Fault Hamiltonicity and Fault Hamiltonian Connectivity of the (n, k)-Star Graphs

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In this paper, we consider the fault Hamiltonicity, and the fault Hamiltonian connectivity of the (n, k)-star graph $S_{n,k}$. Assume that $F \subset V(S_{n,k}) \cup E(S_{n,k})$. For $n - k \geq 2$, we prove that $S_{n,k} - F$ is Hamiltonian if $|F| \leq n - 3$ and $S_{n,k} - F$ is Hamiltonian connected if $|F| \leq n - 4$. For $n - k = 1$, $S_{n,n-1}$ is isomorphic to the n-star graph S_n which is known to be Hamiltonian if and only if $n > 2$ and Hamiltonian connected if and only if $n = 2$. Moreover, all the bounds are tight. © 2003 Wiley Periodicals, Inc.

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1. INTRODUCTION

The architecture of an interconnection network is usually represented by a graph. There are many mutually conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum from all aspects. One has to design a suitable network depending on the requirements of its properties. The Hamiltonian property is one of the major requirements in designing the topology of a network. Fault tolerance is also desirable in massive parallel systems.

In this paper, a network is represented as a loopless undirected graph. For graph definitions and notations, we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) | (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set.* Two vertices u and v are *adjacent* if $(u, v) \in E$. A *path* is represented by $\langle v_0, v_1, v_2, \ldots, v_k \rangle$. The *length* of a path P is the number of edges in P. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ as $\langle v_0, P_1, v_i, v_{i+1}, \ldots, v_j, P_2, v_t, \ldots, v_k \rangle$, where P_1 is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and P_2 is the path $\langle v_j, v_{j+1}, \ldots, v_t \rangle$. Hence, it is possible to write a path as $\langle v_0, v_1, P, v_1, v_2, \ldots, v_k \rangle$ if the length of P is 0. We use $d(u, v)$ to denote the distance between u and v, that is, the length of the shortest path joining u and v. A path is a *Hamiltonian path* if its vertices are distinct and span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a *Hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

In [4], the performance of the Hamiltonian property in faulty networks is discussed. A Hamiltonian graph G is *k-vertex-fault Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subset V(G)$ with $|F| \leq k$. The *vertex fault-tolerant Hamiltonicity* $\mathcal{H}_v(G)$ is defined to be the maximum integer k such that G is k-vertex-fault Hamiltonian if G is Hamiltonian and is undefined otherwise. Obviously, $\mathcal{H}_v(G) \leq \delta(G) - 2$, where $\delta(G) = \min\{\deg(v) | v \in V(G)\}$ if $\mathcal{H}_v(G)$ is defined. Similarly, a Hamiltonian graph G is *k-edge-fault Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subset E(G)$, with $|F| \leq k$. The *edge fault-tolerant Hamiltonicity* $\mathcal{H}_e(G)$ is defined to be the maximum integer k such that G is k-edge-fault Hamiltonian if G is Hamiltonian and is undefined otherwise. Again, $\mathcal{H}_e(G) \leq \delta(G) - 2$ if $\mathcal{H}_e(G)$ is defined. Huang et al. [5] defined a more general parameter: fault-tolerant Hamiltonicity. A Hamiltonian graph G is *k-fault Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$. The *fault-tolerant Hamiltonicity* $\mathcal{H}_f(G)$ is defined to be the maximum integer k such that G is k-fault Hamiltonian if G is Hamiltonian and is undefined otehrwise. Clearly, $\mathcal{H}_f(G) \leq \delta(G) - 2$ if $\mathcal{H}_f(G)$ is defined. Huang et al. [5] also introduced the term fault-tolerant Hamiltonian connected. A graph G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two vertices of G. All Hamiltonian connected graphs except the complete graphs K_1 and K_2 are Hamiltonian. A graph G is *k-fault Hamiltonian connected* if $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$. The *fault-tolerant Hamiltonian connectivity* $\mathcal{H}_f^\kappa(G)$ is defined to be the maximum integer k such that G is k-fault Hamiltonian connected if G is Hamiltonian connected and is undefined otherwise. It can be checked that $\mathcal{H}_f^\kappa(G) \leq \delta(G) - 3$ only if $\mathcal{H}_f^\kappa(G)$ is defined and $|V(G)| \geq 4$.

In this paper, we consider the fault-tolerant Hamiltonic-

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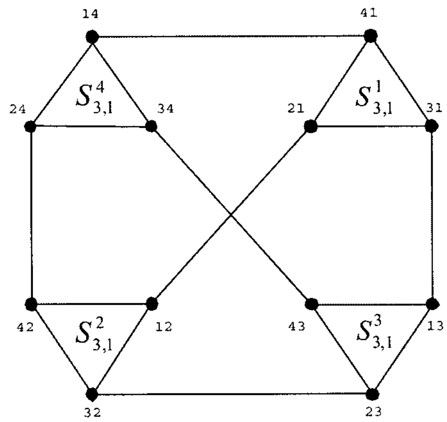

FIG. 1. (4, 2)-star graph.

ity of the (n, k)-star graph. The (n, k)-star graph is an attractive alternative to the n-star graph [1]. However, the growth of vertices is $n!$ for an n-star graph. To remedy this drawback, the (n, k)-star graph was proposed by [3]. The (n, k)-star graph is a generalization of the n-star graph. It has two parameters n and k. When $k = n - 1$, an $(n, n - 1)$-star graph is isomorphic to an n-star graph, and when $k = 1$, an $(n, 1)$-star graph is isomorphic to a complete graph K_n. Then, all the parallel algorithms of the n-star graphs and the complete graphs K_n can be applied to the (n, k)-star graphs.

Throughout this paper, we assume that n and k are positive integers with $n > k$. We use $\langle n \rangle$ to denote the set $\{1, 2, \ldots, n\}$. The (n, k)-star graph, denoted by $S_{n,k}$, is a graph with the vertex set $V(S_{n,k}) = \{u_1u_2 \ldots u_k | u_i \in \langle n \rangle$ and $u_i \neq u_j$ for $i \neq j\}$. Adjacency is defined as follows: A vertex $u_1u_2 \ldots u_i \ldots u_k$ is adjacent to (1) the vertex $u_iu_2u_3 \ldots u_1 \ldots u_k$, where $2 \leq i \leq k$ (i.e., we swap u_i with u_1), and (2) the vertex $xu_2u_3 \ldots u_k$, where $x \in \langle n \rangle - \{u_i | 1 \leq i \leq k\}$. The (4, 2)-star graph is shown in Figure 1. The edges of type (1) are referred to as *i-edges,* and the edges of type (2) are referred to as *1-edges.* By definition, $S(n, k)$ is an $(n - 1)$-regular graph with $n!/(n - k)!$ vertices. Moreover, it is vertex-transitive.

In the following section, we discuss some properties of complete graphs K_n. In Section 3, we discuss some properties of the (n, k)-star graphs. In the final section, we prove that (1) $\mathcal{H}_f(S_{n,k}) = n - 3$ and $\mathcal{H}_f^\kappa(S_{n,k}) = n - 4$ if $n - k \geq 2$; (2) $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}_f^\kappa(S_{2,1}) = 0$; and (3) $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}_f^\kappa(S_{n,n-1})$ is undefined if $n > 2$.

2. SOME PROPERTIES OF COMPLETE GRAPHS

Let $G = (V, E)$ be a graph. We use $\bar{E}$ to denote the edge set of the complement of G. The following theorem was proved by Ore [7]:

Theorem 1 [7]. *Assume that $G = (V, E)$ is a graph with $n \geq 4$ vertices. Then, G is Hamiltonian if $|\bar{E}| \leq n - 3$ and is Hamiltonian connected if $|\bar{E}| \leq n - 4$.*

Let $G = (V, E)$ be a graph with n vertices where $n \geq 4$ and $|\bar{E}| \leq n - 4$. By Theorem 1, there is a Hamiltonian path joining any two different vertices. Actually, there are two Hamiltonian paths of different types joining any two different vertices. Therefore, we have a more profound result.

Theorem 2. *Assume that $G = (V, E)$ is a graph with $V = \langle n \rangle$, $n \geq 4$, and $|\bar{E}| \leq n - 4$. Then, there are two Hamiltonian paths of G joining any two different vertices i and j in V, say $P_1 = \langle i = i_1, i_2, \ldots, i_{n-1}, i_n = j \rangle$ and $P_2 = \langle i = i'_1, i'_2, \ldots, i'_{n-1}, i'_n = j \rangle$, such that $i_2 \neq i'_2$ and $i_{n-1} \neq i'_{n-1}$.*

Proof. We prove this theorem by induction on n. The theorem is true for $n = 4$ because G is the complete graph K_4. Assume that the theorem holds for every integer m with $n > m \geq 4$. Let i and j be any two vertices of G. We want to find two Hamiltonian paths of G joining i and j, say $P_1 = \langle i = i_1, i_2, \ldots, i_{n-1}, i_n = j \rangle$ and $P_2 = \langle i = i'_1, i'_2, \ldots, i'_{n-1}, i'_n = j \rangle$, such that $i_2 \neq i'_2$ and $i_{n-1} \neq i'_{n-1}$. Let X be a subset of $\langle n \rangle$. We use G^X to denote the subgraph of G induced by X and $\bar{E}^X$ to denote the set $\{(i, j) | (i, j) \in \bar{E}, i, j \in X\}$.

Case 1. $|\{x \in \langle n - 1 \rangle | (n, x) \in \bar{E}\}| = 0$. Hence, $|\bar{E}^{\langle n-1 \rangle}| \leq n - 4$ and $(n, l) \in E$ for any $l \in \langle n - 1 \rangle$.

Suppose that $i = n$ or $j = n$. Without loss of generality, we assume that $i = n$. Since $|\bar{E}^{\langle n-1 \rangle}| = n - 4 = (n - 1) - 3$, by Theorem 1, there is a Hamiltonian cycle of $G^{\langle n-1 \rangle}$, say $\langle j = a_1, a_2, a_3, \ldots a_{n-1}, a_1 = j \rangle$. Then, $P_1 = \langle i = n, a_2, a_3, \ldots, a_{n-1}, a_1 = j \rangle$ and $P_2 = \langle i = n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1 = j \rangle$ form two Hamiltonian paths of G, satisfying our requirements. See Figure 2(a) for an example. For illustrative purpose, we draw P_1 and P_2 as internal disjoint paths. Similar situations hold for the remaining figures.

Now, we consider that $i \neq n$ and $j \neq n$. Suppose that $\bar{E}^{\langle n-1 \rangle} = \emptyset$. Then, G is the complete graph K_n and the theorem is obviously true. Suppose that $\bar{E}^{\langle n-1 \rangle} \neq \emptyset$. We can choose any edge e in $\bar{E}^{\langle n-1 \rangle}$. Obviously, $G^{\langle n-1 \rangle} + e$ is a graph with $n - 1$ vertices. Moreover, the complement of $G^{\langle n-1 \rangle} + e$ contains at most $(n - 1) - 4$ edges. By induction, there are two Hamiltonian paths of $G^{\langle n-1 \rangle} + e$ joining i and j, say $P'_1 = \langle i = a^1_1, a^1_2, \ldots, a^1_{n-2}, a^1_{n-1} = j \rangle$ and $P'_2 = \langle i = a^2_1, a^2_2, \ldots, a^2_{n-2}, a^2_{n-1} = j \rangle$, such that $a^1_2 \neq a^2_2$ and $a^1_{n-2} \neq a^2_{n-2}$. For $l \in \{1, 2\}$, we set P_l as $\langle i = a^l_1, a^l_2, \ldots, a^l_t, n, a^l_{t+1}, \ldots, a^l_{n-2}, a^l_{n-1} = j \rangle$, where $t = 2$ if $(a^l_p, a^l_{p+1}) \neq e$ for all $1 \leq p \leq n - 1$, or t is the index p such that $(a^l_p, a^l_{p+1}) = e$. Obviously, P_1 and P_2 form two Hamiltonian paths of G, satisfying our requirements. See Figure 2(b) for an illustration.

Case 2. $|\{x \in \langle n - 1 \rangle | (n, x) \in \bar{E}\}| \geq 1$. Hence, $|\bar{E}^{\langle n-1 \rangle}| \leq n - 5$. Suppose that $i = n$ or $j = n$. Without loss of generality, we assume that $i = n$. Since $|\bar{E}| \leq n - 4$, there are at least three different vertices r, s, and t, such that $\{(r,$

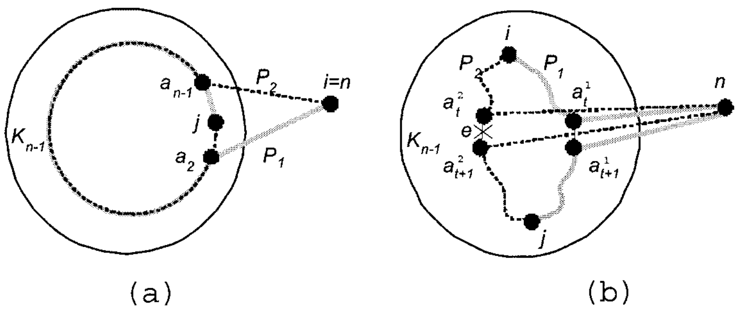

FIG. 2. Illustrations for Case 1.

$n), (s, n), (t, n)\} \subset E$. Without loss of generality, we may assume that $r \neq j$ and $s \neq j$. By induction, there exist two Hamiltonian paths P_1 and P_2 of $G^{\langle n-1\rangle}$ joining r and j, say $P_1 = \langle a_1 = r, a_2, \ldots, a_{n-1} = j\rangle$ and $P_2 = \langle b_1 = r, b_2, \ldots, b_{n-1} = j\rangle$, such that $a_2 \neq b_2$ and $a_{n-2} \neq b_{n-2}$. Similarly, there exist two Hamiltonian paths P_3 and P_4 of $G^{\langle n-1\rangle}$ joining s and j, say $P_3 = \langle c_1 = s, c_2, \ldots, c_{n-1} = j\rangle$ and $P_4 = \langle d_1 = s, d_2, \ldots, d_{n-1} = j\rangle$, such that $c_2 \neq d_2$ and $c_{n-2} \neq d_{n-2}$. Without loss of generality, we may assume that $a_{n-2} \neq c_{n-2}$. Obviously, $\langle i, a_1 = r, a_2, \ldots, a_{n-1} = j\rangle$ and $\langle i, c_1 = s, c_2, \ldots, c_{n-1} = j\rangle$ form the desired Hamiltonian paths of G.

Now, we consider that $i \neq n$ and $j \neq n$. Since $|\bar{E}^{\langle n-1\rangle}| \leq n - 5$, by induction, there are two Hamiltonian paths of $G^{\langle n-1\rangle}$ joining i and j, say $P_1 = \langle a_1 = i, a_2, \ldots, a_{n-1} = j\rangle$ and $P_2 = \langle b_1 = i, b_2, \ldots, b_{n-1} = j\rangle$, such that $a_2 \neq b_2$ and $a_{n-2} \neq b_{n-2}$.

Suppose that $\{(i, n), (j, n)\} \not\subset E$. Without loss of generality, we assume that $(i, n) \in \bar{E}$. Since $|\bar{E}| \leq n - 4$, there exists a smallest index p such that $(n, a_p) \in E$. Obviously, $2 \leq p \leq n - 3$. Set $J = \{a_1, a_2, \ldots, a_{p-1}\}$. Thus, $|\bar{E}^{\langle n\rangle - J}| \leq |\bar{E}| - |J| = n - 4 - (p - 1)$ and $G^{\langle n\rangle - J}$ is a graph with $|\langle n\rangle - J| = n - p + 1 \geq n - (n - 3) + 1 = 4$ vertices. By induction, there are two Hamiltonian paths P_3 and P_4 of $G^{\langle n\rangle - J}$ joining a_p to j, say $P_3 = \langle c_p = a_p, c_{p+1}, \ldots, c_n = j\rangle$ and $P_4 = \langle d_p = a_p, d_{p+1}, \ldots, d_n = j\rangle$, such that $c_{p+1} \neq d_{p+1}$ and $c_{n-1} \neq d_{n-1}$. See Figure 3(a) for an illustration. Similarly, let q be the smallest index such that $(n, b_q) \in E$. Set $J' = \{b_1, b_2, \ldots, b_{q-1}\}$. Again, $2 \leq q \leq n - 3$, $|\bar{E}^{\langle n\rangle - J'}| \leq |\bar{E}| - |J'| = n - 4 - (q - 1)$, and $G^{\langle n\rangle - J'}$ is a graph with

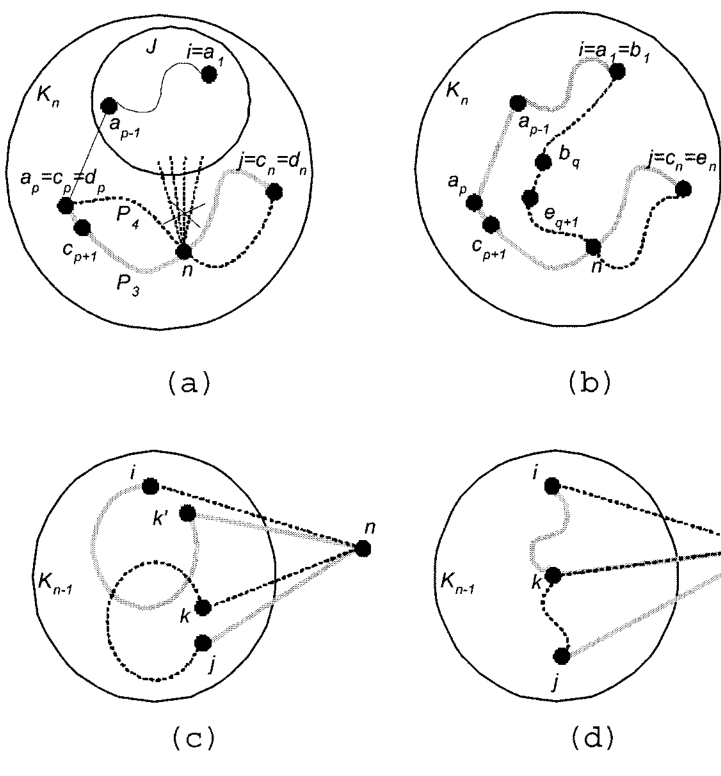

FIG. 3. Illustrations for Case 2.

$n - q + 1 \geq 4$ vertices. By induction, there are two Hamiltonian paths P_5 and P_6 of $G^{\langle n \rangle - J'}$ joining b_q to j, say $P_5 = \langle e_q = b_q, e_{q+1}, \ldots, e_n = j \rangle$ and $P_6 = \langle f_q = b_q, f_{q+1}, \ldots, f_n = j \rangle$, such that $e_{q+1} \neq f_{q+1}$ and $e_{n-1} \neq f_{n-1}$. Without loss of generality, we may assume that $c_{n-1} \neq e_{n-1}$. Then, we set $P_7 = \langle a_1 = i, a_2, \ldots, a_p = c_p, c_{p+1}, \ldots, c_{n-1}, c_n = j \rangle$ and $P_8 = \langle b_1 = i, b_2, \ldots, b_q = e_q, e_{q+1}, \ldots, e_{n-1}, e_n = j \rangle$. Obviously, P_7 and P_8 form the desired Hamiltonian paths of G. See Figure 3(b) for an illustration.

Suppose that $\{(i, n), (j, n)\} \subset E$. We assume that $|\{x \in \langle n \rangle | (n, x) \in \bar{E}\}| \leq 1$. By Theorem 1, there exists a Hamiltonian cycle of $G^{\langle n-1 \rangle - \{i\}}$, say $C = \langle j, k, Q_1, l, j \rangle$. Since the cycle C can be traversed forward and backward, we may assume that $(k, n) \in E$. Then, $P_1 = \langle i, n, k, Q_1, l, j \rangle$ forms a Hamiltonian path of G. Similarly, there also exists a Hamiltonian cycle of $G^{\langle n-1 \rangle - \{j\}}$, say $\langle i, k', Q_2, l', i \rangle$ with $(k', n) \in E$. Then, $P_2 = \langle i, l', Q_2, k', n, j \rangle$ forms another Hamiltonian path of G. Thus, P_1 and P_2 form the desired Hamiltonian paths of G. See Figure 3(c) for an illustration.

Assume that $|\{x \in \langle n \rangle | (n, x) \in \bar{E}\}| \geq 2$. Since $|\bar{E}| \leq n - 4$, $n \geq 6$ and there exists a vertex k in $\langle n \rangle - \{i, j\}$ such that $(k, n) \in E$. Obviously, $|\bar{E}^{\langle n-1 \rangle - \{i\}}| \leq n - 6$ and $G^{\langle n-1 \rangle - \{i\}}$ is a graph with $(n - 2)$ vertices. By Theorem 1, there exists a Hamiltonian path $\langle g_1 = k, g_2, \ldots, g_{n-2} = j \rangle$ of $G^{\langle n-1 \rangle - \{i\}}$ joining k to j. Similarly, there exists a Hamiltonian path $\langle h_1 = i, h_2, \ldots, h_{n-2} = k \rangle$ of $G^{\langle n-1 \rangle - \{j\}}$ joining i to k. Obviously, $\langle i, n, k = g1, \mathrm{g}_2, \ldots, \mathrm{g}_{n-2} = \mathrm{j} \rangle$ and $\langle h_1 = i, h_2, \ldots, h_{n-2} = k, n, j \rangle$ form the desired Hamiltonian paths of G. See Figure 3(d) for an illustration. ∎

Lemma 1. *Assume that $n \geq 4$. Then, K_n is $(n - 3)$-fault Hamiltonian and $(n - 4)$-fault Hamiltonian connected.*

Proof. Let F be any subset of $V(K_n) \cup E(K_n)$. We use F_V to denote $F \cap V(K_n)$. Then, $K_n - F$ is isomorphic to $K_{n-f} - F'$, where $f = |F_V|$ and F' is a subset of edges in the subgraph of K_n induced by $\langle n \rangle - F_V$. Obviously, $|F'| \leq |F| - f$. Thus, if $|F| \leq n - i$, then $\bar{E}(K_{n-f} - F') = |F'| \leq |F| - f \leq (n - f) - i$. Since $n - f$ is the number of vertices of $K_{n-f} - F'$, the lemma follows from Theorem 1. ∎

The following theorem was proved by Hung et al. [6]:

Theorem 3 [6]. *Let $K_n = (V, E)$ be the complete graph with n vertices. Let $F \subset (V \cup E)$ be a faulty set with $|F| \leq n - 2$. Then, there exists a vertex set $V' \subseteq V(K_n) - F$ with $|V'| = n - |F|$ such that there exists a Hamiltonian path of $K_n - F$ joining every pair of vertices in V'.*

3. BASIC PROPERTIES OF THE (*N*, *K*)-STAR GRAPHS

Let $\mathbf{u} = u_1 u_2 \ldots u_k$ be any vertex of the (n, k)-star graph. We say u_i is the ith coordinate of $\mathbf{u}$, denoted as $(\mathbf{u})_i$, for $1 \leq i \leq k$. Let $\mathbf{v}$ be a neighbor of $\mathbf{u}$. We say that $\mathbf{v}$ is an *i-neighbor* of $\mathbf{u}$ if $u_i \neq v_i$. By the definition of $S_{n,k}$, there is exactly one i-neighbor of $\mathbf{u}$ for $2 \leq i \leq k$ and there are $(n - k)$ 1-neighbors of $\mathbf{u}$. We use $i(\mathbf{u})$ to denote the unique i-neighbor of $\mathbf{u}$ if $i \neq 1$. Hence, $(k(\mathbf{u}))_k = (\mathbf{u})_1$. For $1 \leq i \leq n$, let $S^i_{n-1,k-1}$ be the subgraph of $S_{n,k}$ induced by those vertices $\mathbf{u}$ with $(\mathbf{u})_k = i$. In [3], Chiang and Chen proved that $S_{n,k}$ can be decomposed into n subgraphs $S^i_{n-1,k-1}$, $1 \leq i \leq n$, such that each subgraph $S^i_{n-1,k-1}$ is isomorphic to $S_{n-1,k-1}$. Thus, the (n, k)-star graph can be constructed recursively.

Lemma 2. *Let $n > k > 1$ and $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices in $S^l_{n-1,k-1}$ with $d(\mathbf{u}, \mathbf{v}) \leq 2$ for some $1 \leq l \leq n$. Then, $(\mathbf{u})_1 \neq (\mathbf{v})_1$.*

Proof. Let $\mathbf{u} = u_1 u_2 \ldots u_k$. Suppose that $d(\mathbf{u}, \mathbf{v}) = 1$. Since every edge in $S^l_{n-1,k-1}$ is an i-edge with $1 \leq i < k$, $\mathbf{v}$ is either $i(\mathbf{u})$ for some $2 \leq i < k$ or $x u_2 \ldots u_k$ for some $x \in \langle n \rangle - \{u_j | 1 \leq j \leq k\}$. Obviously, $(\mathbf{v})_1$ is either u_i with $2 \leq i < k$ or x. Hence, $(\mathbf{u})_1 \neq (\mathbf{v})_1$.

Suppose that $d(\mathbf{u}, \mathbf{v}) = 2$. Let $\mathbf{w} = w_1 w_2 \ldots w_k$ be the common neighbor of $\mathbf{u}$ and $\mathbf{v}$ in $S^l_{n-1,k-1}$. Then, $\mathbf{u}$ is either $i(\mathbf{w})$ or $x_1 w_2 w_3 \ldots w_k$ for some $x_1 \in \langle n \rangle - \{w_r | 1 \leq r \leq k\}$.

Assume that $\mathbf{u}$ is $i(\mathbf{w})$. Then, $\mathbf{v}$ is either $j(\mathbf{w})$ for some $2 \leq j \neq i < k$ or $x w_2 \ldots w_k$ for some $x \in \langle n \rangle - \{w_r | 1 \leq r \leq k\}$. Thus, $(\mathbf{u})_1 = w_i$. Moreover, $(\mathbf{v})_1 = w_j$ or $(\mathbf{v})_1 = x$ with $2 \leq i \neq j < k$. Hence, $(\mathbf{u})_1 \neq (\mathbf{v})_1$.

Assume that $\mathbf{u}$ is $x_1 w_2 \ldots w_k$ for some $x_1 \in \langle n \rangle - \{w_r | 1 \leq r \leq k\}$. Then, $\mathbf{v}$ is $x_2 w_2 \ldots w_k$ for some $x_2 \in \langle n \rangle - \{w_r | 1 \leq r \leq k\}$, with $x_1 \neq x_2$. Thus, $(\mathbf{u})_1 = x_1$ and $(\mathbf{v})_1 = x_2$. Hence, $(\mathbf{u})_1 \neq (\mathbf{v})_1$.

Thus, the lemma is proved. ∎

For $1 \leq i \neq j \leq n$, we use $E^{i,j}$ to denote the set of edges between $S^i_{n-1,k-1}$ and $S^j_{n-1,k-1}$. Let $(\mathbf{u}, \mathbf{v})$ be any edge in $E^{i,j}$. We assume that $\mathbf{u} \in S^i_{n-1,k-1}$ and $\mathbf{v} \in S^j_{n-1,k-1}$. Thus, $(\mathbf{u}, \mathbf{v}) \in E^{i,j}$ implies that $(\mathbf{v}, \mathbf{u}) \in E^{j,i}$. However, $(\mathbf{u}, \mathbf{v}) \notin E^{j,i}$ if $(\mathbf{u}, \mathbf{v}) \in E^{i,j}$. In [3], it was proved that $|E^{i,j}| = [(n - 2)!]/[(n - k)!]$. Thus, $|E^{i,j}| = 1$ if $k = 2$. The following lemma can be easily obtained from the definition of $S_{n,k}$.

Lemma 3. *Let $(\mathbf{u}, \mathbf{v})$ and $(\mathbf{u}', \mathbf{v}')$ be any two distinct edges in $E^{i,j}$. Then, $\{\mathbf{u}, \mathbf{v}\} \cap \{\mathbf{u}', \mathbf{v}'\} = \emptyset$.*

Let F be a faulty set of $S_{n,k}$. An edge $(\mathbf{u}, \mathbf{v})$ is *F-fault* if $(\mathbf{u}, \mathbf{v}) \in F$, $\mathbf{u} \in F$, or $\mathbf{v} \in F$, and $(\mathbf{u}, \mathbf{v})$ is *F-fault free* if $(\mathbf{u}, \mathbf{v})$ is not F-fault. Let $H = (V', E')$ be a subgraph of $S_{n,k}$. We use $F(H)$ to denote the set $(V' \cup E') \cap F$. We associate $S_{n,k}$ with the complete graph K_n with vertex set $\langle n \rangle$ such that vertex l of K_n is associated with $S^l_{n-1,k-1}$ for every $1 \leq l \leq n$. We define a faulty edge set $R(F)$ of K_n as $(i, j) \in R(F)$ if some edge of $E^{i,j}$ is F-fault. Obviously, $|R(F)| \leq |F|$. Assume that I is any subset of $\langle n \rangle$. We use $S^I_{n-1,k-1}$ to denote the subgraph of $S_{n,k}$ induced by $\cup_{i \in I} V(S^i_{n-1,k-1})$. Similarly, we use K^I_n to denote the subgraph of K_n induced by I.

Lemma 4. *Suppose that $k \geq 2$ and $(n - k) \geq 2$. Let $I \subseteq \langle n \rangle$ with $|I| = m \geq 2$ and let $F \subset V(S_{n,k}) \cup E(S_{n,k})$ with $S^i_{n-1,k-1} - F$ being Hamiltonian connected for all $i \in I$. Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be any two vertices of $S^I_{n-1,k-1}$ such that (1) $(\mathbf{u})_k \neq (\mathbf{v})_k$, (2) there exists a Hamiltonian path $P = \langle(\mathbf{u})_k = i_1, i_2, \ldots, i_m = (\mathbf{v})_k\rangle$ of $K^I_n - R(F)$, and (3) $(\mathbf{u})_1 \neq i_2$ and $(\mathbf{v})_1 \neq i_{m-1}$ if $k = 2$. Then, there exists a Hamiltonian path of $S^I_{n-1,k-1} - F$ joining* $\mathbf{u}$ *to* $\mathbf{v}$.

Proof. Let $\mathbf{u}^1 = \mathbf{u}$ and $\mathbf{v}^m = \mathbf{v}$. Suppose that we can choose two different vertices $\mathbf{u}^l$ and $\mathbf{v}^l$ in $S^{i_l}_{n-1,k-1}$ for every $i_l \in I$ such that $(\mathbf{v}^l, \mathbf{u}^{l+1}) \in E_{i_l}^{,i_{l+1}}$. Since $(i_l, i_{l+1}) \notin R(F)$, $(\mathbf{v}^l, \mathbf{u}^{l+1})$ is F-fault free. Since $S^{i_l}_{n-1,k-1} - F$ is Hamiltonian connected, there exists a Hamiltonian path P_l of $S^{i_l}_{n-1,k-1} - F$ joining $\mathbf{u}^l$ and $\mathbf{v}^l$ for all $1 \leq l \leq m$. Using the second condition, then $\langle \mathbf{u} = \mathbf{u}^1, P_1, \mathbf{v}^1, \mathbf{u}^2, P_2, \mathbf{v}^2, \ldots, \mathbf{u}^m, P_m, \mathbf{v}^m = \mathbf{v}\rangle$ forms a Hamiltonian path of $S^I_{n-1,k-1} - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Thus, the lemma is proved as long as such a choice is achievable.

Suppose that $k \geq 3$. Since $|E^{i,j}| \geq (n-2) \geq 3$ for any i and j in I, such a choice is easily achievable. Suppose that $k = 2$. We can choose $\mathbf{u}^l$ and $\mathbf{v}^l$ in $S^{i_l}_{n-1,k-1}$ for every $i_l \in I$ as the only edge $(\mathbf{v}^l, \mathbf{u}^{l+1}) \in E^{i_l,i_{l+1}}$. Since $(\mathbf{u}^l)_1 = (\mathbf{v}^{l-1})_k \neq (\mathbf{v}^l)_1 = (\mathbf{u}^{l+1})_k$, $\mathbf{u}^l \neq \mathbf{v}^l$ for every $1 < l < m$. The conditions $(\mathbf{u})_1 \neq i_2$ and $(\mathbf{v})_1 \neq i_{m-1}$ imply that $\mathbf{u} = \mathbf{u}^1 \neq \mathbf{v}^1$ and $\mathbf{v} = \mathbf{v}^m \neq \mathbf{u}^m$. Hence, the lemma is proved. ∎

4. HAMILTONIAN PROPERTIES OF THE (*N*, *K*)-STAR GRAPHS

Lemma 5. *$S_{4,2}$ is 1-fault Hamiltonian and Hamiltonian connected.*

Proof. Let $F = \{f\} \subset V(S_{4,2}) \cup E(S_{4,2})$. Assume that $f \in V(S_{4,2})$. Since $S_{4,2}$ is vertex-transitive, we may assume that f is the vertex 12. Obviously,

$$\langle 32, 42, 24, 34, 14, 41, 21, 31, 13, 43, 23, 32\rangle$$

forms a Hamiltonian cycle of $S_{4,2} - F$. Suppose that f is an edge. By the symmetric property of $S_{4,2}$, we may assume that f is either the edge (42, 32) or the edge (12, 21). Obviously,

$$\langle 32, 12, 42, 24, 34, 14, 41, 21, 31, 13, 43, 23, 32\rangle$$

forms a Hamiltonian cycle of $S_{4,2} - F$. Hence, $S_{4,2}$ is 1-fault Hamiltonian.

Let $\mathbf{x}$ and $\mathbf{y}$ be any two vertices of $S_{4,2}$. By the symmetric property of $S_{4,2}$, we may assume that $\mathbf{x}$ is the vertex 12 and $\mathbf{y} \in \{32, 21, 41, 14, 34\}$. Thus,

$$\langle 12, 42, 24, 34, 14, 41, 21, 31, 13, 43, 23, 32\rangle,$$

$$\langle 12, 42, 32, 23, 13, 43, 34, 24, 14, 41, 31, 21\rangle,$$

$$\langle 12, 32, 42, 24, 14, 34, 43, 23, 13, 31, 21, 41\rangle,$$

$$\langle 12, 21, 41, 31, 13, 43, 23, 32, 42, 24, 34, 14\rangle, \text{ and}$$

$$\langle 12, 21, 41, 31, 13, 43, 23, 32, 42, 24, 14, 34\rangle$$

are the corresponding Hamiltonian paths of $S_{4,2}$. Hence, $S_{4,2}$ is Hamiltonian connected. ∎

Lemma 6. *Suppose that $S_{n-1,k-1}$ is $(n-4)$-fault Hamiltonian and $(n-5)$-fault Hamiltonian connected, for some $k \geq 2$, $n \geq 5$, and $n - k \geq 2$. Then, $S_{n,k}$ is $(n-3)$-fault Hamiltonian.*

Proof. Assume that F is any faulty set of $S_{n,k}$ with $|F| \leq n - 3$. Without loss of generality, we assume that $|F(S^1_{n-1,k-1})| \geq |F(S^2_{n-1,k-1})| \geq \ldots \geq |F(S^n_{n-1,k-1})|$. Let $F' = F - F(S^1_{n-1,k-1})$.

Case 1. $|F(S^1_{n-1,k-1})| \leq n - 5$. By the assumption of this lemma, $S^i_{n-1,k-1} - F$ is Hamiltonian connected for every $i \in \langle n \rangle$. Since $|R(F)| \leq |F| \leq n - 3$, by Lemma 1, $K_n - R(F)$ is Hamiltonian. Let $C = \langle t_1, t_2, \ldots, t_n, t_1\rangle$ be a Hamiltonian cycle of $K_n - R(F)$. Thus, all edges in E^{t_1,t_n} are F-fault free. We choose any edge $(\mathbf{u}, \mathbf{v})$ in E^{t_1,t_n}. Obviously, $\langle t_1, t_2, \ldots, t_n\rangle$ is a Hamiltonian path of $K_n - R(F)$, $(\mathbf{u})_1 = (\mathbf{v})_k = t_n$, and $(\mathbf{v})_1 = (\mathbf{u})_k = t_1$. By Lemma 4, there exists a Hamiltonian path P_1 of $S_{n,k} - F$ joining $\mathbf{u}$ to $\mathbf{v}$. Thus, $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{u}\rangle$ forms a Hamiltonian cycle of $S_{n,k} - F$.

Case 2. $|F(S^1_{n-1,k-1})| = n - 4$. Thus, $|F'| \leq 1$ and $|R(F')| \leq 1$. By the assumption of this lemma, $S^1_{n-1,k-1} - F$ is Hamiltonian.

Suppose that $S^i_{n-1,k-1} - F$ is Hamiltonian connected for every $i \neq 1$. Let C be a Hamiltonian cycle of $S^1_{n-1,k-1} - F$. Since the length of C is at least 3, there exists an edge $(\mathbf{u}, \mathbf{v})$ in C such that both $(\mathbf{u}, k(\mathbf{u}))$ and $(\mathbf{v}, k(\mathbf{v}))$ are F-fault free. We can write C as $\langle \mathbf{u}, P_1, \mathbf{v}, \mathbf{u}\rangle$. Since $|R(F')| \leq 1$, $K_n^{\langle n \rangle - \{1\}} - R(F')$ is Hamiltonian connected. Obviously, $(k(k(\mathbf{u})))_k = (\mathbf{u})_k = (\mathbf{v})_k = 1$. By Lemma 4, there is a Hamiltonian path P_2 of $S^{\langle n \rangle - \{1\}}_{n-1,k-1} - F'$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Hence, $\langle \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{v}), \mathbf{v}, P_1, \mathbf{u}\rangle$ forms a Hamiltonian cycle of $S_{n,k} - F$.

Suppose that $S^i_{n-1,k-1} - F$ is not Hamiltonian connected for every $i \neq 1$. We claim that $n = 5$ and $|F(S^2_{n-1,k-1})| = 1$. Suppose that $n \geq 6$ or ($n = 5$ and $|F(S^2_{n-1,k-1})| = 0$). Since $|F'| \leq 1$, $S^i_{n-1,k-1} - F$ is Hamiltonian connected by the assumption of this lemma. We get a contradiction.

Hence, we consider $n = 5$. Obviously, $k \in \{2, 3\}$. Moreover, $|F| = 2$, $|F(S^1_{4,k-1})| = |F(S^2_{4,k-1})| = 1$, and $|F(S^{\{3,4,5\}}_{4,k-1})| = 0$. Suppose that $k = 2$. We use brute force to construct such Hamiltonian cycles for $S_{5,2} - F$. (See Appendix)

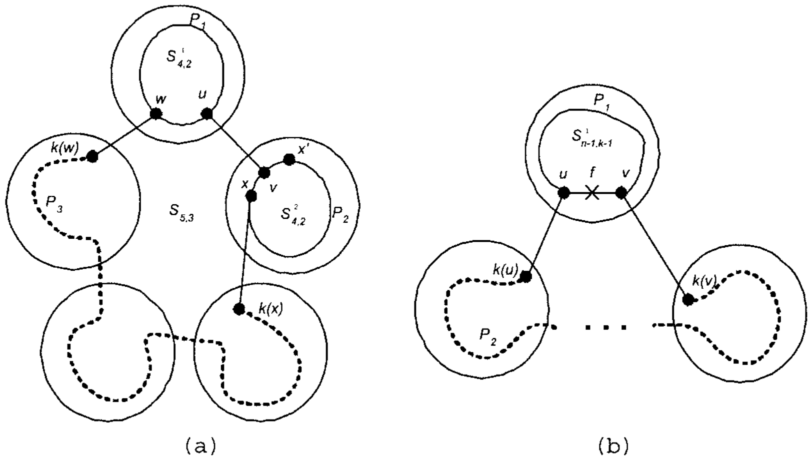

FIG. 4. Illustrations for Lemma 6.

Suppose that $k = 3$. Since $|E^{i,j}| = 3$ for any $1 \le i, j \le 5$ in $S_{5,3}$, there is an F-fault-free edge $(\mathbf{u}, \mathbf{v})$ in $E^{1,2}$. Since $S_{4,2}$ is 1-fault Hamiltonian, there is a Hamiltonian cycle $C_1 = \langle \mathbf{u}, P_1, \mathbf{w}, \mathbf{u}\rangle$ in $S^1_{4,2} - F$ and there is a Hamiltonian cycle $C_2 = \langle \mathbf{v}, \mathbf{x}, P_2, \mathbf{x}', \mathbf{v}\rangle$ in $S^2_{4,2} - F$. By Lemma 2, $(\mathbf{x})_1 \neq (\mathbf{x}')_1$. Since cycle C_2 can be traversed backward and forward, we may assume that $(\mathbf{w})_1 \neq (\mathbf{x})_1$. Since $K_n^{\{3,4,5\}}$ is Hamiltonian connected, by Lemma 4, there exists a Hamiltonian path P_3 of $S^{\{3,4,5\}}_{4,2}$ joining $k(\mathbf{w})$ and $k(\mathbf{x})$. Then, $\langle \mathbf{u}, P_1, \mathbf{w}, k(\mathbf{w}), P_3, k(\mathbf{x}), \mathbf{x}, P_2, \mathbf{x}', \mathbf{v}, \mathbf{u}\rangle$ forms a Hamiltonian cycle of $S_{5,3} - F$. See Figure 4(a) for an illustration.

Case 3. $|F(S^1_{n-1,k-1})| = n - 3$. Thus, $|F - F(S^1_{n-1,k-1})| = 0$. Choose any element f in $F(S^1_{n-1,k-1})$. By the assumption of this lemma, there exists a Hamiltonian cycle of $S^1_{n-1,k-1} - F + \{f\}$. By deleting f from $S^1_{n-1,k-1} - F$, we can find a Hamiltonian path of $S^1_{n-1,k-1} - F$ joining $\mathbf{u}$ and $\mathbf{v}$ such that $d(\mathbf{u}, \mathbf{v}) \le 2$, no matter whether f is a vertex or an edge. By Lemma 2, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Since $K_n^{\langle n\rangle - \{1\}}$ is Hamiltonian connected and $(\mathbf{u})_k = (\mathbf{v})_k = 1$, by Lemma 4, there exists a Hamiltonian path P_2 of $S^{\langle n\rangle - \{1\}}_{n-1,k-1}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{v}), \mathbf{v}, P_1, \mathbf{u}\rangle$ forms a Hamiltonian cycle of $S_{n,k} - F$. See Figure 4(b) for an illustration.

Hence, the lemma follows: ∎

Lemma 7. *$S_{n,2}$ is $(n - 4)$-fault Hamiltonian connected for $n \ge 5$.*

Proof. By definition, $S^i_{n-1,1}$ is isomorphic to K_{n-1} for every $i \in \langle n\rangle$. Moreover, $|E^{i,j}| = 1$ for any two $i, j \in \langle n\rangle$. Assume that F is any faulty set of $S_{n,2}$ with $|F| \le n - 4$. Without loss of generality, we assume that $|F(S^1_{n-1,1})| \ge |F(S^2_{n-1,1})| \ge \ldots \ge |F(S^n_{n-1,1})|$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two arbitrary vertices of $S_{n,2} - F$. We need to construct a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Case 1. $|F(S^1_{n-1,1})| \le n - 5$. By Lemma 1, $S^t_{n-1,1} - F(S^t_{n-1,1})$ is Hamiltonian connected for any $t \in \langle n\rangle$.

Subcase 1.1. $(\mathbf{x})_k = (\mathbf{y})_k$. Let $F' = F \cup \{(\mathbf{y}, k(\mathbf{y}))\}$. Then, $|R(F')| \le n - 3$. By Lemma 1, there exists a Hamiltonian cycle C in $K_n - R(F')$, say $C = \langle (\mathbf{x})_k = a_1, a_2, \ldots, a_n, a_1\rangle$. Thus, the only edge $(\mathbf{u}, k(\mathbf{u}))$ in E^{a_1,a_2} and the only edge $(\mathbf{v}, k(\mathbf{v}))$ in E^{a_1,a_n} are F-fault free.

Suppose that $|F(S^{a_1}_{n-1,1})| = 0$. Since $(\mathbf{y}, k(\mathbf{y})) \in F'$, $\mathbf{v} \neq \mathbf{y}$. Obviously, $\langle a_1, a_3, \ldots, a_n\rangle$ is a Hamiltonian path of $K_n^{\langle n\rangle - \{a_1\}} - R(F')$ and $(\mathbf{u})_k = (\mathbf{v})_k = a_1$. By Lemma 4, there exists a Hamiltonian path P_1 of $S^{\langle n\rangle - \{a_1\}}_{n-1,1}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Since $S^{a_1}_{n-1,1}$ is k_{n-1}, there exist two paths P_2 and P_3 covering all vertices in $S^{a_1}_{n-1,1}$ such that P_2 joins $\mathbf{x}$ to $\mathbf{u}$ and P_3 joins $\mathbf{v}$ to $\mathbf{y}$. Then, $\langle \mathbf{x}, P_2, \mathbf{u}, k(\mathbf{u}), P_1, k(\mathbf{v}), \mathbf{v}, P_3, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 5(a) for an illustration.

Suppose that $|F(S^{a_1}_{n-1,1})| \ge 1$. We create a new graph H by setting $V(H) = V(S^{a_1}_{n-1,1}) \cup \{n\}$ and $E(H) = E(S^{a_1}_{n-1,1}) \cup \{(\mathbf{w}, n) | \mathbf{w} \in V(S^{a_1}_{n-1,1})\}$. Hence, H is K_n. Then, we set $F'' = F(S^{a_1}_{n-1,1}) \cup \{(\mathbf{w}, n) |$ the only edge in $E^{a_1,(\mathbf{w})_1}$ is F-fault$\}$. Hence, $|F''| \le n - 4$. By Lemma 1, $H - F''$ is Hamiltonian connected. Thus, there exists a Hamiltonian path P_1 of $H - F''$ joining $\mathbf{x}$ to $\mathbf{y}$. Since n is an internal vertex of P_1, we can write P_1 as $\langle \mathbf{x} = \mathbf{u}^1, Q_1, \mathbf{u}^s, n = \mathbf{u}^{s+1}, \mathbf{u}^{s+2}, Q_2, \mathbf{u}^n = \mathbf{y}\rangle$. Since $|F(S^{a_1}_{n-1,1})| \ge 1$, $|R(F(S^{\langle n\rangle - \{a_1\}}_{n-1,1}))| \le |F - F(S^{a_1}_{n-1,1})| \le n - 5$. By Lemma 1, $K_n^{\langle n\rangle - \{a_1\}} - R(F(S^{\langle n\rangle - \{a_1\}}_{n-1,1}))$ is Hamiltonian connected. Obviously, $(\mathbf{u}^s)_k = (\mathbf{u}^{s+2})_k = a_1$. By Lemma 4, there exists a Hamiltonian path P_2 of $S^{\langle n\rangle - \{a_1\}}_{n-1,1} - F$ joining $k(\mathbf{u}^s)$ to $k(\mathbf{u}^{s+2})$. Then, $\langle \mathbf{x} = \mathbf{u}^1, Q_1, \mathbf{u}^s, k(\mathbf{u}^s), P_2, k(\mathbf{u}^{s+2}), \mathbf{u}^{s+2}, Q_2, \mathbf{u}^n = \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 5(b) for an illustration.

Subcase 1.2. $(\mathbf{x})_k \neq (\mathbf{y})_k$. Since $|F| \le n - 4$, $|R(F)| \le n - 4$. By Theorem 2, there are two Hamiltonian paths of $K_n - R(F)$ joining $(\mathbf{x})_k$ to $(\mathbf{y})_k$, say $P_1 = \langle (\mathbf{x})_k = l_1, l_2, \ldots,$

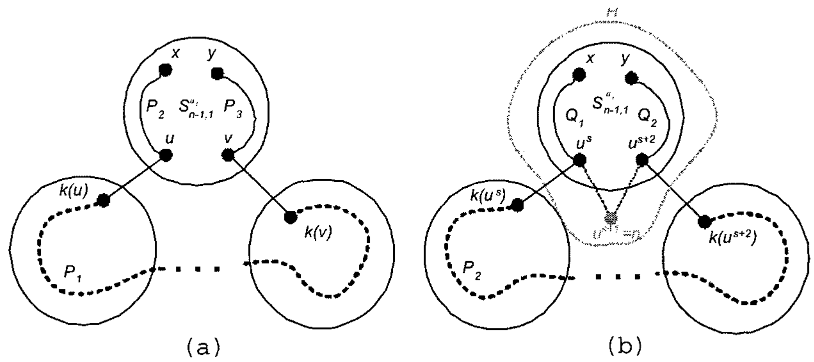

FIG. 5. Illustrations for Subcase 1.1.

$l_n = (\mathbf{y})_k\rangle$ and $P_2 = \langle(\mathbf{x})_k = l'_1, l'_2, \ldots, l'_n = (\mathbf{y})_k\rangle$, such that $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$. Suppose that $((\mathbf{x})_1 \neq l_2$ and $(\mathbf{y})_1 \neq l_{n-1})$ or $((\mathbf{x})_1 \neq l'_2$ and $(\mathbf{y})_1 \neq l'_{n-1})$. By Lemma 4, there is a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. Thus, we consider that $((\mathbf{x})_1 = l_2$ or $(\mathbf{y})_1 = l_{n-1})$ and $((\mathbf{x})_1 = l'_2$ or $(\mathbf{y})_1 = l'_{n-1})$. Since $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$, without loss of generality, we assume that $(\mathbf{x})_1 = l_2$ and $(\mathbf{y})_1 = l'_{n-1}$.

Suppose that $|F(S^{l_1}_{n-1,1})| \geq 1$ or some edges in $\cup_{j\in\langle n\rangle-\{l_1\}}E^{l_1,j}$ are F-fault. Since $|F| \leq n-4$ and $\cup_{j\in\langle n\rangle-\{l_1\}}E^{l_1,j} = n-1$, there exists an index $i \subset \langle n\rangle - \{l_1, l_2, l_n\}$ such that the only edge $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1,i}$ is F-fault free. Since $(\mathbf{x})_1 = l_2 \neq i$, $\mathbf{u} \neq \mathbf{x}$. By Lemma 1, there exists a Hamiltonian path P_6 of $S^{l_1}_{n-1,1} - F$ joining $\mathbf{x}$ to $\mathbf{u}$. Let $F' = F(S^{\langle n\rangle-\{l_1\}}_{n-1,1})$. Then, $|R(F')| \leq n-5$. By Theorem 2, there exist two Hamiltonian paths of $K^{\langle n\rangle-\{l_1\}}_n - R(F')$ joining i to l_n, say $P_3 = \langle i = a_1, a_2, \ldots, a_{n-1} = l_n\rangle$ and $P_4 = \langle i = b_1, b_2, \ldots, b_{n-1} = l_n\rangle$, such that $a_2 \neq b_2$ and $a_{n-2} \neq b_{n-2}$. Without loss of generality, we may assume that $(\mathbf{y})_1 \neq a_{n-2}$. Obviously, $(\mathbf{u})_k = l_1$. By Lemma 4, there exists a Hamiltonian path P_5 of $S^{\langle n\rangle-\{l_1\}}_{n-1,1} - F'$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Then, $\langle\mathbf{x}, P_6, \mathbf{u}, k(\mathbf{u}), P_5, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 6(a) for an illustration.

Suppose that $|F(S^{l_1}_{n-1,1})| = 0$ and all edges in $\cup_{j\in\langle n\rangle-\{l_1\}}E^{l_1,j}$ are F-fault free. Let $\mathbf{u}$ be the only vertex of $S^{l_1}_{n-1,1}$ such that $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1,l_n}$. Since $(\mathbf{y})_1 = l_{n-1} \neq (\mathbf{u})_k = l_1$ and $(\mathbf{x})_1 = l_2 \neq (\mathbf{u})_1 = l_n$, $k(\mathbf{u}) \neq \mathbf{y}$ and $\mathbf{u} \neq \mathbf{x}$. By Lemma 1, there exists a Hamiltonian path P_7 of $S^{l_n}_{n-1,1} - F$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Let $F' = F(S^{\langle n\rangle-\{l_1,l_n\}}_{n-1,1})$. Since $|F| \leq n-4$, $|R(F')| \leq n-4$. Thus, $K^{\langle n\rangle-\{l_1,l_n\}}_n - R(F')$ has a Hamiltonian path, say $\langle c_1, c_2, \ldots, c_{n-2}\rangle$. Since all edges in $\cup_{j\in\langle n\rangle-\{l_1\}}E^{l_1,j}$ are F-fault free, the only edge $(\mathbf{v}, k(\mathbf{v}))$ in E^{l_1,c_1} and the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{l_1,c_{n-2}}$ are F-fault free. Obviously, $(k(k(\mathbf{v})))_k = (\mathbf{v})_k = l_1$ and $(k(k(\mathbf{w})))_k = (\mathbf{w})_k = l_1$. By Lemma 4, there exists a Hamiltonian path P_8 of $S^{\langle n\rangle-\{l_1,l_n\}}_{n-1,1} - F'$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Since $k(\mathbf{v})$ and $k(\mathbf{w})$ are the endpoints of the path P_8, at least one of $\mathbf{v}$ and $\mathbf{w}$ is not $\mathbf{x}$. Without loss of generality, we assume that $\mathbf{v} \neq \mathbf{x}$. Since $S^{l_1}_{n-1,1}$ is K_{n-1}, there exist two disjoint paths P_9 and P_{10} covering all vertices of $S^{l_1}_{n-1,1}$ such that P_9 joins $\mathbf{x}$ to $\mathbf{w}$ and P_{10} joins $\mathbf{u}$ to $\mathbf{v}$. Note that it is possible that $\mathbf{x}$ is $\mathbf{w}$. Hence, $\langle\mathbf{x}, P_9, \mathbf{w}, k(\mathbf{w}), P_8, k(\mathbf{v}), \mathbf{v}, P_{10}, \mathbf{u}, k(\mathbf{u}), P_7, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 6(b) for an illustration.

CASE 2. $|F(S^1_{n-1,1})| = n-4$. Thus, $S^1_{n-1,1} - F$ is Hamiltonian and $|F - F(S^1_{n-1,1})| = 0$.

SUBCASE 2.1. $(\mathbf{x})_k = (\mathbf{y})_k = 1$. Choose any element f in $F(S^1_{n-1,1})$. By Lemma 1, there exists a Hamiltonian path P of $S^1_{n-1,1} - F(S^1_{n-1,1}) + f$ joining $\mathbf{x}$ and $\mathbf{y}$. By deleting f, we can find two paths P_1 and P_2 covering all vertices of

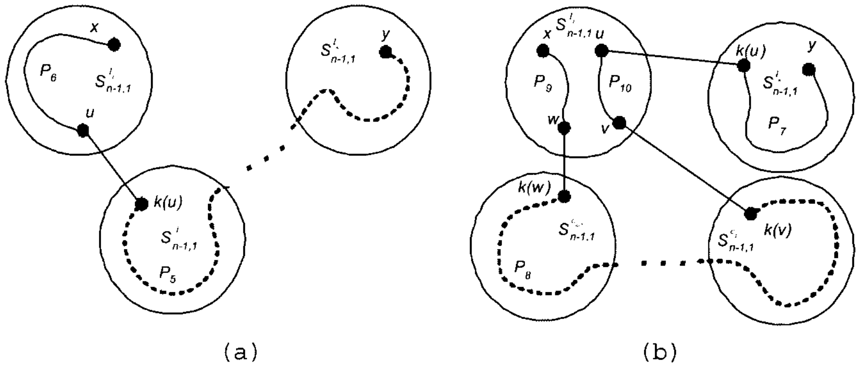

FIG. 6. Illustrations for Subcase 1.2.

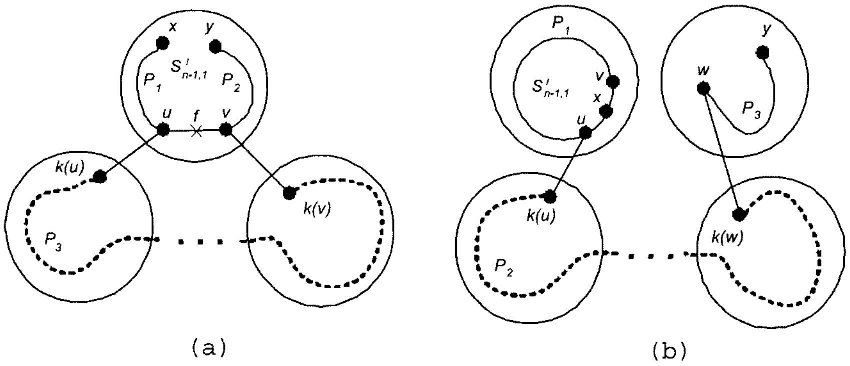

FIG. 7. Illustrations for Subcases 2.1 and 2.2.

$S^1_{n-1,1} - F$ such that P_1 joins $\mathbf{x}$ to $\mathbf{u}$, and P_2 joins $\mathbf{v}$ to $\mathbf{y}$. Since $S^1_{n-1,1}$ is K_{n-1}, $d(\mathbf{u}, \mathbf{v}) = 1$. By Lemma 2, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Obviously, $K_n^{\langle n\rangle - \{1\}}$ is Hamiltonian connected and $(\mathbf{u})_k = (\mathbf{v})_k = 1$. From Lemma 4, there is a Hamiltonian path P_3 of $S_{n-1,1}^{\langle n\rangle - \{1\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{x}, P_1, \mathbf{u}, k(\mathbf{u}), P_3, k(\mathbf{v}), \mathbf{v}, P_2, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 7(a) for an illustration.

SUBCASE 2.2. $(\mathbf{x})_k = 1$ and $(\mathbf{y})_k \neq 1$. Let C be a Hamiltonian cycle of $S^1_{n-1,1} - F$. Write C as $\langle \mathbf{x}, \mathbf{u}, P_1, \mathbf{v}, \mathbf{x}\rangle$. By Lemma 2, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Since the cycle C can be traversed forward and backward, we may assume that $(\mathbf{u})_1 = i \neq (\mathbf{y})_k$. Since $|F - F(S^1_{n-1,1})| = 0$ and $n \geq 5$, there exists an $l \in \langle n\rangle - \{(\mathbf{x})_k, i, (\mathbf{y})_k\}$ such that the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{(y)_k, l}$ satisfies $\mathbf{w} \neq \mathbf{y}$. Obviously, $K_n^{\langle n\rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}$ is Hamiltonian connected, $(\mathbf{u})_k = (\mathbf{x})_k$, and $(\mathbf{w})_k = (\mathbf{y})_k$. From Lemma 4, there exists a Hamiltonian path P_2 of $S_{n-1,1}^{\langle n\rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}$ joining $k(\mathbf{u})$ to $k(\mathbf{w})$. By Lemma 1, there exists a Hamiltonian path P_3 *of* $S^{(\mathbf{y})_k}_{n-1,1}$ joining w to $\mathbf{y}$. Thus, $\langle \mathbf{x}, \mathbf{v}, P_1, \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{w}), \mathbf{w}, P_3, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 7(b) for an illustration.

SUBCASE 2.3. $(\mathbf{x})_k = (\mathbf{y})_k \neq 1$. By Lemma 1, there exists a Hamiltonian cycle C of $S^1_{n-1,1} - F$.

Assume that the only edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k, 1}$ is F-fault free. Write C as $\langle k(\mathbf{u}), \mathbf{v}, P_2, \mathbf{v}', k(\mathbf{u})\rangle$. By Lemma 2, $(\mathbf{v})_1 \neq (\mathbf{v}')_1$. Since $n \geq 5$, we can choose a vertex $\mathbf{w}$ in $S^{(\mathbf{x})_k}_{n-1,1}$ such that $(\mathbf{v})_1 \neq (\mathbf{w})_1$, $\mathbf{w} \neq \mathbf{u}$, and $\mathbf{w} \neq \mathbf{y}$. Since $S^{(\mathbf{x})_k}_{n-1,1}$ is K_{n-1}, a Hamiltonian path of $S^{(\mathbf{x})_k}_{n-1,1}$ can be written as $\langle \mathbf{x}, \mathbf{w}, P_4, \mathbf{y}\rangle$ if $\mathbf{x} = \mathbf{u}$ and $\langle \mathbf{x}, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$ if $\mathbf{x} \neq \mathbf{u}$. Thus, such a Hamiltonian path can be expressed as $\langle \mathbf{x}, P_3, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$. Obviously, $K_n^{\langle n\rangle - \{1, (\mathbf{x})_k\}}$ is Hamiltonian connected, $(\mathbf{v})_k = 1$, and $(\mathbf{w})_k = (\mathbf{x})_k$. By Lemma 4, there exists a Hamiltonian path P_5 of $S_{n-1,1}^{\langle n\rangle - \{1, (\mathbf{x})_k\}}$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Thus, $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), \mathbf{v}', P_2, \mathbf{v}, k(\mathbf{v}), P_5, k(\mathbf{w}), \mathbf{w}, P_4, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 8(a) for an illustration.

Assume that the only edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k, 1}$ is F-fault. Since $|\cup_{j \in \langle n\rangle - \{1\}} E^{1,j}| = n - 1$ and $|F| \leq n - 4$, there are at least three F-fault free edges in $\cup_{j \in \langle n\rangle - \{1\}} E^{1,j}$. Thus, there exists an index $r \in \langle n\rangle - \{(\mathbf{x})_k, 1\}$ such that the only edge $(\mathbf{v}, k(\mathbf{v}))$ in $E^{(\mathbf{x})_k, r}$ and the only edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{1,r}$ are F-fault free. Thus, $k(\mathbf{w}) \neq k(\mathbf{v})$. By Lemma 1, there exists a Hamiltonian path P_1 of $S^r_{n-1,1}$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$ and there exists a Hamiltonian path P_2 of $S^{(\mathbf{x})_k}_{n-1,1}$ joining $\mathbf{x}$ to $\mathbf{y}$. We write P_2 as $\langle \mathbf{x}, \mathbf{v}, \mathbf{z}, P_4, \mathbf{y}\rangle$ and write C as $\langle \mathbf{w}, \mathbf{t}, P_2, \mathbf{t}', \mathbf{w}\rangle$. Since $S^1_{n-1,1}$ is K_{n-1}, $d(\mathbf{t}, \mathbf{t}') = 1$. By Lemma 2, $(\mathbf{t})_1 \neq (\mathbf{t}')_1$. Since the cycle C can be traversed forward and backward, we may assume that $(\mathbf{t})_1 \neq (\mathbf{z})_1$.

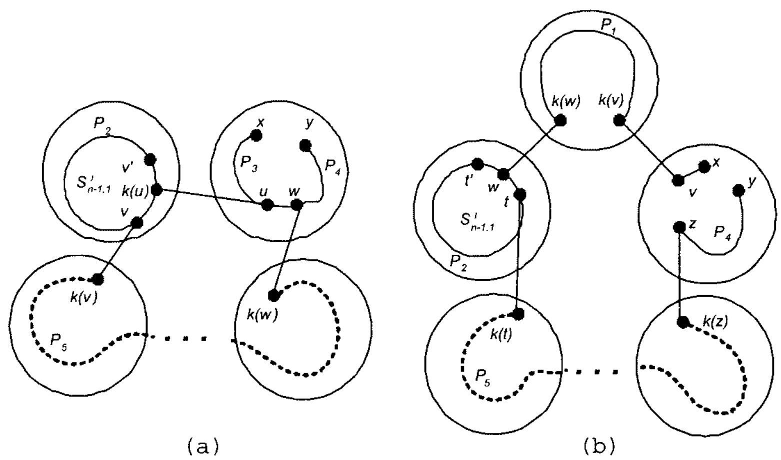

FIG. 8. Illustrations for Subcase 2.3.

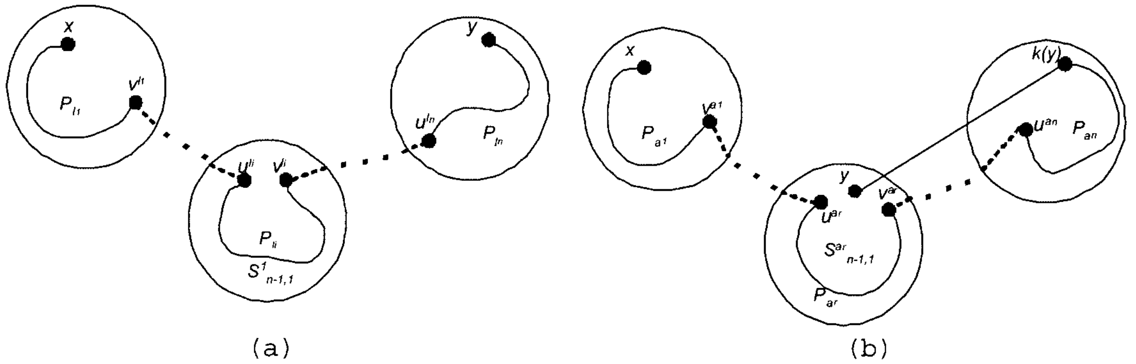

FIG. 9. Illustrations for Subcase 2.4.

Obviously, $K_n^{\langle n\rangle - \{(\mathbf{x})_k, 1, r\}}$ is Hamiltonian connected, $(\mathbf{t})_k = 1$, and $(\mathbf{z})_k = (\mathbf{x})_k$. By Lemma 4, there exists a Hamiltonian path P_5 of $S^{\langle n\rangle - \{(\mathbf{x})_k, 1, r\}}$ joining $k(\mathbf{t})$ to $k(\mathbf{z})$. Then, $\langle \mathbf{x}, \mathbf{v}, k(\mathbf{v}), P_1, k(\mathbf{w}), \mathbf{w}, \mathbf{t}', P_2, \mathbf{t}, k(\mathbf{t}), P_5, k(\mathbf{z}), \mathbf{z}, P_4, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 8(b) for an illustration.

SUBCASE 2.4. $(\mathbf{x})_k$, $(\mathbf{y})_k$, and 1 are distinct. By Theorem 3, there exists a vertex set V' of $S^1_{n-1,1}$ with $|V'| \geq 3$ such that there exists a Hamiltonian path of $S^1_{n-1,1} - F$ joining every pair of vertices in V'. We define $F^* = \{(1, l) | (\mathbf{u}, \mathbf{v}) \in E^{1,l}$ and $\mathbf{u} \notin V'\}$. Since $|V'| \geq 3$, $|F^*| \leq n - 4$. By Theorem 2, there are two Hamiltonian paths of $K_n - F^*$ joining $(\mathbf{x})_k$ to $(\mathbf{y})_k$, say $P_1 = \langle (\mathbf{x})_k = l_1, l_2, \ldots, l_n = (\mathbf{y})_k\rangle$ and $P_2 = \langle (\mathbf{x})_k = l'_1, l'_2, \ldots, l'_n = (\mathbf{y})_k\rangle$, such that $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$.

Suppose that $((\mathbf{x})_1 \neq l_2$ and $(\mathbf{y})_1 \neq l_{n-1})$ or $((\mathbf{x})_1 \neq l'_2$ and $(\mathbf{y})_1 \neq l'_{n-1})$. Without loss of generality, we assume that $(\mathbf{x})_1 \neq l_2$ and $(\mathbf{y})_1 \neq l_{n-1}$. Obviously, we can choose the only F-fault-free edge $(\mathbf{v}^{l_t}, \mathbf{u}^{l_{t+1}})$ in $E^{l_t, l_{t+1}}$ for any $1 \leq t \leq n - 1$. Since 1 is an internal vertex of P_1, we assume that $1 = l_i$. Since $k(\mathbf{u}^{l_i}) = \mathbf{v}^{l_{i-1}}$ and $k(\mathbf{v}^{l_i}) = \mathbf{u}^{l_{i+1}}$, $\mathbf{u}^{l_i} \neq \mathbf{v}^{l_i}$. Since $\mathbf{u}^{l_i}$ and $\mathbf{v}^{l_i}$ are in V' and $\mathbf{u}^{l_i} \neq \mathbf{v}^{l_i}$, there exists a Hamiltonian path P_{l_i} of $S^{l_i}_{n-1,1} - F$ joining $\mathbf{u}^{l_i}$ to $\mathbf{v}^{l_i}$. Since $k(\mathbf{u}^{l_r}) = \mathbf{v}^{l_{r-1}}$ and $k(\mathbf{v}^{l_r}) = \mathbf{u}^{l_{r+1}}$, $\mathbf{u}^{l_r} \neq \mathbf{v}^{l_r}$. Since $S^{l_r}_{n-1,1}$ is K_{n-1} for any $l_r \in \langle n\rangle - \{1\}$ and $\mathbf{u}^{l_r} \neq \mathbf{v}^{l_r}$, there exists a Hamiltonian path P_{l_r} of $S^{l_r}_{n-1,1}$ joining $\mathbf{u}^{l_r}$ to $\mathbf{v}^{l_r}$. Then, $\langle \mathbf{x} = \mathbf{u}^{l_1}, P_{l_1}, \mathbf{v}^{\mathbf{l}_1}, \mathbf{u}^{l_2}, P_{l_2}, \mathbf{v}^{l_2}, \ldots, \mathbf{u}^{l_n}, P_{l_n}, \mathbf{v}^{l_n} = \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 9(a) for an illustration.

Thus, we consider that $((\mathbf{x})_1 = l_2$ or $(\mathbf{y})_1 = l_{n-1})$ and $((\mathbf{x})_1 = l'_2$ or $(\mathbf{y})_1 = l'_{n-1})$. Since $l_2 \neq l'_2$ and $l_{n-1} \neq l'_{n-1}$, without loss of generality, we assume that $(\mathbf{x})_1 = l'_2$ and $(\mathbf{y})_1 = l_{n-1}$.

Suppose that there exists an index t such that $1 \leq t < n - 2$, $1 \neq l_t$, and $1 \neq l_{t+1}$. Since $|F - F(S^1_{n-1,1})| = 0$, the only edge $(\mathbf{p}, k(\mathbf{p}))$ in E^{l_n, l_t} and the only edge $(\mathbf{q}, k(\mathbf{q}))$ in $E^{l_n, l_{t+1}}$ are F-fault free. Obviously, $P_3 = \langle l_1, \ldots, l_t, l_n, l_{t+1}, \ldots, l_{n-1}\rangle$ is also a Hamiltonian path of $K_n - R(F^*)$. We rewrite P_3 as $\langle a_1, a_2, \ldots, a_n\rangle$. We set $\mathbf{x} = \mathbf{u}^{a_1}$ and $k(\mathbf{y}) = \mathbf{v}^{a_n}$. Then, we choose the only edge in $E^{a_r, a_{r+1}}$ as $(\mathbf{v}^{a_r}, \mathbf{u}^{a_{r+1}})$ for any $1 \leq r \leq n$. Since $k(\mathbf{v}^{a_r}) = \mathbf{u}^{a_{r+1}}$ and $k(\mathbf{u}^{a_r}) = \mathbf{v}^{a_{r-1}}$, $\mathbf{v}^{a_r} \neq \mathbf{u}^{a_r}$. Let i be the index such that $a_i = 1$. Obviously, $\mathbf{u}^{a_i}$ and $\mathbf{v}^{a_i}$ are in V' and there exists a Hamiltonian path P_{a_i} of $S^1_{n-1,1} - F$ joining $\mathbf{u}^{a_i}$ to $\mathbf{v}^{a_i}$. Since $S^r_{n-1,1}$ is K_{n-1} for any $r \in \langle n\rangle$, there exists a Hamiltonian path P_{a_r} of $S^{a_r}_{n-1,1} - \{\mathbf{y}\}$ joining $\mathbf{u}^{a_r}$ to $\mathbf{v}^{a_r}$ for any $a_r \in \langle n\rangle - \{1\}$. Then, $P_4 = \langle \mathbf{x} = \mathbf{u}^{a_1}, P_{a_1}, \mathbf{v}^{a_1}, \mathbf{u}^{a_2}, P_{a_2}, \mathbf{v}^{a_2}, \ldots, \mathbf{u}^{a_n}, P_{a_n}, \mathbf{v}^{a_n} = k(\mathbf{y})\rangle$ forms a Hamiltonian path of $S_{n,2} - F - \{\mathbf{y}\}$ joining $\mathbf{x}$ to $k(\mathbf{y})$. Then, $\langle \mathbf{x}, P_4, k(\mathbf{y}), \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 9(b) for an illustration.

Suppose that there is no index t such that $1 \leq t < n - 2$, $1 \neq l_t$, and $1 \neq l_{t+1}$. We claim that $n = 5$ and $l_2 = 1$. Let p be the index such that $l_p = 1$. Suppose that $n \geq 6$. We can choose t to be 1 if $p \geq 3$ and choose t to be 3 otherwise. Suppose that $n = 5$ and $l_2 \neq 1$. We can choose t to be 1. Obviously, $1 \neq l_t$ and $1 \neq l_{t+1}$. We get a contradiction.

Thus, we only consider the case that $n = 5$ and $l_2 = 1$. Since $(\mathbf{x})_1 = l'_2 \neq l'_5 = l_5$ and $(\mathbf{y})_1 = l_4 \neq l_1$, the only edge $(\mathbf{u}, k(\mathbf{u})) \in E^{l_1, l_5}$ satisfies $\mathbf{u} \neq \mathbf{x}$ and $k(\mathbf{u}) \neq \mathbf{y}$. By Lemma 1, there exists a Hamiltonian path P_7 of $S^{l_5}_{n-1,1}$ joining $k(\mathbf{u})$ to $\mathbf{y}$. Since $|F - F(S^1_{n-1,1})| = 0$, the only edge $(\mathbf{w}, k(\mathbf{w}))$ in E^{l_1, l_4} is F-fault free. Since $(l_1, l_2) \in P_1$, the only edge $(\mathbf{v}, \mathbf{v}^{l_2})$ in E^{l_1, l_2} is F-fault free. Since $S^{l_1}_{n-1,1}$ is K_{n-1}, there exist two paths P_5 and P_6 covering all vertices in $S^{l_1}_{n-1,1}$ such that P_5 joins $\mathbf{x}$ to $\mathbf{w}$ and P_6 joins $\mathbf{v}$ to $\mathbf{u}$. Since $\langle l_2, l_3, l_4\rangle$ is a subpath of P_1, the only edge $(\mathbf{u}^{l_2}, \mathbf{v}^{l_3})$ in E^{l_2, l_3} and the only edge $(\mathbf{u}^{l_3}, \mathbf{v}^{l_4})$ in E^{l_3, l_4} are F-fault free. Since $\mathbf{v}^{l_2}$ and $\mathbf{u}^{l_2}$ are in V', by Lemma 3, there exists a Hamiltonian path P_2 of $S^{l_2}_{n-1,1} - F$ joining $\mathbf{u}^{l_2}$ to $\mathbf{v}^{l_2}$. By Lemma 1, there exists a Hamiltonian path P_{l_3} of $S^{l_3}_{n-1,1}$ joining $\mathbf{u}^{l_3}$ to $\mathbf{v}^{l_3}$ and a Hamiltonian path P_{l_4} of $S^{l_2}_{n-1,1}$ joining $k(\mathbf{w})$ to $\mathbf{v}^{l_4}$. Then, $\langle \mathbf{x}, P_5, \mathbf{w}, k(\mathbf{w}), P_{l_4}, \mathbf{v}^{l_4}, \mathbf{u}^{l_3}, P_{l_3}, \mathbf{v}^{l_3}, \mathbf{u}^{l_2}, P_{l_2}, \mathbf{v}^{l_2}, \mathbf{v}, P_6, \mathbf{u}, k(\mathbf{u}), P_7, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{5,2} - F$ joining $\mathbf{x}$ to $\mathbf{y}$.

Thus, the lemma is proved. ■

Lemma 8. *Suppose that $S_{n-1,k-1}$ is $(n - 4)$-fault Hamiltonian and $(n - 5)$-fault Hamiltonian connected, for some $k \geq 3$ and $n - k \geq 2$. Then, $S_{n,k}$ is $(n - 4)$-fault Hamiltonian connected.*

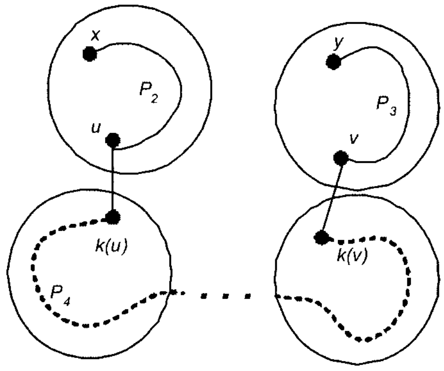

FIG. 10. Illustration for Subcase 1.2.

Proof. Since $k \geq 3$, $n \geq 5$, and $(n-k) \geq 2$, $|E^{r,s}| = [(n-2)!]/[(n-k)!] \geq (n-2)$ for all $1 \leq r \neq s \leq n$. By Lemma 3, all edges in $E^{r,s}$ are independent. Assume that F is any faulty set of $S_{n,k}$ with $|F| \leq n-4$. Without loss of generality, we assume that $|F(S^1_{n-1,k-1})| \geq |F(S^2_{n-1,k-1})| \geq \ldots \geq |F(S^n_{n-1,k-1})|$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two arbitrary vertices of $S_{n,k} - F$. We want to construct a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Case 1. $|F(S^1_{n-1,k-1})| \leq n-5$. By the assumption of this lemma, $S^t_{n-1,k-1} - F(S^t_{n-1,k-1})$ is Hamiltonian connected for any $t \in \langle n \rangle$.

Subcase 1.1. $(\mathbf{x})_k \neq (\mathbf{y})_k$. Since $|R(F)| \leq n-4$, by Lemma 1, $K_n - R(F)$ is Hamiltonian connected. By Lemma 4, there exists a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ and $\mathbf{y}$.

Subcase 1.2. $(\mathbf{x})_k = (\mathbf{y})_k$. By the assumption of this lemma, there exists a Hamiltonian path P_1 of $S^{(\mathbf{x})_k}_{n-1,k-1} - F(S^{(\mathbf{x})_k}_{n-1,k-1})$ joining $\mathbf{x}$ to $\mathbf{y}$. We claim that there exists an edge $(\mathbf{u}, \mathbf{v})$ of P_1 such that $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k,(\mathbf{u})_1}$ and $(\mathbf{v}, k(\mathbf{v}))$ in $E^{(\mathbf{x})_k,(\mathbf{v})_1}$ are F-fault free. Let F' denote the set of F-fault edges in $\cup_{j \in \langle n \rangle - \{(\mathbf{x})_k\}} E^{(\mathbf{x})_k,j}$. Suppose that no such edge exists. Then, $|F'| \geq |P_1|/2$. Thus, $|F' \cup F(S^{(\mathbf{x})_k}_{n-1,k-1})| \geq [(n-1)!]/[2(n-k)!] > |F|$ when $n \geq 5$ and $k \geq 3$. We get a contradiction.

Thus, we can write P_1 as $\langle \mathbf{x}, P_2, \mathbf{u}, \mathbf{v}, P_3, \mathbf{y} \rangle$. Since $d(\mathbf{u}, \mathbf{v}) = 1$, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Let $\langle (\mathbf{u})_1 = l_1, l_2, \ldots, l_{n-1} = (\mathbf{v})_1 \rangle$ be any Hamiltonian path of $K_n^{\langle n \rangle - \{(\mathbf{x})_k\}}$. We set $k(\mathbf{u}) = \mathbf{v}^{l_1}$ and $k(\mathbf{v}) = \mathbf{u}^{l_{n-1}}$. Since $|E^{r,s}| - |F| \geq 2$ for any $r, s \in \langle n \rangle$, we can choose any F-fault-free edges $(\mathbf{u}^{l_i}, \mathbf{v}^{l_{i+1}})$ in $E^{l_i,l_{i+1}}$ for all $1 \leq i \leq n-1$. By the assumption of this lemma, there exists a Hamiltonian path P_{l_i} of $S^{l_i}_{n-1,k-1} - F$ joining $\mathbf{v}^{l_i}$ to $\mathbf{u}^{l_i}$. Then, $P_4 = \langle \mathbf{v}^{l_1}, P_{l_1}, \mathbf{u}^{l_1}, \mathbf{v}^{l_2}, \ldots, P_{l_{n-1}}, \mathbf{u}^{l_{n-1}} \rangle$ forms a Hamiltonian path of $S^{\langle n \rangle - \{(\mathbf{x})_k\}}_{n-1,k-1} - F$ joining $k(\mathbf{u})$ to $k(\mathbf{v})$. Thus, $\langle \mathbf{x}, P_2, \mathbf{u}, k(\mathbf{u}), P_4, k(\mathbf{v}), \mathbf{v}, P_3, \mathbf{y} \rangle$ forms a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 10 for an illustration.

Case 2. $|F(S^1_{n-1,k-1})| = n-4$. In this case, all faults are in $S^1_{n-1,k-1}$.

Subcase 2.1. $(\mathbf{x})_k = (\mathbf{y})_k = 1$. Choose any element f in $F(S^1_{n-1,k-1})$. By the assumption of this lemma, we can find a Hamiltonian path P of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1}) + f$ joining $\mathbf{x}$ to $\mathbf{y}$. By deleting f, we can find two vertices $\mathbf{u}$ and $\mathbf{v}$ with $d(\mathbf{u}, \mathbf{v}) \leq 2$ such that (1) there are two paths P_1 and P_2 covering all the vertices of $S^1_{n-1,k-1} - F$, (2) P_1 joins $\mathbf{x}$ to $\mathbf{u}$, and (3) P_2 joins $\mathbf{v}$ to $\mathbf{y}$. By Lemma 2, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Then, there exists a Hamiltonian path of $K_n^{\langle n \rangle - \{1\}}$ joining $(\mathbf{u})_1$ to $(\mathbf{v})_1$. By Lemma 4, there is a Hamiltonian path P_3 joining $k(\mathbf{u})$ and $k(\mathbf{v})$ in $S^{\langle n \rangle - \{1\}}_{n-1,k-1}$. Thus, $\langle \mathbf{x}, P_1, \mathbf{u}, k(\mathbf{u}), P_3, k(\mathbf{v}), \mathbf{v}, P_2, \mathbf{y} \rangle$ forms a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 11(a) for an illustration.

Subcase 2.2. $(\mathbf{x})_k = 1$ and $(\mathbf{y})_k \neq 1$. Let C be a Hamiltonian cycle of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1})$. Write C as $\langle \mathbf{x}, \mathbf{u}, P_1, \mathbf{v}, \mathbf{x} \rangle$. Thus, $d(\mathbf{u}, \mathbf{v}) \leq 2$. By Lemma 2, $(\mathbf{u})_1 \neq (\mathbf{v})_1$. Since the cycle C can be traversed forward and backward, we may assume that $(\mathbf{u})_1 \neq (\mathbf{y})_k$. Since $|F - F(S^1_{n-1,k-1})| = 0$ and $|E^{r,s}| \geq (n-2)$ for any $1 \leq r < s \leq n$, there exists an F-fault-free edge $(\mathbf{w}, k(\mathbf{w}))$ in $E^{(\mathbf{y})_k,l}$ for some $l \in \langle n \rangle - \{(\mathbf{x})_k, (\mathbf{u})_1, (\mathbf{y})_k\}$ such that $\mathbf{w} \neq \mathbf{y}$. Obviously, there exists a Hamiltonian path of $K_n^{\langle n \rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}$ joining $(\mathbf{u})_1$ to $(\mathbf{w})_1$, $(k(k(\mathbf{u})))_k = (\mathbf{u})_k = (\mathbf{x})_k$, and $(\mathbf{w})_k = (\mathbf{y})_k$. By Lemma 4, there exists a Hamiltonian path P_2 of $S^{\langle n \rangle - \{(\mathbf{x})_k, (\mathbf{y})_k\}}_{n-1,k-1}$ joining $k(\mathbf{u})$ to $k(\mathbf{w})$. By the assumption of this lemma, there exists a Hamiltonian path P_3 of $S^{(\mathbf{y})_k}_{n-1,k-1}$

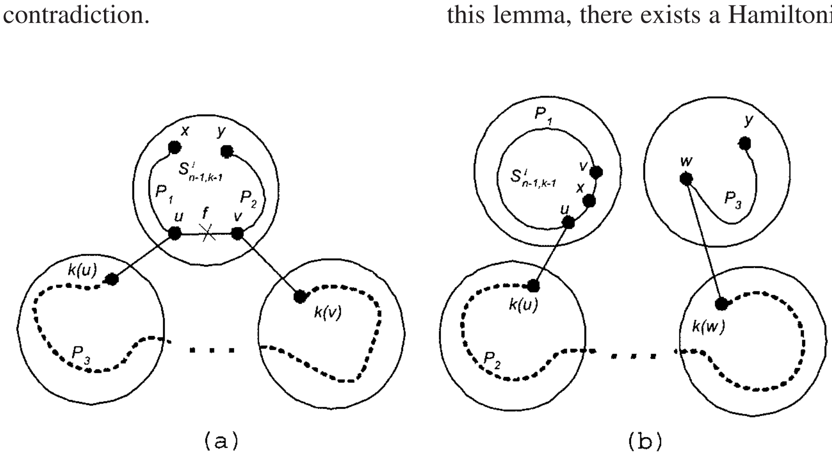

FIG. 11. Illustrations for Subcases 2.1 and 2.2.

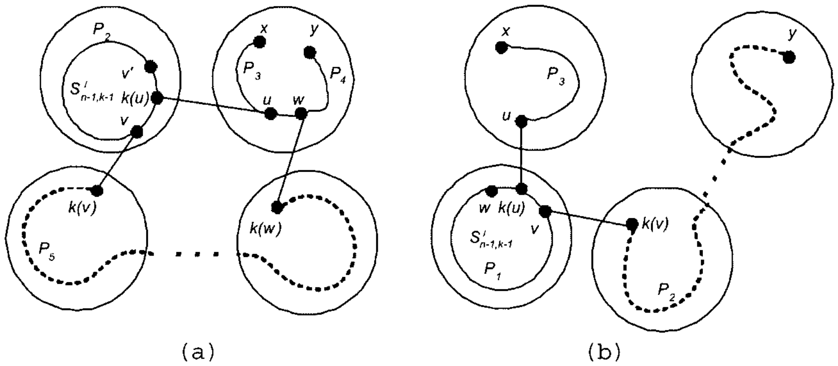

FIG. 12. Illustrations for Subcases 2.3 and 2.4.

joining $\mathbf{w}$ to $\mathbf{y}$. Then, $\langle \mathbf{x}, \mathbf{v}, P_1, \mathbf{u}, k(\mathbf{u}), P_2, k(\mathbf{w}), \mathbf{w}, P_3, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 11(b) for an illustration.

SUBCASE 2.3. $(\mathbf{x})_k = (\mathbf{y})_k \neq 1$. By the assumption of this lemma, there exists a Hamiltonian path P_1 of $S^{(\mathbf{x})_k}_{n-1,k-1}$ joining $\mathbf{x}$ to $\mathbf{y}$ and there exists a Hamiltonian cycle C of $S^1_{n-1,k-1} - F(S^1_{n-1,k-1})$. Since $|E^{(\mathbf{x})_k,1}| - |F| \geq 2$, there exists an F-fault-free edge $(\mathbf{u}, k(\mathbf{u})) \in E^{(\mathbf{x})_k,1}$ with $\mathbf{x} \neq \mathbf{u}$. Thus, we can write C as $\langle k(\mathbf{u}), \mathbf{v}, P_2, \mathbf{v}', k(\mathbf{u})\rangle$ and write P_1 as $\langle \mathbf{x}, P_3, \mathbf{u}, \mathbf{w}, P_4, \mathbf{y}\rangle$. Since $d(\mathbf{v}, \mathbf{v}') \leq 2$, $(\mathbf{v})_1 \neq (\mathbf{v}')_1$. Without loss of generality, we assume that $(\mathbf{v})_1 \neq (\mathbf{w})_1$. Obviously, there exists a Hamiltonian path of $K_n^{\langle n\rangle - \{1,(\mathbf{x})_k\}}$ joining $(\mathbf{u})_1$ to $(\mathbf{w})_1$. By Lemma 4, there exists a Hamiltonian path P_5 of $S^{\langle n\rangle - \{1,(\mathbf{x})_k\}}_{n-1,k-1}$ joining $k(\mathbf{v})$ to $k(\mathbf{w})$. Thus, $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), \mathbf{v}', P_2, \mathbf{v}, k(\mathbf{v}), P_5, k(\mathbf{w}), \mathbf{w}, P_4, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 12(a) for an illustration.

SUBCASE 2.4. $(\mathbf{x})_k$, $(\mathbf{y})_k$, and 1 are distinct. Since $|E^{(\mathbf{x})_k,1}| \geq (n-2)$, there exists an F-fault-free edge $(\mathbf{u}, k(\mathbf{u}))$ in $E^{(\mathbf{x})_k,1}$ and $\mathbf{u} \neq \mathbf{x}$. By the assumption of this lemma, there exists a Hamiltonian path P_3 of $S^{(\mathbf{x})_k}_{n-1,k-1}$ joining $\mathbf{x}$ to $\mathbf{u}$ and there is a Hamiltonian cycle C in $S^1_{n-1,k-1} - F$. We can write C as $\langle k(\mathbf{u}), \mathbf{v}, P_1, \mathbf{w}, k(\mathbf{u})\rangle$. Since $d(\mathbf{v}, \mathbf{w}) \leq 2$, $(\mathbf{v})_1 \neq (\mathbf{w})_1$. Without loss of generality, we assume that $(\mathbf{v})_1 \neq (\mathbf{y})_k$. Then, there exists a Hamiltonian path of $K_n^{\langle n\rangle - \{(\mathbf{x})_k,1\}}$ joining $(\mathbf{v})_1$ to $(\mathbf{y})_k$. By Lemma 4, there exists a Hamiltonian path P_2 of $S^{\langle n\rangle - \{(\mathbf{x})_k,1\}}$ joining $k(\mathbf{v})$ to $\mathbf{y}$. Then, $\langle \mathbf{x}, P_3, \mathbf{u}, k(\mathbf{u}), \mathbf{w}, P_1, \mathbf{v}, k(\mathbf{v}), P_2, \mathbf{y}\rangle$ forms a Hamiltonian path of $S_{n,k} - F$ joining $\mathbf{x}$ to $\mathbf{y}$. See Figure 12(b) for an illustration.

Thus, the lemma is proved. ∎

Theorem 4. *Let n and k be two positive integers with $n > k \geq 1$. Then,*

(1) $\mathcal{H}_f(S_{n,k}) = n - 3$ and $\mathcal{H}^{\kappa}_f(S_{n,k}) = n - 4$ if $n - k \geq 2$;
(2) $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}^{\kappa}_f(S_{2,1}) = 0$; and
(3) $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}^{\kappa}_f(S_{n,n-1})$ is undefined if $n > 2$.

Proof. We first consider the case $k = n - 1$. It is proved in [3] that $S_{n,n-1}$ is isomorphic to the n-star graph S_n. In [1], it is proved that S_n is bipartite for every n and S_n is Hamiltonian if and only if $n > 2$. The graph S_2 is K_2 which is Hamiltonian connected. It is known that the number of vertices in both partite sets of any bipartite Hamiltonian graph are the same. For these reasons, any bipartite Hamiltonian graph is not Hamiltonian connected. Thus, $\mathcal{H}_f(S_{n,n-1}) = 0$ and $\mathcal{H}^{\kappa}_f(S_{n,n-1})$ is undefined if $n > 2$. Moreover, $\mathcal{H}_f(S_{2,1})$ is undefined and $\mathcal{H}^{\kappa}_f(S_{2,1}) = 0$.

Now, we consider the case $n > k \geq 1$. By Lemma 1, the theorem is true for $S_{n,1}$. According to Lemma 5, the theorem is true for $S_{4,2}$. Based on the Lemmas 6 and 7, the theorem is true for $S_{n,2}$ with $n \geq 5$. By Lemmas 6 and 8, the theorem is true for all $S_{n,k}$ with $n \geq 5$, $k \geq 3$, and $(n - k) \geq 2$. Hence, the theorem is proved. ∎

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APPENDIX

Fact 1. *$S_{5,2} - F$ is Hamiltonian for any $F \subseteq V(S_{5,2}) \cup E(S_{5,2})$ with $|F| = 2$ and $|F(S^1_{4,1})| = |F(S^2_{4,1})| = 1$ (see Fig. A.1).*

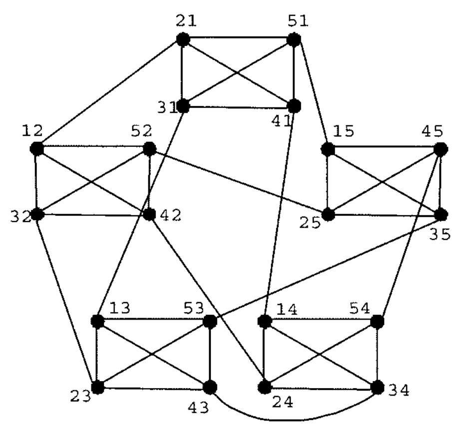

FIG. A.1. $S_{5,2}$.

Proof. Assume that F consists of two vertices. Then, we have the following 16 cases, namely: (1) {21, 12}, (2) {21, 32}, (3) {21, 42}, (4) {21, 52}, (5) {31, 12}, (6) {31, 32}, (7) {31, 42}, (8) {31, 52}, (9) {41, 12}, (10) {41, 32}, (11) {41, 42}, (12) {41, 52}, (13) {51, 12}, (14) {51, 32}, (15) {51, 42}, and (16) {51, 52}. The corresponding Hamiltonian cycles of $S_{5,2} - F$ are listed below:

(1)	⟨31, 13, 23, 32, 42, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 51, 31⟩
(2)	⟨31, 13, 23, 43, 53, 35, 15, 45, 25, 52, 12, 42, 24, 34, 54, 14, 41, 51, 31⟩
(3)	⟨31, 13, 23, 32, 12, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 51, 31⟩
(4)	⟨31, 13, 23, 32, 12, 42, 24, 14, 54, 34, 43, 53, 35, 25, 45, 15, 51, 41, 31⟩
(5)	⟨21, 41, 14, 24, 42, 32, 52, 25, 35, 53, 13, 23, 43, 34, 54, 45, 15, 51, 21⟩
(6)	⟨21, 12, 42, 52, 25, 15, 45, 35, 53, 13, 23, 43, 34, 24, 54, 14, 41, 51, 21⟩
(7)	⟨21, 12, 32, 52, 25, 15, 45, 35, 53, 13, 23, 43, 34, 24, 54, 14, 41, 51, 21⟩
(8)	⟨21, 12, 32, 42, 24, 14, 54, 34, 43, 13, 23, 53, 35, 25, 45, 15, 51, 41, 21⟩
(9)	⟨21, 31, 13, 23, 32, 42, 52, 25, 35, 53, 43, 34, 14, 24, 54, 45, 15, 51, 21⟩
(10)	⟨21, 12, 42, 52, 25, 15, 35, 45, 54, 14, 24, 34, 43, 23, 53, 13, 31, 51, 21⟩
(11)	⟨21, 12, 32, 52, 25, 15, 35, 45, 54, 14, 24, 34, 43, 23, 53, 13, 31, 51, 21⟩
(12)	⟨21, 12, 32, 42, 24, 14, 34, 54, 45, 15, 25, 35, 53, 23, 43, 13, 31, 51, 21⟩
(13)	⟨21, 31, 13, 23, 32, 42, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 21⟩
(14)	⟨21, 12, 42, 52, 25, 15, 35, 45, 54, 14, 24, 34, 43, 23, 53, 13, 31, 41, 21⟩
(15)	⟨21, 12, 32, 52, 25, 15, 35, 45, 54, 14, 24, 34, 43, 23, 53, 13, 31, 41, 21⟩
(16)	⟨21, 12, 32, 42, 24, 14, 34, 54, 45, 15, 25, 35, 53, 23, 43, 13, 31, 41, 21⟩

Assume that F consists of two edges, Then, we have 36 cases. We divide these 36 cases into four classes, namely:

(1)	{(21, 31), (12, 32)}, {(21, 31), (42, 52)}, {(21, 41), (12, 32)}, {(21, 41), (42, 52)},
	{(31, 51), (12, 32)}, {(31, 51), (42, 52)}, {(41, 51), (12, 32)}, {(41, 51), (42, 52)}.
(2)	{(21, 31), (12, 42)}, {(21, 31), (12, 52)}, {(21, 31), (32, 42)}, {(21, 31), (32, 52)},
	{(21, 41), (12, 42)}, {(21, 41), (12, 52)}, {(21, 41), (32, 42)}, {(21, 41), (32, 52)},
	{(31, 51), (12, 42)}, {(31, 51), (12, 52)}, {(31, 51), (32, 42)}, {(31, 51), (32, 52)},
	{(41, 51), (12, 42)}, {(41, 51), (12, 52)}, {(41, 51), (32, 42)}, {(41, 51), (32, 52)}.
(3)	{(21, 51), (12, 32)}, {(21, 51), (42, 52)}, {(31, 41), (12, 32)}, {(31, 41), (42, 52)}.
(4)	{(21, 51), (12, 42)}, {(21, 51), (12, 52)}, {(21, 51), (32, 42)}, {(21, 51), (32, 52)},
	{(31, 41), (12, 42)}, {(31, 41), (12, 52)}, {(31, 41), (32, 42)}, {(31, 41), (32, 52)}.

The corresponding Hamiltonian cycles of $S_{5,2} - F$ are listed below:

(1)	⟨21, 12, 42, 24, 14, 41, 31, 13, 23, 32, 52, 25, 35, 53, 43, 34, 54, 45, 15, 51, 21⟩
(2)	⟨21, 12, 32, 23, 13, 31, 41, 14, 24, 42, 52, 25, 35, 53, 43, 34, 54, 45, 15, 51, 21⟩
(3)	⟨21, 12, 42, 24, 14, 41, 51, 15, 25, 52, 32, 23, 43, 34, 54, 45, 35, 53, 13, 31, 21⟩
(4)	⟨21, 12, 32, 23, 13, 31, 51, 15, 25, 52, 42, 24, 34, 43, 53, 35, 45, 54, 14, 41, 21⟩

Assume that F consists of one vertex and one edge. Then, we have 48 cases. We divide these 48 cases into 18 classes, namely:

(1)	{21, (12, 32)}, {21, (42, 52)}.
(2)	{21, (12, 42)}.
(3)	{21, (12, 52)}, {21, (32, 42)}, {21, (32, 52)}.
(4)	{31, (12, 32)}, {31, (42, 52)}.
(5)	{31, (12, 42)}, {31, (12, 52)}, {31, (32, 42)}, {31, (32, 52)}.
(6)	{41, (12, 32)}, {41, (42, 52)}.
(7)	{41, (12, 42)}, {41, (12, 52)}, {41, (32, 42)}, {41, (32, 52)}.
(8)	{51, (12, 32)}, {51, (42, 52)}.
(9)	{51, (12, 42)}, {51, (12, 52)}, {51, (32, 42)}, {51, (32, 52)}.
(10)	{12, (21, 31)}, {12, (41, 51)}.
(11)	{12, (21, 41)}, {12, (31, 41)}, {12, (31, 51)}.
(12)	{12, (21, 51)}.
(13)	{32, (21, 31)}, {32, (21, 51)}, {32, (31, 41)}, {32, (41, 51)}.
(14)	{32, (21, 41)}, {32, (31, 51)}.
(15)	{42, (21, 31)}, {42, (21, 51)}, {42, (31, 41)}, {42, (41, 51)}.
(16)	{42, (21, 41)}, {42, (31, 51)}.
(17)	{52, (21, 31)}, {52, (21, 41)}, {52, (31, 51)}, {52, (41, 51)}.
(18)	{52, (21, 51)}, {52, (31, 41)}.

The corresponding Hamiltonian cycles of $S_{5,2} - F$ are listed below:

(1)	$\langle 31, 13, 23, 32, 42, 12, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 51, 31 \rangle$
(2)	$\langle 31, 13, 23, 32, 12, 52, 42, 24, 14, 54, 34, 43, 53, 35, 25, 45, 15, 51, 41, 31 \rangle$
(3)	$\langle 31, 13, 23, 32, 12, 42, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 51, 31 \rangle$
(4)	$\langle 21, 12, 42, 24, 14, 34, 54, 45, 25, 52, 32, 23, 13, 43, 53, 35, 15, 51, 41, 21 \rangle$
(5)	$\langle 21, 12, 32, 23, 13, 43, 53, 35, 15, 45, 25, 52, 42, 24, 34, 54, 14, 41, 51, 21 \rangle$
(6)	$\langle 21, 12, 42, 24, 14, 34, 54, 45, 15, 35, 25, 52, 32, 23, 43, 53, 13, 31, 51, 21 \rangle$
(7)	$\langle 21, 12, 32, 23, 13, 43, 53, 35, 25, 52, 42, 24, 14, 34, 54, 45, 15, 51, 31, 21 \rangle$
(8)	$\langle 21, 12, 42, 24, 14, 34, 54, 45, 15, 35, 25, 52, 32, 23, 43, 53, 13, 31, 41, 21 \rangle$
(9)	$\langle 21, 12, 32, 23, 13, 43, 53, 35, 15, 45, 25, 52, 42, 24, 34, 54, 14, 41, 31, 21 \rangle$
(10)	$\langle 21, 41, 14, 24, 42, 32, 52, 25, 15, 35, 45, 54, 34, 43, 23, 53, 13, 31, 51, 21 \rangle$
(11)	$\langle 21, 31, 13, 23, 32, 42, 52, 25, 15, 45, 35, 53, 43, 34, 24, 54, 14, 41, 51, 21 \rangle$
(12)	$\langle 21, 31, 13, 23, 32, 42, 52, 25, 35, 53, 43, 34, 14, 24, 54, 45, 15, 51, 41, 21 \rangle$
(13)	$\langle 21, 12, 42, 52, 25, 15, 51, 31, 13, 23, 43, 53, 35, 45, 54, 24, 34, 14, 41, 21 \rangle$
(14)	$\langle 21, 12, 42, 52, 25, 15, 51, 41, 14, 24, 34, 54, 45, 35, 53, 23, 43, 13, 31, 21 \rangle$
(15)	$\langle 21, 12, 32, 52, 25, 15, 51, 31, 13, 23, 43, 53, 35, 45, 54, 24, 34, 14, 41, 21 \rangle$
(16)	$\langle 21, 12, 32, 52, 25, 15, 51, 41, 14, 24, 34, 54, 45, 35, 53, 23, 43, 13, 31, 21 \rangle$
(17)	$\langle 21, 12, 32, 42, 24, 14, 41, 31, 13, 23, 53, 43, 34, 54, 45, 25, 35, 15, 51, 21 \rangle$
(18)	$\langle 21, 12, 32, 42, 24, 14, 41, 51, 15, 25, 35, 45, 54, 34, 43, 23, 53, 13, 31, 21 \rangle$

Hence, $S_{5,2} - F$ is Hamiltonian when $|F| = 2$ and $|F(S^1_{4,1})| = |F(S^2_{4,1})| = 1$. ■

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