sum_{n=-\infty}^{\infty} e^{2\pi n z i} f(z). \quad (9)$$

So the expression (3) when includes the NPF can be written as

$$\int dz \sum_{n=-\infty}^{\infty} \left\{ E + \left[\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{z^2}{\rho^2} \right) \right] - V(\rho) \right\} G_z(\rho, \rho'; E) \frac{e^{i(z-\mu_0)(\varphi+2n\pi-\varphi')}}{2\pi}, \quad (10)$$

where the superscript 0 in G_n^0 has been suppressed to denote that the AB effect is included. Obviously, the number n in the right-hand side is precisely the winding number by which we want to classify the Green's function. Employing the special case of the Poisson formula $\sum_n \exp\{ik(\varphi + 2n\pi - \varphi')\} = \sum_{m=-\infty}^{\infty} \delta(k - m) \exp\{im(\varphi - \varphi')\}$, the summation over all indices n forces $z = \mu_0$ modulo an arbitrary integer number. Thus, we obtain

$$\sum_{m=-\infty}^{\infty} \left\{ E + \left[\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{|m + \mu_0|^2}{\rho^2} \right) \right] - V(\rho) \right\} G_{|m+\mu_0|}(\rho, \rho'; E) \frac{e^{im\varphi}}{2\pi}. \quad (11)$$

We see that the influence of the AB effect to the radial Green's function is to replace the integer quantum number m with a real one $|m + \mu_0|$ which depends on the magnitude of magnetic flux. Applying the Fourier expansion of δ function,

$$\delta(\varphi - \varphi') = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\varphi - \varphi')}, \quad (12)$$

to the r.h.s. of Eq. (1) and defining $\alpha = |m + \mu_0|$ for convenience, we reduce the radial Green's function to

$$\left\{ E + \left[\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{\alpha^2}{\rho^2} \right) \right] - V(\rho) \right\} G_\alpha(\rho, \rho'; E) = \delta(\rho - \rho'). \quad (13)$$

As a result, the corresponding radial wave equation reads

$$\left\{ E + \left[\frac{\hbar^2}{2\mu} \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{\alpha^2}{\rho^2} \right) \right] - V(\rho) \right\} R_{\alpha k}(\rho) = 0, \quad (14)$$

where the subscript set (α, k) with $k \equiv \sqrt{2\mu E}/\hbar$ denotes the state of scattering particle.

For a short range potential, say $V(\rho)$ vanishes as $\rho > a$, the exterior solution is the linear combination of 1st and 2nd kind Bessel functions $J_\alpha(k\rho)$, and $N_\alpha(k\rho)$, and may be given by

$$R_{\alpha k}(\rho) = \sqrt{k} [\cos \delta_\alpha(k) J_\alpha(k\rho) - \sin \delta_\alpha(k) N_\alpha(k\rho)], \quad (15)$$

where $k = \sqrt{2\mu E}/\hbar$ and $\delta_\alpha(k)$ is the phase shift which can be used to measure the interaction strength of potential. Thus the general solution $\Psi_k(\mathbf{x})$ of a scattering particle is given by superposition of the partial waves $\Psi_{\alpha k}(\mathbf{x}) = R_{\alpha k}(\rho) e^{im\varphi}$, which reads

$$\Psi_k(\mathbf{x}) = \sum_{m=-\infty}^{\infty} \sqrt{k} [\cos \delta_\alpha(k) J_\alpha(k\rho) - \sin \delta_\alpha(k) N_\alpha(k\rho)] e^{im\varphi}. \quad (16)$$

Since it must describe both the incident and the scattered waves at large distance, we naturally expect it to become

$$\Psi_k(\mathbf{x}) \xrightarrow{|\mathbf{x}| \rightarrow \infty} \mathcal{F}_\infty \left(\exp\{i\mathbf{k} \cdot \mathbf{x}\} \exp \left\{ \frac{ie}{\hbar c} \int_C^{\mathbf{x}} \mathbf{A}(\mathbf{x}') \cdot d\mathbf{x}' \right\} \right) + f(\varphi) \sqrt{\frac{i}{\rho}} \exp\{ik\rho\}, \quad (17)$$

where $\exp\{i\mathbf{k} \cdot \mathbf{x}\}$ describes the incident plane wave of a charged particle with momentum $\mathbf{p} = \mu\mathbf{k}$ and $\mathcal{F}_\infty(\cdot)$ stands for its asymptotic form. The phase modulation of the NPF comes from the fact that the field $\mathbf{A}(\mathbf{x})$ of AB magnetic flux affects the charged particle globally. The subscript C in the integral is used to represent the nature of the NPF which depends on the different paths. To find the amplitude $f(\varphi)$ we first note that the plane wave in Eq. (17) can be expanded in terms of the partial waves

$$e^{i\mathbf{k} \cdot \mathbf{x}} = \sum_{m=-\infty}^{\infty} i^m J_m(k\rho) e^{im\varphi}. \quad (18)$$

Using the same procedure as in Eqs. (9)–(11), we combine the nonlocal flux effect into the partial wave expansion, and obtain the result

$$e^{i\mathbf{k} \cdot \mathbf{x}} e^{\frac{ie}{\hbar c} \int_C^{\mathbf{x}} \mathbf{A}(\mathbf{x}') \cdot d\mathbf{x}'} = \sum_{m=-\infty}^{\infty} i^\alpha J_\alpha(k\rho) e^{im\varphi}. \quad (19)$$

Taking the asymptotic approximations of Bessel functions, and comparing both asymptotic forms of Eqs. (16) and (17), we find the scattering amplitude

$$f(\varphi) = \frac{1}{\sqrt{2\pi k}} \sum_{m=-\infty}^{\infty} e^{i(\delta_\alpha - \pi/4)} 2i \sin \delta_\alpha e^{im\varphi}. \quad (20)$$

It is noteworthy that if the flux is quantized for integer μ_0 , the result reduces to the flux-free case [12]. In most cases, the total cross section of our major concern is defined by

$$\sigma_t = \int_{-\pi}^{\pi} |f(\varphi)|^2 d\varphi. \quad (21)$$

Thus, the partial wave representation of total cross section for a charged particle scattered by a short range potential plus the nonlocal AB effect is given by

$$\sigma_t = \frac{4}{k} \sum_{m=-\infty}^{\infty} \sin^2 \delta_\alpha. \quad (22)$$

It is obvious that the cross section is completely determined by the scattering phase shifts which are concluded by the potential of different types. Furthermore, when a nonlocal AB magnetic flux exists, both the phase shift and the cross section are affected globally. A relation between the total cross section σ_t and the scattering amplitude is obtained if we set $\varphi = 0$, and then take the imaginary part. It gives $\sigma_t = (2\sqrt{2\pi}/\sqrt{k}) \text{Im} f(0)$. This is the optical theorem and is essentially a consequence of the conservation of particles. For the case of identical bosons (fermions) carrying the magnetic flux in two dimensions, the differential cross section is given by $\sigma(\varphi) = |f(\varphi) \pm f(\varphi + \pi)|^2$, where the plus sign is for bosons as usual. The total cross sections are given by the integral $\int_{-\pi}^{\pi} \sigma(\varphi) d\varphi$, and yield

$$\sigma_t(\text{bosons}) = \frac{16}{k} \sum_{m=-\infty, \text{even}}^{\infty} \sin^2 \delta_\alpha, \quad (23)$$

and

$$\sigma_t(\text{fermions}) = \frac{16}{k} \sum_{m=-\infty, \text{odd}}^{\infty} \sin^2 \delta_\alpha, \tag{24}$$

where the subscript “odd” (“even”) is used to indicate the summation over odd (even) numbers only. As a realization of the nonlocal influence of the AB flux on the cross section, let us consider a charged particle scattered by a hard disk potential and a magnetic flux. The potential is given by $V(\rho) = \infty$, for $\rho \leq a$, and $V(\rho) = 0$, for $\rho > a$. Using the boundary condition of the wave function $R_{\alpha k}(a^+) = 0$, we find that the phase shift is given by

$$\tan \delta_\alpha(k) = \frac{J_\alpha(ka)}{N_\alpha(ka)}, \tag{25}$$

where $J_\alpha(z)$ ($N_\alpha(z)$) is the Bessel function of first (second) kind. Substituting this expression into Eq. (22), the total cross-section is found to be

$$\sigma_t = \frac{4}{k} \sum_{m=-\infty}^{\infty} \frac{J_\alpha^2(ka)}{J_\alpha^2(ka) + N_\alpha^2(ka)}. \tag{26}$$

Note that the result will reduce to the pure disk case if the flux is quantized for $\mu_0 = n\Phi_0$. In this case the low energy limit $k \rightarrow 0$ (assuming the radius a is finite) can be found by the asymptotic expansion of Bessel functions, and only index $m = 0$ survives. So the phase shift becomes

$$\tan \delta_0(k) = J_0(ka)/N_0(ka) \approx \frac{\pi}{2 \ln(ka/2)}. \tag{27}$$

This implies the total cross section at the low energy limit is

$$\sigma_t \approx \frac{8a}{\pi} \frac{1}{ka \ln(ka)} \rightarrow \infty. \tag{28}$$

At the high energy limit $k \rightarrow \infty$, we may use the formulas of Bessel functions of the large argument to turn Eq. (26) into

$$\begin{aligned} \sigma_t &= \frac{4}{k} \sum_{m=-\infty}^{\infty} \cos^2 \left[ka - \left(m + \frac{1}{2} \right) \frac{\pi}{2} \right] \\ &= \lim_{ka \rightarrow \infty} \frac{4}{k} \left\{ \sum_{m=-[ka], \text{even}}^{[ka]} \cos^2 \left(ka - \frac{\pi}{4} \right) + \sum_{m=-[ka], \text{odd}}^{[ka]} \sin^2 \left(ka - \frac{\pi}{4} \right) \right\} = 4a. \end{aligned} \tag{29}$$

The value $4a$ explains that the quantum result does not go over to the actual classical result $\sigma_t \rightarrow 2a$ even though the wave length of de Broglie is much less than a . The numerical result for α with noninteger value is plotted in Fig. 1, where the normalization σ_0 is chosen as $4a$. There are two main results: (1) the cross section σ_t is drastically suppressed at the low energy limit (equivalently, the short range potential), say $ka \leq 1$, at quantized magnetic flux $\Phi = (2n + 1)\Phi_0/2$, $n = 0, 1, 2, \dots$, with Φ_0 periodicity as shown in Figs. 1 and 2. (2) A more interesting consideration is given by the scattering of identical particles simulated by the hard disks carrying the magnetic flux. In Fig. 3, we plot the total cross sections of identical bosons carrying the magnetic flux via Eq. (23). The outcome shows that the cross section approaches zero ($\sigma_t \rightarrow 0$) when the value $ka \rightarrow 0$ if the magnetic flux is at quantized value $(2n + 1)\Phi_0$. On the contrary, if the magnetic flux is equal to $2n\Phi_0$, the cross section becomes maximum and the effect of magnetic flux disappears. Since the decay rate of a current \mathbf{j} traveling a distance \mathbf{x} is given by $\mathbf{j}(\mathbf{x}) = \mathbf{j}(0) \exp(-\sigma_t n_0 \mathbf{x})$, where n_0 is the number of the scattering center, the total cross section $\sigma_t \rightarrow 0$ at the low energy limit at $\Phi = (2n + 1)\Phi_0$ means that the resistance $R \rightarrow 0$ and results in the persistence of current. This phenomenon is consistent with the picture of composite boson in fractional quantum Hall states located at the

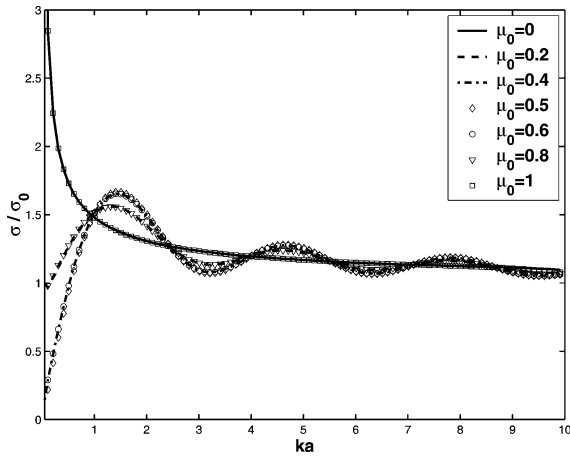


Fig. 1. The total cross section for a charged particle scattered by a hard disk with radius a and a magnetic flux along the z -axis. The normalization $\sigma_0 = 4a$ has been selected. Due to the existence of magnetic flux, at the limit of the long wave (equivalently, the short range potential), say $ka \leq 1$, the total cross section is drastic suppressed at quantized magnetic flux $\Phi = (2n + 1)\Phi_0/2$, where $n = 0, 1, 2, \dots$, with Φ_0 periodicity, see Fig. 2. The magnetic flux effect disappears when the flux is quantized at $\Phi = n\Phi_0$.

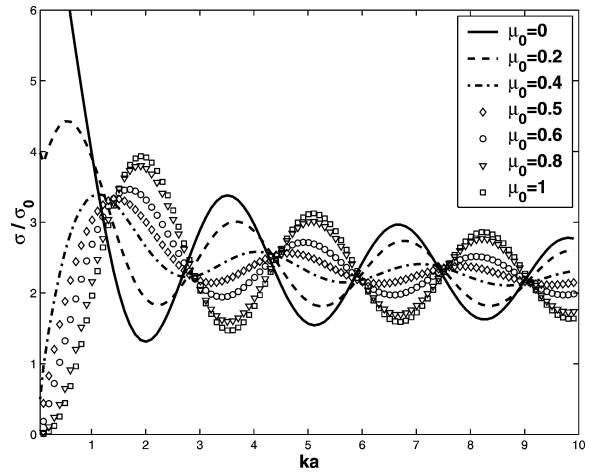


Fig. 3. Total cross sections for identical bosons carrying the magnetic flux with various μ_0 . The cross section at the long wave length limit (equivalently, the sufficient short range potential), say $ka \leq 0.5$, approaches zero at the quantized magnetic flux $\Phi = (2n + 1)\Phi_0$. On the contrary, the cross section becomes maximum and the effect of magnetic flux disappears when $\Phi = 2n\Phi_0$. The periodic structure is $2\Phi_0$ as shown in Fig. 4.

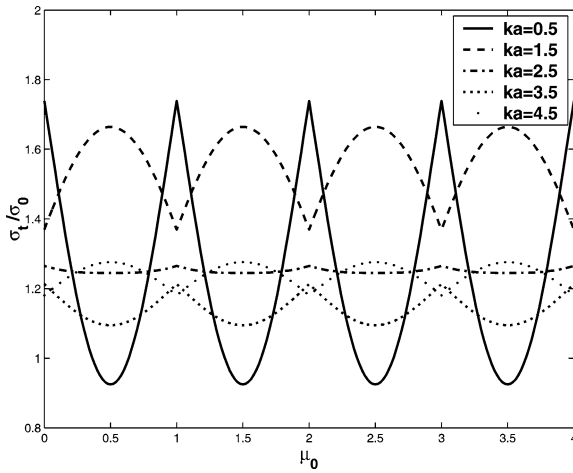


Fig. 2. Periodic structures of total cross sections of a charged particle scattered by a hard disk plus a magnetic flux along the z -axis. At quantized values of magnetic flux $\Phi = (2n + 1)\Phi_0/2$, $n = 0, 1, 2, \dots$, the cross section reduces to the minimum for $ka \leq 0.5$.

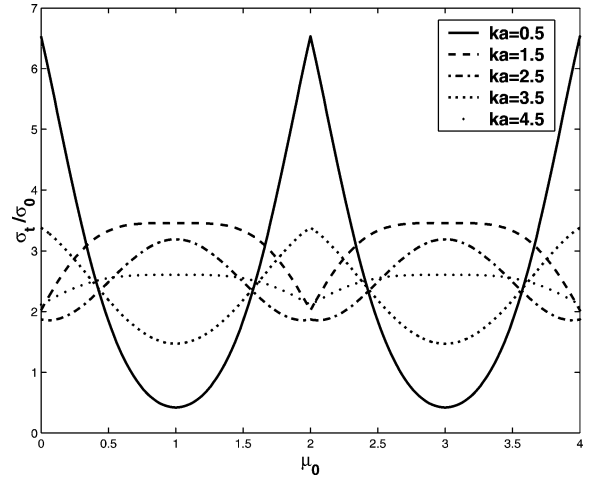


Fig. 4. Periodic structures of cross sections of identical bosons carrying the magnetic flux. The cross section approaches zero when the magnetic flux is quantized at $\Phi = (2n + 1)\Phi_0$ for $ka \leq 0.5$.

filling factor with odd denominator such as $\nu = 1/3$. The composite boson is pictured by an electron carrying the quantized magnetic flux $\Phi = (2n + 1)\Phi_0$. It dictates the quantized Hall states which exhibit the perfect conduction in the longitudinal direction, i.e., the resistance originated from the collisions between composite bosons disappear [10]. The global structure of the total cross section is given by $2\Phi_0$ periodicity as shown in Fig. 4. In the case of

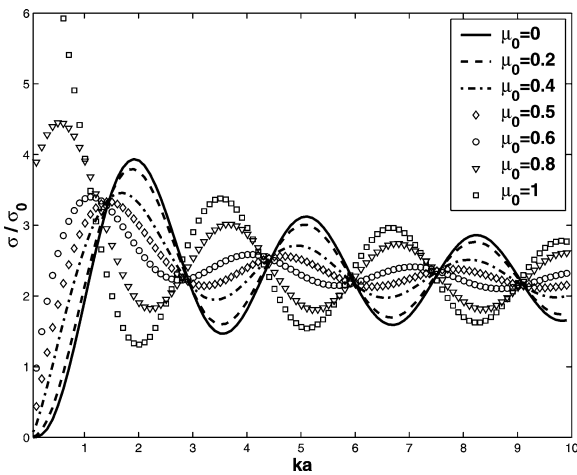


Fig. 5. Total cross sections of identical fermions carrying the magnetic flux with various μ_0 . The cross section approaches zero for $ka \leq 0.5$ when the flux becomes $2n\Phi_0$. The magnetic flux effect disappears when the magnitude of flux is at $(2n + 1)\Phi_0$. The global periodic structures in cross sections is $2\Phi_0$ as shown in Fig. 6.

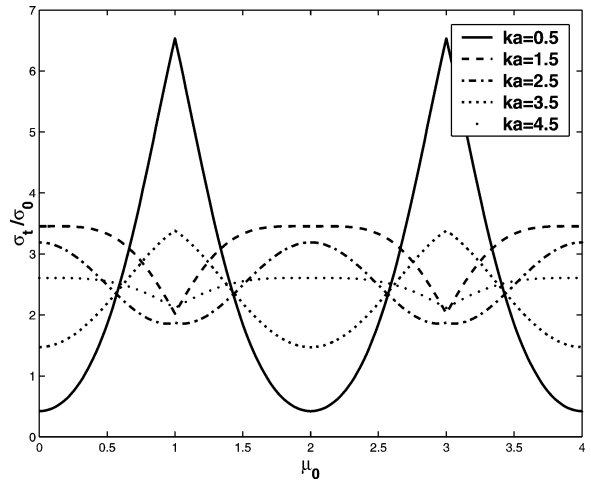


Fig. 6. Periodic structures of total cross sections for identical fermions carrying the magnetic flux. The cross section approaches zero when the magnetic flux is quantized at $\Phi = 2n\Phi_0$ for $ka \leq 0.5$.

identical fermions, the total cross section $\sigma_t \rightarrow 0$ is found at the quantized magnetic flux $\Phi = 2n\Phi_0$ as shown in Fig. 5. Such effect is consistent with the model of composite fermion in the quantum Hall state located at the filling factor with even denominator $\nu = 5/2$. The composite fermion is described by an electron carrying the quantized magnetic flux $\Phi = 2n\Phi_0$. In Ref. [14], a quantitative explanation of quantum Hall state at the filling factor $\nu = 5/2$ is given by the existence of a shorter range potential between the composite fermions than the case of the filling factor $\nu = 1/2$. Here we can see that, in Fig. 5, a sufficiently short range potential, say $ka < 0.5$, between the fermions carrying the quantized magnetic flux $\Phi = 2n\Phi_0$ will cause negligible cross section and thus agree with the composite fermions model. Similar to the boson case, the oscillating period is given by $2\Phi_0$ as shown in Fig. 6.

In this Letter, we study the partial wave method of scattering theory for a short range potential and a magnetic flux. As an illustration, the hard disk potential plus a magnetic flux is calculated in detail. The nonlocal influence of the magnetic flux is discussed. Since the nonlocal effect of magnetic flux on the charged particle is universal, the effect should be general in similar systems. Although we assume that the potential must be $V(\rho) = 0$ for $\rho > a$, we do not specify the radius a beyond which $V(\rho) = 0$. Hence we expect that the method given in this Letter should be valid for a very general potential as long as the potential decreases rapidly enough when $r \rightarrow \infty$. On the other hand, though in our discussion the magnetic flux is placed at the origin, it can be moved to the other points as long as it still locates in the potential region. This is due to the fact that the final outcome just relates to the flux via homotopy classes. We hope the discussions would be helpful in understanding mesoscopic systems and strongly correlated systems.

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