Arch. Rational. Mech. Anal. 127 (1994) 297–318.

On the Dynamics of a Tethered Satellite System

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Communicated by P. HOLMES

Abstract

The Hamiltonian structure for a fundamental model of a tethered satellite system is constructed. The model is composed of two point masses connected by a string with no restrictions on the motions of the two masses. A certain symmetry with respect to the special orthogonal group $SO(3)$ for such a system is observed. The classical station-keeping mode for the tethered system is found to be nothing more than the relative equilibrium corresponding to the reduction of the system by the symmetry. The microgravity forces on the two point masses are responsible for the possible configurations of the string at the so-called *radial relative equilibrium.* A stability analysis is performed on the basis of the reduced energy-momentum method. Criteria for stability are derived, which could find potential applications in space technology.

1. Introduction

A Tethered Satellite System (TSS) basically contains two spacecraft, such as a space shuttle and a satellite, connected by a long rod or tether, with the whole assembly moving in a central gravitational field. Since the idea of a TSS was introduced around 1973 [Co, G], problems involving dynamics and control of such systems have been investigated by many researchers; cf. [F, KP, VK, PPL, LA, LB], and the references therein. A TSS has several potential applications in space technologies, for example, scientific experiments, deployment or retrieval of satellites, power generation, measurement of aerodynamic forces, etc.; cf. [PA].

The dynamical behavior of a TSS is quite complicated. It is essentially a coupled-body system moving in a complex environment which includes gravity forces, electromagnetic forces, and aerodynamic forces. The operational modes of such a system can be roughly divided into three categories: deployment, station-keeping, and retrieval. Stability is one of the primary concerns during the station-keeping mode. However, the analysis in the above-mentioned works was mostly based on simplified models, such as that of two point masses connected by a massless rigid rod, while one point mass, say the space shuttle, is assumed to be rather massive and to be the center of mass of the assembly; cf. [LA, F]. Some other models take the mass and flexibility of the tether into consideration; cf. [LB, PPL]. But a basic assumption in these models is that the shuttle is restricted to a circular motion, with emphasis placed on the problem of control at different modes. There are not many mathematically rigorous analyses of more complex or natural models. One such analysis is that of BELETSKY & LEVIN [BL], who used the energy method to prove the stability of radial equilibria of the above-mentioned string model and computed some natural modes. They also considered the effects of aero- and electro-dynamic forces. On the other hand, ANTMAN & WOLFE [AW] discussed the multiple equilibria of elastic strings near the singular point of the central force field. Many analytical problems in this field remain to be solved.

The dynamics of a TSS are discussed in this paper in the Hamiltonian framework; only gravity forces are considered. Since the tether proposed for space applications is quite long (roughly 20–100km in the station-keeping mode), its mass and flexibility definitely cannot be assumed to be negligible. Accordingly, it is unnatural (or costly) to assume both that the space shuttle is in a circular orbit and that it remains unaffected by the motions of the tether and the other satellite. As a result, a fundamental and intuitive model is developed here; it is a two-mass system connected by a massive string with the coupling between each element appropriately included. The Hamiltonian structure is also determined. We show that such a system admits a natural symmetry corresponding to the $SO(3)$ action. Within this framework, reduction can be performed and the relative equilibria can be defined. The configurations in the station-keeping mode used in engineering circles are actually the same as those at the relative equilibria corresponding to this reduction.

The aforementioned coupling effects enter into the dynamics through the microgravity forces which are exerted on the string by the point masses at relative equilibria and through their reactions exerted by the point masses on the string. (Due to material constraints, the point masses do not move on circular Keplerian orbits at relative equilibria, where the centrifugal and gravitational forces are balanced. The microgravity forces are the differences between these two forces.) Possible configurations for such relative equilibria, or *steady motions*, are proved to exist. To perform the stability analysis for the relative equilibria, we adopt the reduced energy-momentum method (cf. [SLM, WK]) which respects the symmetry structure. However, the associated locked-inertia tensor for the radial case fails to be invertible. Thus the block-diagonalization technique is not directly applicable here, and the complete reduced energy-momentum method must be invoked. Stability conditions are then derived. The rather complicated nature of the TSS problem is revealed by the present analysis. Those conditions on the relative equilibria and the associated stability criteria obtained in this paper may potentially have some engineering applications.

The physical system under consideration is described in Section 2. Here the equations of motion and their Hamiltonian structure are derived. Our problem is consequently a nonlinear dynamic boundary-value problem. A brief description of simple mechanical systems with symmetry and the reduced energy-momentum method is provided in Section 3. The intrinsic symmetry structure in our Hamiltonian system and the associated reduction process are then discussed in Section 4. Here the relative equilibria are defined and the existence of the so-called *radial relative equilibria* is discussed. For such radial relative equilibria, or radial steady motions, the stability analysis is performed in Section 5 by applying the reduced energy-momentum method. Compared to the classical energy method, the reduced energy-momentum method indeed leads to weaker stability conditions, which can be obtained by solving a Sturm-Liouville problem. Some concluding remarks are given in Section 6.

In our analysis, the constitutive law for the string, characterized by a general stored-energy density function, is intrinsically nonlinear. However, the Cauchy problem for the nonlinear string is difficult and has not yet been solved. A simpler case is treated in Appendix I, where semigroup theory is applied to solve the Cauchy problem for a string with a quadratic stored-energy density. Moreover, the time-map analysis discussed in Section 4 is specialized in Appendix II to one particular form of the stored-energy density which characterizes the case of linearly elastic strings. Methodologies discussed for the (general) nonlinear case can be more easily grasped for such linear strings.

2. Equations of Motion and the Hamiltonian Structure

The satellite and the shuttle are considered here as point masses and the tether is modeled as an elastic string; cf. Figure 1. Let e_1, e_2, e_3 be the coordinate axes of the inertial frame. Let $\boldsymbol{r}$ denote the vectors from the origin of the inertial frame to the points of the tethered system. The equations of motion can be derived from the action principle. Let $Q = \{\boldsymbol{r} : [0, L] \to \mathbb{R}^3 | \boldsymbol{r} \text{ is a smooth embedding}\}$ be the configuration space, and let $s \in [0, L]$ be the reference parameter for the string. Consider paths $t \mapsto \boldsymbol{r}(t, \cdot)$ in Q, and let $\dot{\boldsymbol{r}} = \partial \boldsymbol{r}/\partial t$. The Lagrangian function is $\mathscr{L} = T(\dot{\boldsymbol{r}}) - V(\boldsymbol{r})$, where the kinetic energy T and the potential energy V are

$$T(\dot{\boldsymbol{r}}) = \int_0^L \frac{\rho}{2}\langle \dot{\boldsymbol{r}}, \dot{\boldsymbol{r}}\rangle ds + \frac{m_1}{2}\langle \dot{\boldsymbol{r}}, \dot{\boldsymbol{r}}\rangle\Big|_{s=0} + \frac{m_2}{2}\langle \dot{\boldsymbol{r}}, \dot{\boldsymbol{r}}\rangle\Big|_{s=L}, \tag{1}$$

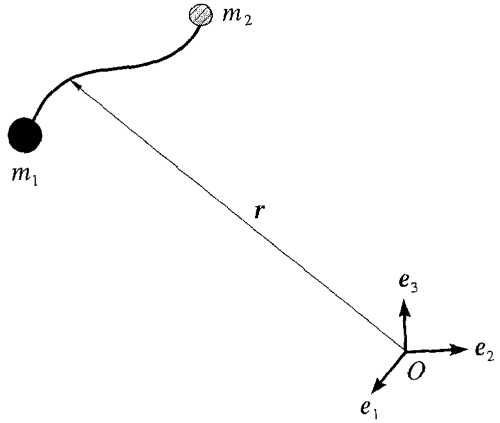

Fig. 1. A tethered satellite system.

$$V(r) = -\int_0^L \frac{\mu\rho}{|r|}ds - \frac{\mu m_1}{|r|}\Big|_{s=0} - \frac{\mu m_2}{|r|}\Big|_{s=L} + \int_0^L W(|r_s|)\,ds. \tag{2}$$

Here $\rho(s)$ is the mass density for the string with $s \in (0,L)$; m_1, m_2 are the masses at $r(0)$ and $r(L)$, respectively; μ is the gravitational constant; the stored-energy density of the string W is a smooth function on $\mathbb{R}$; and $r_s = \partial r/\partial s$. Let $W'(\Lambda) = dW(\Lambda)/d\Lambda$. Then the Euler-Lagrange equations for the action functional $\int \mathscr{L}(r,\dot r)\,dt$ can be written as

$$\begin{aligned} \rho\ddot r &= \left(W'(|r_s|)\frac{r_s}{|r_s|}\right)_s - \frac{\mu\rho}{|r|^3}r, \quad s \in (0,L), \\ m_1\ddot r(0) &= -\frac{\mu m_1}{|r(0)|^3}r(0) + W'(|r_s(0)|)\frac{r_s}{|r_s|}(0), \\ m_2\ddot r(L) &= -\frac{\mu m_2}{|r(L)|^3}r(L) - W'(|r_s(L)|)\frac{r_s}{|r_s|}(L). \end{aligned} \tag{3}$$

This is a system comprising a quasilinear (strictly or nonstrictly) hyperbolic system and two second-order nonlinear differential equations which are coupled through boundary conditions. The system should be interpreted in a weak sense, i.e., in terms of the *principle of virtual work* (cf. [AN]).

The Cauchy problem for (3) with general stored-energy density function W is challenging; shock waves may appear (cf. [S1, S2], where the strictly hyperbolic case with fixed boundary values and no potentials has been discussed). However, for the case that the terms involving W in (3) are linear, e.g., $W(\Lambda) = \frac{1}{2}(\Lambda^2)$, the Cauchy problem can be solved via the semigroup method; cf. Appendix I.

We now examine the geometry of the phase space, along with the Hamiltonian structure of the system under consideration. See [AM] for a general reference and [M] for infinite-dimensional geometric settings. An inner product on the tangent space T_rQ is defined as

$$\langle\delta r_1, \delta r_2\rangle_{TQ} = \int_0^L \rho\langle\delta r_1, \delta r_2\rangle ds + m_1\langle\delta r_1, \delta r_2\rangle\Big|_{s=0} + m_2\langle\delta r_1, \delta r_2\rangle\Big|_{s=L}, \tag{4}$$

where $\delta r_1, \ \delta r_2 \in T_rQ$. To this inner product there corresponds a pairing on $T^*Q \times TQ$ and an inner product on T^*Q (generated by a Legendre transformation from the computation of $D_{\dot r}\mathscr{L} \cdot \delta\dot r$):

$$\begin{aligned} \langle\langle p, w\rangle\rangle &:= \int_0^L \langle p, w\rangle ds + \langle p, w\rangle\Big|_{s=0} + \langle p, w\rangle\Big|_{s=L}, \\ \langle p_1, p_2\rangle_{T^*Q} &:= \int_0^L \frac{1}{\rho}\langle p_1, p_2\rangle ds + \frac{1}{m_1}\langle p_1, p_2\rangle\Big|_{s=0} + \frac{1}{m_2}\langle p_1, p_2\rangle\Big|_{s=L}. \end{aligned}$$

With these operations, the Legendre transformation is found to be $p = \rho\dot r$, for $s \in (0,L)$ and $p(0) = m_1\dot r(0)$, $p(L) = m_2\dot r(L)$. The associated Hamiltonian function is then

$$H(\boldsymbol{r}, \boldsymbol{p}) = \langle\langle \boldsymbol{p}, \dot{\boldsymbol{r}} \rangle\rangle - \mathscr{L} = \tfrac{1}{2} \langle \boldsymbol{p}, \boldsymbol{p} \rangle_{T^*Q} + V(\boldsymbol{r}). \tag{5}$$

To find Hamilton's equation, we construct the *symplectic structure* on the infinite-dimensional space T^*Q. This construction can be obtained from the canonical one in the context of our pairings. In fact, the symplectic form σ on T^*Q is simply

$$\sigma_{(\boldsymbol{r},\boldsymbol{p})}((\delta \boldsymbol{r}_1, \delta \boldsymbol{p}_1), (\delta \boldsymbol{r}_2, \delta \boldsymbol{p}_2)) = \langle\langle \delta \boldsymbol{p}_2, \delta \boldsymbol{r}_1 \rangle\rangle - \langle\langle \delta \boldsymbol{p}_1, \delta \boldsymbol{r}_2 \rangle\rangle,$$

where $\delta \boldsymbol{p}_1$ and $\delta \boldsymbol{p}_2$ are in $T^*_{\boldsymbol{r}}Q$. The Hamiltonian vector field corresponding to the Hamiltonian H is the vector field X_H on T^*Q satisfying

$$\sigma(X_H, Y) = dH(Y) \quad \text{for all } Y \in TT^*Q.$$

Thus Hamilton's equation is $d/dt(\boldsymbol{r},\boldsymbol{p}) = X_H(\boldsymbol{r},\boldsymbol{p})$ interpreted in the sense that

$$\sigma(X_H, Y) = \sigma\left(\frac{d}{dt}(\boldsymbol{r}, \boldsymbol{p}), Y\right) \quad \text{for all } Y \in TT^*Q. \tag{6}$$

With this symplectic structure, the Poisson bracket on $C^\infty(T^*Q, \mathbb{R})$ can be defined by

$$\{F, G\} = \sigma(X_F, X_G) = \boldsymbol{D_r}F \cdot \boldsymbol{D_p}G - \boldsymbol{D_p}F \cdot \boldsymbol{D_r}G,$$

where $F, G \in C^\infty(T^*Q, \mathbb{R})$, and $\boldsymbol{D_r}$, $\boldsymbol{D_p}$ are the Fréchet derivatives with respect to $\boldsymbol{r}$ and $\boldsymbol{p}$. The equations of motion can be further written in the Poisson bracket form as

$$\dot{F} = \{F, H\} \quad \text{for all } F \in C^\infty(T^*Q, \mathbb{R}). \tag{7}$$

The symmetry of the system and the corresponding reduction can be then discussed in terms of the Hamiltonian structure.

3. The Reduced Energy-Momentum Method

In this section, we review some basic notions about simple mechanical systems with symmetry and the reduced energy-momentum method.

A simple mechanical system with symmetry (cf. [AM]) is a quadruple (Q, K, V, G), where Q is the configuration manifold, K is a Riemannian metric, V is the potential function, and G is the symmetry group acting on Q. With this symmetry, the dynamics on T^*Q can be reduced to a reduced dynamics on T^*Q/G by the equivalence classes of group orbits. Let $\pi : T^*Q \to T^*Q/G$ be the canonical projection. A point $\boldsymbol{p_r} \in T^*Q$ is a *relative equilibrium* if $\pi(\boldsymbol{p_r})$ is an equilibrium of the reduced dynamics. A relative equilibrium $(\boldsymbol{p_r})_e$ is *relatively stable* if $\pi((\boldsymbol{p_r})_e)$ is a stable equilibrium of the reduced dynamics in the sense of Lyapunov. The relative stability of relative equilibria can be determined by applying the energy-momentum method. For simple mechanical systems with symmetry, we may further take advantage of the geometric structure of the system and apply the *reduced energy-momentum method*; cf. [SLM, SPM].

For a simple mechanical system with symmetry, the associated energy function $H : T^*Q \to \mathbb{R}$ and the momentum map $J : T^*Q \to \mathscr{G}^*$, where $\mathscr{G}$ is the Lie algebra corresponding to the Lie group G and $\mathscr{G}^*$ is its dual space, are

$$H(p_x) = \tfrac{1}{2}\langle p_x, p_x\rangle + V(x),$$
$$\langle J(p_x), \xi\rangle = \langle\langle p_x, \xi_Q(x)\rangle\rangle,$$

respectively, where $\langle\cdot,\cdot\rangle$, $\langle\langle\cdot,\cdot\rangle\rangle$ are defined through the Riemannian metric K, where $\xi \in \mathscr{G}$, and where ξ_Q is the associated infinitesimal generator of the group action on Q. It can be further verified that both H and $\langle J, \xi\rangle$ are conserved quantities along the trajectories of motion. Moreover, because of the symmetry, they are invariant under the group action. Thus, if the projected function $\bar{H}_\xi$ on T^*Q/G, defined by

$$\bar{H}_\xi \circ \pi = H + \langle J, \xi\rangle \triangleq H_\xi, \tag{8}$$

is positive-definite at relative equilibria, then the relative stability can be determined by constructing a Lyapunov function. The reduced energy-momentum method is designed to check this condition in a systematic manner. This process can actually be greatly simplified by respecting the inherent structure of simple mechanical systems with symmetry. We first introduce some notation.

Let the Legendre transformation be denoted by $\boldsymbol{FL} : TQ \to T^*Q$. With the embedding $\mathscr{Z} : Q \times \mathscr{G} \to T^*Q$ defined by

$$\mathscr{Z}(x,\eta) = (x, \boldsymbol{FL}(\eta_Q(x))),$$

the *induced energy-momentum map* can be defined by $\tilde{H}_\xi = H_\xi \circ \mathscr{Z}$. The essence of the reduced energy-momentum method lies in studying the positive-definiteness of this induced energy-momentum map on the space $Q \times \mathscr{G}$. In fact, corresponding to the group action on Q and the adjoint action of G on $\mathscr{G}$, a symmetry may be constructed by a G-action on $Q \times \mathscr{G}$ defined as

$$\begin{aligned} \Psi : G \times (Q \times \mathscr{G}) &\to Q \times \mathscr{G}, \\ (g, (x,\eta)) &\mapsto (g \cdot x, \mathrm{Ad}_g\,\eta), \end{aligned} \tag{9}$$

where Ad is the adjoint action. The invariance of the induced energy-momentum map under this action can be checked.

The *locked inertia tensor* $\boldsymbol{I}_{\text{lock}}(x) : \mathscr{G} \to \mathscr{G}^*$ of a simple mechanical system with symmetry can be defined by

$$\langle \xi, \boldsymbol{I}_{\text{lock}}(x)\eta\rangle \triangleq K(x)(\xi_Q(x), \eta_Q(x)) \tag{10}$$

for $\xi, \eta \in \mathscr{G}$. For the Lie group G, the *isotropy subgroup* associated with $\mu \in \mathscr{G}^*$ is defined by

$$G_\mu = \{g \in G : \mathrm{Ad}_g^*\mu = \mu\}, \tag{11}$$

with the associated Lie algebra

$$\mathscr{G}_\mu = \{\eta \in \mathscr{G} : \mathrm{ad}_\eta^*\mu = 0\}. \tag{12}$$

The block-diagonalization technique described in [SLM, WK] cannot be directly applied to the system considered in this paper, since the locked-inertia tensor is not invertible. Instead, the reduced energy-momentum method must be used in its more general form. We now outline the method.

0. Pick $\xi \in \mathscr{G}$.
1. Find x_e such that $DV_\xi(x_e) = 0$, where V_ξ is the augmented potential defined by

$$V_\xi(x) = V(x) - \tfrac{1}{2}\langle \xi_Q(x), \xi_Q(x)\rangle_{TQ}. \tag{13}$$

2. Compute the *premomentum map* $\tilde{J} : Q \times \mathscr{G} \to \mathscr{G}^*$,

$$\tilde{J}(x,\eta) = \boldsymbol{I}_{\text{lock}}(x)\eta,$$

and let $\mu_e = \tilde{J}(x_e, \xi)$.
3. Find the kernel of $D\tilde{J}$ at (x_e, ξ), i.e., $\ker D\tilde{J}(x_e,\xi) \in T_{(x_e,\,\xi)}(Q \times \mathscr{G})$.
4. Compute the isotropy subgroup G_{μ_e} and the associated tangent space of the group orbit $T_{(x_e,\,\xi)}(G_{\mu_e} \cdot (x_e, \xi))$.
5. Find the subspace $\mathscr{S}$ such that

$$\ker D\tilde{J}(x_e,\xi) = \mathscr{S} \oplus T_{(x_e,\,\xi)}(G_{\mu_e} \cdot (x_e,\xi)).$$

6. Check the positive-definiteness of the second variation of the induced energy-momentum map $\tilde{H}_\xi$ on the space $\mathscr{S}$. The second variation of $\tilde{H}_\xi$ can be computed from the formula

$$D^2\tilde{H}_\xi(x_e,\xi)\cdot(\delta x_1,\eta_1)\cdot(\delta x_2,\eta_2) = \langle \eta_1, \boldsymbol{I}_{\text{lock}}(x_e)\eta_2\rangle + D^2V_\xi(x_e)\cdot\delta x_1\cdot\delta x_2. \tag{14}$$

The relative equilibrium $(x_e, \boldsymbol{FL}(\xi_Q(x_e)))$ is relatively stable if the associated quadratic form on $\mathscr{S}$ is positive-definite.

4. Symmetry and Relative Equilibria

The canonical action of the rotation group $SO(3)$ on the phase space is considered in this section and the reduced dynamics is examined. Let $A \in SO(3)$ act on Q according to the rule: $A \cdot \boldsymbol{r} = A\boldsymbol{r}$ for $\boldsymbol{r} \in Q$. This action can be lifted to the phase space T^*Q by $A(\boldsymbol{r},\boldsymbol{p}) = (A\boldsymbol{r}, A\boldsymbol{p})$. The Hamiltonian in (5) is easily seen to be invariant under this action. Hence, the theory of *mechanical systems with symmetry* can be applied. In fact, the system under consideration is a simple mechanical system with symmetry (Q, K, V, G), where the Riemannian metric K is defined through the inner product in (4), the potential energy V is in (2), and $G = SO(3)$.

Since, for our case, the symmetry group is the rotation group, the reduced dynamics is the dynamics of (3) reduced to the reduced phase space which 'ignores' uniformly rotating motions. The relative equilibrium is the orbit of some phase point in uniformly rotating motion, which has been called the *steady motion* in engineering literature.

Let $\xi \in \mathbb{R}^3$ be an arbitrary vector. The *augmented potential* defined in (13) can be computed as

$$V_\xi(\boldsymbol{r}) = V(\boldsymbol{r}) - \tfrac{1}{2}\langle \xi \times \boldsymbol{r}, \xi \times \boldsymbol{r}\rangle_{TQ}. \tag{15}$$

It has been shown that relative equilibria can be characterized by the critical points of V_ξ for some ξ; cf. [P]. The first variation of V_ξ is computed next; it leads to the following conditions for relative equilibria:

$$DV_\xi(\boldsymbol{r}_e) \equiv 0 \Longleftrightarrow \begin{cases} \frac{\mu\rho}{|\boldsymbol{r}|^3}\boldsymbol{r} - \left(W'(|\boldsymbol{r}_s|)\frac{\boldsymbol{r}_s}{|\boldsymbol{r}_s|}\right)_s - \rho\hat{\xi}^T\hat{\xi}\boldsymbol{r} & = 0, \quad s \in (0, L), \\ \left(\frac{\mu m_1}{|\boldsymbol{r}|^3}\boldsymbol{r} - W'(|\boldsymbol{r}_s|)\frac{\boldsymbol{r}_s}{|\boldsymbol{r}_s|} - m_1\hat{\xi}^T\hat{\xi}\boldsymbol{r}\right)\Big|_{s=0} & = 0, \\ \left(\frac{\mu m_2}{|\boldsymbol{r}|^3}\boldsymbol{r} + W'(|\boldsymbol{r}_s|)\frac{\boldsymbol{r}_s}{|\boldsymbol{r}_s|} - m_2\hat{\xi}^T\hat{\xi}\boldsymbol{r}\right)\Big|_{s=L} & = 0, \end{cases} \tag{16}$$

where ^{T} denotes the transpose of a matrix. Here the operator $\hat{}$ represents the natural isomorphism between $\mathbb{R}^3$ and the space of 3×3 skew-symmetric matrices $so(3)$ according to the rule

$$\widehat{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \tag{17}$$

Conditions (16) can be also obtained by substituting

$$\boldsymbol{r}(t) = e^{t\hat{\xi}}\boldsymbol{r}_e$$

into (3) and deriving conditions on $\boldsymbol{r}_e$, i.e., the relative equilibrium corresponds to the orbit of $\boldsymbol{r}_e$ rotating about ξ uniformly. This is yet another way of observing the nature of relative equilibria. Finding the configurations for relative equilibria necessitates solving (16), which is a nonlinear boundary-value problem for ordinary differential equations. Some special solutions can be obtained by also assuming that the solution is radial, and that $\xi = \omega e_3$, $\omega > 0$. Let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ be a (body) frame which rotates about ξ uniformly. It is then assumed for the radial case that

$$\boldsymbol{r}_e(s) = \alpha(s)\boldsymbol{b}_2, \quad \boldsymbol{r}_e(0) = a\boldsymbol{b}_2, \quad \boldsymbol{r}_e(L) = b\boldsymbol{b}_2, \quad 0 < a < b, \tag{18}$$

which are fixed relative to that body frame. For such a specific case, the problem of finding relative equilibria reduces to

$$\begin{aligned} W'(\alpha_s)_s - \left(\frac{\mu}{|\alpha|^3} - \omega^2\right)\rho\alpha &= 0, \quad s \in (0, L), \\ W'(\alpha_s(0)) &= m_1\left(\frac{\mu}{|a|^3} - \omega^2\right)a, \\ W'(\alpha_s(L)) &= -\, m_2\left(\frac{\mu}{|b|^3} - \omega^2\right)b. \end{aligned} \tag{19}$$

Solutions of (19) give rise to the so-called *radial relative equilibria*. In the following discussion we assume that

$$\rho(s) \equiv \text{a constant}. \tag{20}$$

For simplicity, we let

$$x = \alpha, \quad y = \alpha_s, \quad \kappa = \rho\mu, \quad \tau = \rho\omega^2,$$
$$\kappa_1 = m_1\mu, \quad \tau_1 = m_1\omega^2, \quad \kappa_2 = m_2\mu, \quad \tau_2 = m_2\omega^2.$$

Equation (19) can be rewritten as

$$x_s = y,$$
$$(W'(y))_s = \kappa x^{-2} - \tau x, \quad s \in (0, L), \tag{21}$$
$$L(x, y) = (0,0), \tag{22}$$

where

$$L(x, y) := (W'(y(0)) - \kappa_1 x(0)^{-2} + \tau_1 x(0), W'(y(L)) + \kappa_2 x(L)^{-2} - \tau_2 x(L)).$$

Equation (21) is an integrable system. However, determining whether any of its solutions also satisfies the nonlinear boundary condition (22) is not an easy task. A phase-plane analysis is attempted here. A first integral for (21) can be found to be

$$U(y) + \kappa x^{-1} + \tfrac{1}{2}\tau x^2 = \text{constant}, \tag{23}$$

where $U(y)$ is such that $U'(y) = yW''(y)$. To seek a solution of (21), (22) in the first quadrant of the (x, y)-plane, we further assume that

$$x > 0, \ y > 0, \ W'(1) = 0, \ W'' > 0. \tag{24}$$

Based on these assumptions, we make the following observations:

OB1. The two curves $C_1 : W'(y) - \kappa_1 x^{-2} + \tau_1 x = 0$, $C_2 : W'(y) + \kappa_2 x^{-2} - \tau_2 x = 0$, which contain the boundary values, intersect at $(\sqrt[3]{\mu/\omega^2}, 1)$. Note that W' is strictly monotonic, so that $(W')^{-1}$ (the inverse function) exists.

OB2. On C_1, y is strictly decreasing in x, and on C_2, y is strictly increasing in x because

$$W''(y)\frac{dy}{dx} = -2\kappa_1 x^{-3} - \tau_1 < 0 \quad \text{on } C_1,$$

$$W''(y)\frac{dy}{dx} = 2\kappa_2 x^{-3} + \tau_2 > 0 \qquad \text{on } C_2.$$

OB3. On the portion $\{(x, y) \in C_1 | x < \sqrt[3]{\mu/\omega^2}\}$, the vector field is in the $(+,+)$ direction (i.e., $\nearrow$); while on $\{(x, y) \in C_2 | x > \sqrt[3]{\mu/\omega^2}\}$, the vector field is in the $(+,-)$ direction (i.e., $\searrow$).

From these observations, it can be verified that a trajectory starting at $(x_0, y_0) \in C_1$, along some integral curve, moves to a point $(x_f, y_f) \in C_2$ (see Figure 2). Such a trajectory with travel time L is a solution of (21), (22). To justify the existence of this solution, one needs to examine the time-map further. Denoting the inverse function of U by U^{-1} and integrating (23) from a point $(p, (W')^{-1}(\kappa_1 p^{-2} - \tau_1 p)) \in C_1$ along the integral curve, we obtain the time-map

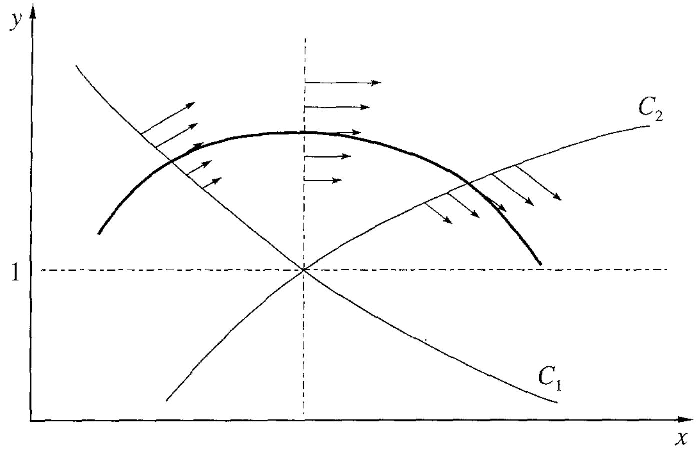

Fig. 2. The phase curve corresponding to the solution.

$$s_{B(p)} - s_p = \int_p^{B(p)} \frac{dx}{U^{-1}(c - \frac{\kappa}{x} - \frac{1}{2}\tau x^2)}, \tag{25}$$

where $c = U(p) + \kappa p^{-1} + \frac{1}{2}\tau p^2$ and $B(p)$ satisfies

$$c = U \circ (W')^{-1}(-\kappa_2 B(p)^{-2} + \tau_2 B(p)) + \kappa B(p)^{-1} + \tfrac{1}{2}\tau B(p)^2.$$

The time-map analysis is complicated even for simple forms of W. Moreover, the parameter ω varies with respect to the position of the string. Thus, the first integral (23) changes for different initial conditions. In Appendix II, numerical computations are performed for the specific W which characterizes linearly elastic strings. The reference length L of the string can be chosen so that the end point reaches C_2; hence a solution arises.

Some interesting problems here require further investigations:

(i) The existence of solutions for (16). Specifically, a concrete method such as a variational method, instead of the degree-theory method, is desired.

(ii) An analytical result on the time-map (25).

5. Stability Analysis

The existence of some radial $SO(3)$-relative equilibria was shown in the previous section. We now study their stability properties. The reduced energy-momentum method discussed in Section 3 is employed here to explore the stability properties of the radial relative equilibria. The momentum map $\boldsymbol{J} : T^*Q \longrightarrow so(3)^*$ is canonical:

$$\boldsymbol{J}(\boldsymbol{p_r})(\hat{\xi}) = \langle\langle \boldsymbol{p_r}, \hat{\xi}_Q(\boldsymbol{r})\rangle\rangle,$$

where $\hat{\xi} \in so(3)$ and its infinitesimal generator on Q is $\hat{\xi}_Q(\boldsymbol{r}) = d/d\varepsilon|_{\varepsilon=0} \exp(\varepsilon\hat{\xi})\, \boldsymbol{r} = \xi \times \boldsymbol{r}$. Hence the *augmented Hamiltonian* is

$$H_\xi(\boldsymbol{r}, \boldsymbol{p}) = H(\boldsymbol{r},\boldsymbol{p}) + \boldsymbol{J}(\boldsymbol{p_r})(\hat{\xi}) = H(\boldsymbol{r},\boldsymbol{p}) + \langle\langle \boldsymbol{p_r}, \xi \times \boldsymbol{r}\rangle\rangle.$$

The *locked inertia tensor* $\boldsymbol{I}_{\text{lock}}(\boldsymbol{r}) : so(3) \longrightarrow so(3)^*$ is found by the following computation:

$$\begin{aligned}
\langle\langle \boldsymbol{I}_{\text{lock}}(\boldsymbol{r})(\hat{\eta}), \hat{\xi}\rangle\rangle_{so(3)} &= \langle \hat{\eta}_Q(\boldsymbol{r}), \hat{\xi}_Q(\boldsymbol{r})\rangle_{TQ} \\
&= \langle \eta \times \boldsymbol{r}, \xi \times \boldsymbol{r}\rangle_{TQ} \\
&= \int_0^L \rho\langle \eta \times \boldsymbol{r}, \xi \times \boldsymbol{r}\rangle ds + m_1\langle \eta \times \boldsymbol{r}, \xi \times \boldsymbol{r}\rangle\Big|_{s=0} \\
&\quad + m_2\langle \eta \times \boldsymbol{r}, \xi \times \boldsymbol{r}\rangle\Big|_{s=L} \\
&= \int_0^L \rho\langle \hat{\boldsymbol{r}}\eta, \hat{\boldsymbol{r}}\xi\rangle ds + m_1\langle \hat{\boldsymbol{r}}\eta, \hat{\boldsymbol{r}}\xi\rangle\Big|_{s=0} + m_2\langle \hat{\boldsymbol{r}}\eta, \hat{\boldsymbol{r}}\xi\rangle\Big|_{s=L} \\
&= \left\langle\left\langle \eta, \left(\int_0^L \rho\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}} ds + m_1\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=0} + m_2\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=L}\right)\xi\right\rangle\right\rangle_{\mathbb{R}^3} \\
&= \langle\langle \eta, \boldsymbol{I}^0_{\text{lock}}(\boldsymbol{r}) \cdot \xi\rangle\rangle_{\mathbb{R}^3},
\end{aligned}$$

where

$$\boldsymbol{I}^0_{\text{lock}}(\boldsymbol{r}) = \int_0^L \rho\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}} ds + m_1\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=0} + m_2\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=L},$$

and $\xi, \eta \in \mathbb{R}^3$. In particular, for the radial equilibria (18), we have

$$\boldsymbol{I}^0_{\text{lock}}(\boldsymbol{r}_e) = \left(\int_0^L \rho\alpha^2 ds + m_1a^2 + m_2b^2\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Accordingly, the locked inertia tensor is *not* invertible at such radial equilibria. The degeneracy of the locked inertia tensor comes from the special configuration of the tether system at the radial relative equilibria. In fact, the moment of inertia of the system with respect to the rotation about the $\boldsymbol{b}_2$-axis is zero at these relative equilibria. We note that this situation is different from that for systems possessing body symmetry. In general, the TSS does not have body symmetry, and the techniques for further reducing the system corresponding to body symmetry cannot be used. On the other hand, due to this degeneracy, the block-diagonalization result in [SLM] cannot be directly applied; however, the reduced energy-momentum method outlined in Section 3 is still applicable.

Define

$$I_e = \int_0^L \rho\alpha^2 ds + m_1a^2 + m_2b^2. \tag{27}$$

The momentum mapping at a radial equilibrium can be expressed as

$$\mu_e = I_e \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \tilde{J}(\boldsymbol{r}_e, \xi),$$

where $\tilde{J}$ is the premomentum map (cf. Section 3). We now need to find the kernel space of $D\tilde{J}$ at radial equilibria. It can be verified that for $\delta \boldsymbol{r} \in T_{\boldsymbol{r}}Q$ and $\widehat{\delta\eta} \in \mathscr{G}$,

$$D\tilde{J}(\boldsymbol{r},\eta)(\delta\boldsymbol{r},\widehat{\delta\eta}) = \int_0^L \rho(2\hat{\boldsymbol{r}}\hat{\eta} - \hat{\eta}\hat{\boldsymbol{r}})\delta\boldsymbol{r}ds + m_1[(2\hat{\boldsymbol{r}}\hat{\eta} - \hat{\eta}\hat{\boldsymbol{r}})\delta\boldsymbol{r}]\Big|_{s=0} + m_2[(2\hat{\boldsymbol{r}}\hat{\eta} - \hat{\eta}\hat{\boldsymbol{r}})\delta\boldsymbol{r}]\Big|_{s=L}$$
$$+ \left(\int_0^L \rho\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}ds + m_1\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=0} + m_2\hat{\boldsymbol{r}}^T\hat{\boldsymbol{r}}\Big|_{s=L}\right)\delta\eta. \tag{28}$$

At radial equilibria, we use $\delta\boldsymbol{r} = (\delta r_1, \delta r_2, \delta r_3)$ and $\delta\eta = (\delta\eta_1, \delta\eta_2, \delta\eta_3)$ to write (28) as

$$D\tilde{J}(\boldsymbol{r}_e,\xi)(\delta\boldsymbol{r},\widehat{\delta\eta}) = \begin{pmatrix} I_e\delta\eta_1 \\ -\int_0^L \rho\alpha\omega\delta r_3 ds - m_1 a\omega\delta r_3(0) - m_2 b\omega\delta r_3(L) \\ I_e\delta\eta_3 + 2\int_0^L \rho\alpha\omega\delta r_2 ds + 2m_1 a\omega\delta r_2(0) + 2m_2 b\omega\delta r_2(L) \end{pmatrix}.$$

As a result, the kernel of $D\tilde{J}(\boldsymbol{r}_e,\xi)$ can be expressed as

$$\ker(D\tilde{J}(\boldsymbol{r}_e,\xi)) = \Bigg\{(\delta\boldsymbol{r},\delta\eta) : \delta\eta_1 = 0,\ m_1 a\delta r_3(0) + m_2 b\delta r_3(L) + \int_0^L \rho\alpha\delta r_3 ds = 0,$$
$$\delta\eta_3 = -\frac{2\omega}{I_e}\left(m_1 a\delta r_2(0) + m_2 b\delta r_2(L) + \int_0^L \rho\alpha\delta r_2 ds\right)\Bigg\}. \tag{29}$$

Next we need to find the tangent space of the group orbit $T_{(\boldsymbol{r},\eta)}(G_\mu\cdot(\boldsymbol{r},\eta))$, which is a subspace of $T_{(\boldsymbol{r},\eta)}(Q\times\mathscr{G})$. The isotropy subgroup G_μ can be found from

$$G_\mu = \{B \in SO(3) : Ad^*_B\hat{\mu} = \hat{\mu}\}.$$

Since $\mu_e = I_e\omega\boldsymbol{e}_3$, we have

$$G_{\mu_e} = \{e^{\hat{\boldsymbol{q}}} : \boldsymbol{q} \text{ is parallel to } \boldsymbol{e}_3\}.$$

Thus we immediately obtain the tangent space

$$T_{(\boldsymbol{r}_e,\xi)}(G_{\mu_e}\cdot(\boldsymbol{r}_e,\xi)) = \{(\beta\boldsymbol{e}_3\times\boldsymbol{r}_e,\ 0) : \beta\in\mathbb{R}\} = \{(-\beta\alpha\boldsymbol{e}_1,\ 0) : \beta\in\mathbb{R}\}. \tag{30}$$

Obviously, $T_{(\boldsymbol{r}_e,\xi)}(G_{\mu_e}\cdot(\boldsymbol{r}_e,\xi))$ is a subspace of $\ker(D\tilde{J}(\boldsymbol{r}_e,\xi))$. In light of the symmetry, the augmented Hamiltonian is invariant on the orbit generated by the isotropy subgroup. We therefore need only check the second variation of the augmented Hamiltonian on a subspace $\mathcal{S}$ such that

$$\ker(D\tilde{J}(\boldsymbol{r}_e,\xi)) = \mathcal{S} \oplus T_{(\boldsymbol{r}_e,\xi)}(G_{\mu_e}\cdot(\boldsymbol{r}_e,\xi)).$$

From (29) and (30), the space $\mathcal{S}$ can be written as

$$\mathcal{S} = \Bigg\{ (\delta\boldsymbol{r},\delta\eta) = (0,\delta r_2,\delta r_3,0,\delta\eta_2,\delta\eta_3) :$$

$$m_1 a\delta r_3(0) + m_2 b\delta r_3(L) + \int_0^L \rho\alpha\delta r_3 ds = 0,$$

$$\delta\eta_3 = -\frac{2\omega}{I_e}\left(m_1 a\delta r_2(0) + m_2 b\delta r_2(L) + \int_0^L \rho\alpha\delta r_2 ds\right)\Bigg\}. \qquad (31)$$

Although the variation $\delta\eta_2$ is arbitrary in $\mathcal{S}$, the degeneracy of the locked inertia tensor prohibits it from entering into our considerations regarding stability.

The last step in the reduced energy-momentum method is to check the positive-definiteness of the second variation of $\tilde{H}_\xi$ on $\mathcal{S}$. The second variation is now computed by the formula (cf. (14))

$$D^2\tilde{H}_\xi(\boldsymbol{r}_e,\xi)\cdot(\delta\boldsymbol{r}_1,\delta\eta)\cdot(\delta\boldsymbol{r}_2,\delta\zeta) = \langle\delta\eta,\ \boldsymbol{I}_{\text{lock}}(\boldsymbol{r}_e)\delta\zeta\rangle + D^2V_\xi(\boldsymbol{r}_e)\cdot\delta\boldsymbol{r}_1\cdot\delta\boldsymbol{r}_2. \qquad (32)$$

From (15), the second variation of the augmented potential is

$$\begin{aligned}
&D^2V_\xi(\boldsymbol{r}_e)\cdot\delta\boldsymbol{r}_1\cdot\delta\boldsymbol{r}_2\\
&= \frac{d}{d\varepsilon_1}\bigg|_{\varepsilon_1=0}\frac{d}{d\varepsilon_2}\bigg|_{\varepsilon_2=0} V_\xi(\boldsymbol{r}_e+\varepsilon_1\delta\boldsymbol{r}_1+\varepsilon_2\delta\boldsymbol{r}_2)\\
&= \left\langle\frac{\mu}{|\boldsymbol{r}_e|^3}\delta\boldsymbol{r}_1,\delta\boldsymbol{r}_2\right\rangle_{TQ} - \langle\hat{\xi}^T\hat{\xi}\delta\boldsymbol{r}_1,\delta\boldsymbol{r}_2\rangle_{TQ}\\
&\quad -\int_0^L \frac{3\mu\rho}{|\boldsymbol{r}_e|^5}\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_1\rangle\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_2\rangle ds - \frac{3\mu m_1}{|\boldsymbol{r}_e|^5}\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_1\rangle\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_2\rangle\bigg|_{s=0}\\
&\quad -\frac{3\mu m_2}{|\boldsymbol{r}_e|^5}\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_1\rangle\langle\boldsymbol{r}_e,\delta\boldsymbol{r}_2\rangle\bigg|_{s=L}\\
&\quad +\int_0^L\left(\frac{W''(|(\boldsymbol{r}_e)_s|)}{|(\boldsymbol{r}_e)_s|^2} - \frac{W'(|(\boldsymbol{r}_e)_s|)}{|(\boldsymbol{r}_e)_s|^3}\right)\langle(\boldsymbol{r}_e)_s,(\delta\boldsymbol{r}_1)_s\rangle\langle(\boldsymbol{r}_e)_s,(\delta\boldsymbol{r}_2)_s\rangle ds\\
&\quad +\int_0^L\frac{W'(|(\boldsymbol{r}_e)_s|)}{|(\boldsymbol{r}_e)_s|}\langle(\delta\boldsymbol{r}_1)_s,(\delta\boldsymbol{r}_2)_s\rangle ds.
\end{aligned} \qquad (33)$$

We now examine the stability properties. By substituting the conditions for radial equilibria into (33), we obtain the second variation of $\tilde{H}_\xi$ on $\mathcal{S}$:

$$
\begin{aligned}
&D^2\tilde{H}_\xi(\boldsymbol{r}_e,\xi)\cdot(\delta\boldsymbol{r},\delta\eta)\cdot(\delta\boldsymbol{r},\delta\eta)\\
&=\frac{4\omega^2}{I_e}\left(\int_0^L \rho\alpha\delta r_2 ds + m_1 a\delta r_2(0) + m_2 b\delta r_2(L)\right)^2\\
&-m_1\left(\frac{2\mu}{a^3}+\omega^2\right)(\delta r_2(0))^2 - m_2\left(\frac{2\mu}{b^3}+\omega^2\right)(\delta r_2(L))^2\\
&-\int_0^L \rho\left(\frac{2\mu}{|\alpha|^3}+\omega^2\right)(\delta r_2)^2 ds\\
&+\frac{m_1\mu}{a^3}(\delta r_3(0))^2+\frac{m_2\mu}{b^3}(\delta r_3(L))^2+\int_0^L \rho\frac{\mu}{|\alpha|^3}(\delta r_3)^2 ds\\
&+\int_0^L\left(W''(\alpha_s)-\frac{W'(\alpha_s)}{\alpha_s}\right)(\delta r_2)_s^2\, ds+\int_0^L \frac{W'(\alpha_s)}{\alpha_s}|\delta\boldsymbol{r}_s|^2 ds. \qquad (34)
\end{aligned}
$$

The relation of $\delta\boldsymbol{r}$ and $\delta\eta$ on $\mathscr{S}$ (cf. (31)) has been used here to express $\langle\delta\eta,\ \boldsymbol{I}_{\text{lock}}(\boldsymbol{r}_e)\delta\eta\rangle$ in terms of $\delta\boldsymbol{r}$. It is this positive term which makes the reduced energy-momentum method yield conditions weaker than those of the classical energy analysis.

According to the reduced energy-momentum method, conditions on the system's parameters can be obtained by requiring the quadratic form (34) to be positive-definite. The second variation can be rewritten by performing the integration by parts on δr_2-terms of the last integral in (34), i.e.,

$$
\begin{aligned}
&D^2\tilde{H}_\xi(\boldsymbol{r}_e,\xi)\cdot(\delta\boldsymbol{r},\delta\eta)\cdot(\delta\boldsymbol{r},\delta\eta)\\
&=\frac{4\omega^2}{I_e}\left(\int_0^L \rho\alpha\delta r_2 ds + m_1 a\delta r_2(0) + m_2 b\delta r_2(L)\right)^2\\
&-\left(m_1\left(\frac{2\mu}{a^3}+\omega^2\right)\delta r_2(0)+W''(\alpha_s)(\delta r_2)_s(0)\right)\delta r_2(0)\\
&-\left(m_2\left(\frac{2\mu}{b^3}+\omega^2\right)\delta r_2(L)-W''(\alpha_s)(\delta r_2)_s(L)\right)\delta r_2(L)\\
&-\int_0^L\left(\rho\left(\frac{2\mu}{|\alpha|^3}+\omega^2\right)\delta r_2+\left(W''(\alpha_s)(\delta r_2)_s\right)_s\right)\delta r_2 ds\\
&+\frac{m_1\mu}{a^3}(\delta r_3(0))^2+\frac{m_2\mu}{b^3}(\delta r_3(L))^2+\int_0^L \rho\frac{\mu}{|\alpha|^3}(\delta r_3)^2 ds\\
&+\int_0^L \frac{W'(\alpha_s)}{\alpha_s}|(\delta r_1)_s|^2 ds+\int_0^L \frac{W'(\alpha_s)}{\alpha_s}|(\delta r_3)_s|^2 ds. \qquad (35)
\end{aligned}
$$

Note that $\delta r_1 = 0$ on $\mathcal{S}$. Hence, the terms of δr_2 and δr_3 are left to be studied. From the definition of $\langle\ ,\ \rangle_{TQ}$, an appropriate Hilbert space is defined for δr_3 (or δr_2) by:

Definition 5.1. The Hilbert space $BH([0,L])$ is the completion of $C([0,L])$ under the norm $|\cdot|_b$ induced by the inner product

$$\langle f,g\rangle_b = \int_0^L \rho f g\, ds + m_1 f(0)g(0) + m_2 f(L)g(L) \quad \forall f,\, g. \tag{36}$$

Since $W'(\alpha_s)/\alpha_s > 0$ (recall that $\alpha_s > 1$), the terms including δr_3 are clearly positive-definite on $BH([0,L])$. For δr_2, the following Sturm-Liouville problem must be solved:

$$\begin{aligned} -\left(\frac{2\mu}{|\alpha|^3} + \omega^2\right)\delta r_2 - \left(\frac{1}{\rho}W''(\alpha_s)(\delta r_2)_s\right)_s &= \lambda\delta r_2, \quad s \in (0,L), \\ -\left(\frac{2\mu}{a^3} + \omega^2\right)\delta r_2(0) - \frac{1}{m_1}W''(\alpha_s)(\delta r_2)_s(0) &= \lambda\delta r_2(0), \\ -\left(\frac{2\mu}{b^3} + \omega^2\right)\delta r_2(L) + \frac{1}{m_2}W''(\alpha_s)(\delta r_2)_s(L) &= \lambda\delta r_2(L), \end{aligned} \tag{37}$$

where λ is the eigenvalue to be found. This type of Sturm-Liouville problem has been previously studied in [Ch] and is known to have real eigenvalues $\lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$. Problem (37) is denoted here by $A(\delta r_2) = \lambda(\delta r_2)$ and the δr_2 terms in (35) are rewritten as

$$4\omega^2\left(\left\langle\frac{\alpha}{|\alpha|_b}, \delta r_2\right\rangle_b\right)^2 + \langle A(\delta r_2), (\delta r_2)\rangle_b. \tag{38}$$

Let $\phi_0, \phi_1, \phi_2, \ldots$ be the associated eigenfunctions for $\lambda_0, \lambda_1, \lambda_2, \ldots$. Define $V_0 =$ span $\{\phi_0\}$ and set

$$\delta r_2 = a_0\phi_0 + \phi, \qquad \frac{\alpha}{|\alpha|_b} = b_0\phi_0 + \phi_\alpha,$$

where $\phi = \sum_{i=1}^{\infty} a_i\phi_i$ and $\phi_\alpha \in V_0^\perp$ (the orthogonal complement of V_0 in $BH([0,L])$). Then

$$\begin{aligned} 4\omega^2\left(\left\langle\frac{\alpha}{|\alpha|_b}, \delta r_2\right\rangle_b\right)^2 &= 4\omega^2(a_0 b_0 + \langle\phi_\alpha, \phi\rangle_b)^2 \\ &= 4\omega^2(a_0 b_0 + |\phi|_b K_\alpha)^2, \end{aligned}$$

where $K_\alpha = |\phi_\alpha|_b \cos\theta$, and $\cos\theta = \langle\phi_\alpha, \phi\rangle_b / |\phi_\alpha|_b \cdot |\phi|_b$. On the other hand, if we assume that $\lambda_1 > 0$, then

$$\langle A(\delta r_2), (\delta r_2)\rangle_b = \sum_{i=0}^{\infty}\lambda_i a_i^2 \geqq \lambda_0 a_0^2 + \lambda_1|\phi|_b^2.$$

Hence (38) is greater than or equal to

$$4\omega^2(a_0, |\phi|_b) \begin{pmatrix} b_0^2 + \dfrac{\lambda_0}{4\omega^2} & K_\alpha b_0 \\ K_\alpha b_0 & K_\alpha^2 + \dfrac{\lambda_1}{4\omega^2} \end{pmatrix} \begin{pmatrix} a_0 \\ |\phi|_b \end{pmatrix}, \tag{39}$$

which is positive-definite if

$$\lambda_0 > \max\left(-4\omega^2 b_0^2, \frac{-b_0^2\lambda_1}{K_\alpha^2 + \dfrac{\lambda_1}{4\omega^2}} \right).$$

Since $K_\alpha^2 \leqq 1$, a sufficient condition for this inequality is

$$\lambda_0 > \frac{-4\omega^2 b_0^2 \lambda_1}{4\omega^2 + \lambda_1}.$$

Consequently, we have the following stability result.

Theorem 5.2. *Let λ_0 and λ_1 be the first and second eigenvalues of* (37), *respectively. If $\lambda_1 > 0$ and $\lambda_0 > -4\omega^2 b_0^2 \lambda_1/(4\omega^2 + \lambda_1)$, then the radial relative equilibrium obtained in Section 4 is formally relatively stable.*

Remarks. 1. Since the global smooth solution may not exist, this formal stability result is rigorous only in the time interval where the solution is smooth.
2. The stability obviously holds when $W''(\alpha_s)$ is so large that $\lambda_0 > 0$. This means that the radial relative equilibrium is relatively stable if the string is stiff enough.
3. In contrast to the analysis by the classical energy method, the method adopted here requires weaker conditions to conclude stability. It can be checked that the first term of (38) (which is always positive) does not appear in the classical energy analysis (cf. [L], where the classical analysis was applied to prove the stability of a radial equilibrium in a uniformly rotating frame).

6. Conclusions

We have studied the equations of motion for a fundamental model of the tethered satellite system. The Cauchy problem was briefly discussed and the Hamiltonian structure for such a system was constructed. The system was found to have an intrinsic symmetry with respect to the group $SO(3)$. Via this symmetry, reduction was performed and the associated relative equilibria were defined. Some relative equilibria were obtained by finding the critical points of the augmented potential function, which consequently led to a nonlinear boundary-value problem. The reduced energy-momentum method was applied to some particular radial equilibria in order to prove their relative stability. The method was found to produce weaker conditions than those obtained via the classical energy method. We proved that if the string is stiff enough, there is stability. The results of this paper on the relative equilibria configuration and its associated stability properties may be applied towards the design and control of tethered satellite systems.

Appendix I. The Cauchy Problem

In this Appendix, we study the Cauchy problem for (3) with

$$W(\Lambda) = \tfrac{1}{2}\Lambda^2. \tag{40}$$

The hyperbolic system becomes

$$\begin{aligned} \rho\ddot{\boldsymbol{r}} &= \boldsymbol{r}_{ss} - \frac{\mu\rho}{|\boldsymbol{r}|^3}\boldsymbol{r}, \quad s \in (0,L), \\ m_1\ddot{\boldsymbol{r}}(0) &= -\frac{\mu m_1}{|\boldsymbol{r}(0)|^3}\boldsymbol{r}(0) + \boldsymbol{r}_s(0), \\ m_2\ddot{\boldsymbol{r}}(L) &= -\frac{\mu m_2}{|\boldsymbol{r}(L)|^3}\boldsymbol{r}(L) - \boldsymbol{r}_s(L), \end{aligned} \tag{41}$$

which is semilinear. To explore the Cauchy problem for the semilinear case, we assume for simplicity that $\rho(s) = 1$. The operator $\diamond$ is defined by

$$\diamond\boldsymbol{r} = \begin{pmatrix} \partial_t^2 - \partial_s^2 \\ \left(\partial_t^2 - \frac{1}{m_1}\partial_s\right)\Big|_{s=0} \\ \left(\partial_t^2 + \frac{1}{m_2}\partial_s\right)\Big|_{s=L} \end{pmatrix} \boldsymbol{r}, \tag{42}$$

which is the linear part of the problem (41). We first show that the inhomogeneous problem $\diamond\boldsymbol{r} = \boldsymbol{f}$ with initial conditions $\boldsymbol{r}(0,s) = \boldsymbol{r}_0(s)$, $\boldsymbol{r}_t(0,s) = \boldsymbol{r}_1(s)$, $s \in [0,L]$, can be uniquely solved. Semigroup theory is then applied to give a local solution of the nonlinear problem. Since the operator $\diamond$ is decoupled for each component of $\boldsymbol{r}$, we only need to study the scalar case, that is,

$$\begin{aligned} \partial_t^2 r - \partial_s^2 r &= f(s,t), \quad s \in (0,L), \\ \left(\partial_t^2 - \frac{1}{m_1}\partial_s\right)\Big|_{s=0}(r) &= f_1(t) = f(0,t), \\ \left(\partial_t^2 + \frac{1}{m_2}\partial_s\right)\Big|_{s=L}(r) &= f_2(t) = f(L,t), \end{aligned} \tag{43}$$

with initial conditions $r(0,s) = r_0(s), r_t(0,s) = r_1(s)$.

The function spaces BH^k are now constructed for problem (43). An inner product on the space of functions having continuous derivatives, $C^k([0,L])$, is defined by

$$\langle f,g\rangle_{bk} \equiv \sum_{i=0}^{k}\int_0^L f^{(i)}g^{(i)}ds + m_1\sum_{i=0}^{k^*} f^{(i)}(0)g^{(i)}(0) + m_2\sum_{i=0}^{k^*} f^{(i)}(L)g^{(i)}(L),$$

where $f,g \in C^k([0,L])$ and $k^* = 0$ if $k = 0$, $k^* = k-1$ if $k = 1,2,\ldots$. Its associated norm is

$$\| f \|_{bk} \equiv (\langle f, \ f\rangle_{bk})^{1/2}.$$

The completion of $C^k([0,L])$ with respect to this norm is denoted by $BH^k([0,L])$. Define the operator $\mathscr{A} : D(\mathscr{A}) \to BH^0([0,L])$, where $D(\mathscr{A}) = BH^2([0,L])$, by

$$\mathscr{A} f(s) = \begin{cases} \partial_s^2 f(s), & s \in (0,L), \\ \frac{1}{m_1}\partial_s f(0), & s = 0, \\ -\frac{1}{m_2}\partial_s f(L), & s = L. \end{cases}$$

Let

$$\tilde{\mathscr{A}} = \begin{pmatrix} 0 & I \\ \mathscr{A} & 0 \end{pmatrix},$$

which is defined on $D(\tilde{\mathscr{A}}) = BH^2([0,L]) \times BH^1([0,L])$. The equation $\partial_t^2 r - \mathscr{A} r = f$ can be written in the form

$$\partial_t \begin{pmatrix} r \\ \partial_t r \end{pmatrix} = \tilde{\mathscr{A}} \begin{pmatrix} r \\ \partial_t r \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

We proceed to show that the operator $\tilde{\mathscr{A}}$ generates a strongly continuous semigroup on the Hilbert space $H = BH^1([0,L]) \times BH^0([0,L])$, where the norm on H is $\| f \|_H^2 = \| f \|_{b1}^2 + \| f \|_{b0}^2$. This is similar to the standard wave equation (cf. [Pz]) and the following similar properties can be obtained:

1. If $\lambda > 0$ and $f \in BH^k([0,L])$, $k \geqq 0$, then $u - \lambda \mathscr{A} u = f$ is uniquely solvable in $BH^{k+2}([0,L])$.
2. For $G = (g_1, g_2) \in C^\infty([0,L]) \times C^\infty([0,L])$ and $\lambda \neq 0$, the equation $(u_1, u_2) - \lambda(u_1, u_2)\tilde{\mathscr{A}}^T = G$ is uniquely solvable in $BH^k([0,L]) \times BH^{k-1}([0,L])$, for all $k \geqq 2$, and

$$\|(u_1,u_2)\|_H \leqq (1 - 2|\lambda|)\|(g_1,g_2)\|_H, \quad 0 < |\lambda| < \tfrac{1}{2}.$$

3. The operator $\tilde{\mathscr{A}}$ is the generator of a strongly continuous semigroup $S(t)$ on H, which satisfies

$$\|S(t)\| \leqq e^{2t}.$$

Remark. In fact, the eigenvalue problem

$$-\frac{d^2}{ds^2}\varphi = \lambda\varphi, \quad s \in (0,L),$$

$$-\frac{1}{m_1}\frac{d}{ds}\bigg|_{s=0} \varphi = \lambda\varphi(0), \quad \frac{1}{m_2}\frac{d}{ds}\bigg|_{s=L} \varphi = \lambda\varphi(L) \tag{44}$$

can be explicitly solved. The eigenvalues λ can be found by solving

$$\sqrt{\lambda}\left(\left(\frac{1}{m_1}\right) + \left(\frac{1}{m_2}\right)\right) \cos\sqrt{\lambda}\, L = \left(\lambda - \left(\frac{1}{m_1 m_2}\right)\right) \sin\sqrt{\lambda}\, L,$$

and the associated eigenfunction is simply a linear combination of $\sin\sqrt{\lambda}s$ and $\cos\sqrt{\lambda}s$. Let φ_i denote the eigenfunction associated with the eigenvalue λ_i. Substituting $r(s,t) = \sum_1^\infty \tau_i(t)\varphi_i(s)$ in (43), we obtain the weighting function $\tau_i(t)$ by solving

$$\frac{d^2}{dt^2}\tau_i + \lambda_i\tau_i = \int_0^L f(s,t)\varphi_i(s)\, ds + m_1 f_1(t)\varphi_i(0) + m_2 f_2(t)\varphi_i(L), \tag{45}$$

with initial conditions. This process produces the unique solution (hence the kernel) for (43).

We can now rewrite (41) in an integral form and can apply the typical fixed-point argument to obtain local existence of the solution if the initial condition satisfies $|r_0| \geqq M > 0$. Note that our potential is singular at $\boldsymbol{r} = 0$; hence, the global existence of finite-speed motion cannot be guaranteed. However, the solution exists as long as the motion is bounded away from zero. For a general density ρ, the same argument follows by properly modifying the eigenstructures. The result is summarized in

Proposition A1. *The Cauchy problem for* (41), *with smooth initial data* $\boldsymbol{r}(0,s) \neq 0$ for all $s \in [0,L]$ *and* $\boldsymbol{r}_t(0,s)$, *has a unique smooth solution as long as the solution is bounded away from* 0.

Appendix II. Linearly Elastic String

We now consider the case of a linearly elastic string, which can be characterized by the stored-energy density function

$$W(|\boldsymbol{r}_s|) = \tfrac{1}{2}EA(|\boldsymbol{r}_s| - 1)^2,$$

where E denotes the modulus of elasticity of the string and A is the area of the cross section. The condition for radial relative equilibria can now be written as (cf. (19))

$$\begin{aligned} EA\alpha_{ss} - \left(\frac{\mu}{|\alpha|^3} - \omega^2\right)\rho\alpha &= 0, \quad s \in (0,L), \\ EA(\alpha_s(0) - 1) &= m_1\left(\frac{\mu}{|a|^3} - \omega^2\right)a, \\ EA(\alpha_s(L) - 1) &= -m_2\left(\frac{\mu}{|b|^3} - \omega^2\right)b, \end{aligned} \tag{46}$$

with the continuity conditions $\alpha(0) = a$ and $\alpha(L) = b$. The time-map method discussed in Section 4 can be used to prove the existence of radial relative equilibria for linearly elastic strings. In particular, the first integral is

$$\tfrac{1}{2}EAy^2 + \kappa x^{-1} + \tfrac{1}{2}\tau x^2 = \text{constant } c, \tag{47}$$

and the time-map becomes

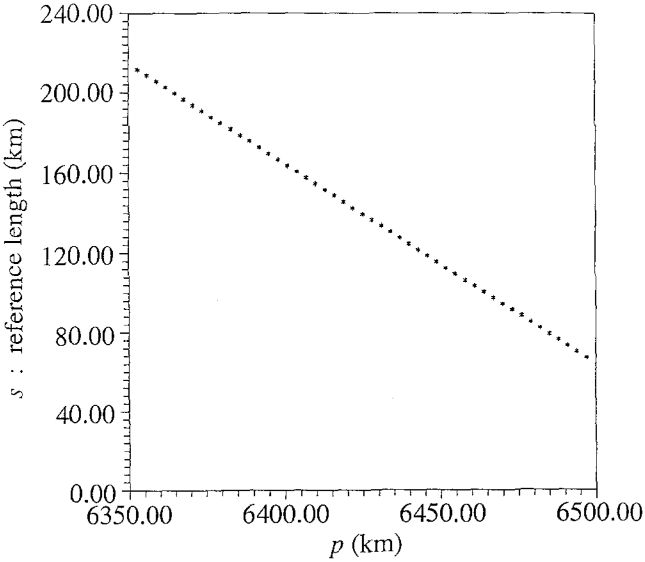

Fig. 3. Some solutions from time-map analysis. The angular rate is $\omega = 1.18936 \times 10^{-3}$ and $m_1 = 10^3$, $m_2 = 10^5$, $\rho = 10$, $EA = 100$.

$$s_{B(p)} - s_p = \int_p^{B(p)} \frac{dx}{\sqrt{\frac{2}{EA}}\sqrt{c - \frac{\kappa}{x} - \frac{1}{2}\tau x^2}}. \tag{48}$$

For a tether with constants m_1, m_2, ρ, and a linearly elastic string of reference length $s_{B(p)}$, constants E, A, we can prove that there is a radial relative equilibrium moving in the gravitational field with angular rate ω with one end of the string on an orbit of radius p and the other on an orbit of radius $B(p)$, by performing the following procedure:

(1) For a given ω, select an appropriate p in the neighborhood of $\sqrt[3]{\mu/\omega^2}$.

(2) Compute

$$c = \frac{1}{2}EA\left[1 + \frac{m_1}{EA}\left(\frac{\mu}{p^3} - \omega^2\right)p\right]^2 + \frac{\kappa}{p} + \frac{1}{2}\tau p^2.$$

(3) Solve

$$\frac{1}{2}EA\left[1 - \frac{m_2}{EA}\left(\frac{\mu}{z^3} - \omega^2\right)z\right]^2 + \frac{\kappa}{z} + \frac{1}{2}\tau z^2 = c$$

for $z = B(p)$.

(4) Compute $s_{B(p)}$ from the time-map formula, assuming $s_p = 0$.

Some numerical solutions are provided in Figure 3. For the stability of these radial relative equilibria, the Sturm-Liouville problem (37) becomes

$$\begin{aligned} -\rho\left(\frac{2\mu}{|\alpha|^3}+\omega^2\right)\delta r_2 - EA(\delta r_2)_{ss} &= \lambda\delta r_2, \\ -m_1\left(\frac{2\mu}{a^3}+\omega^2\right)\delta r_2(0) - EA(\delta r_2)_s(0) &= \lambda\delta r_2(0), \\ -m_2\left(\frac{2\mu}{b^3}+\omega^2\right)\delta r_2(L) + EA(\delta r_2)_s(L) &= \lambda\delta r_2(L), \end{aligned} \tag{49}$$

where λ is the eigenvalue to be obtained. The strategy of designing a stable radial tether therefore lies in finding a suitable EA such that the conditions in Theorem 5.2 are satisfied.

Acknowledgement. This work was partially supported by the National Science Council, R.O.C., under the grants NSC 82-0208-M-002-028 and 82-0208-M-007-049, 82-0208-M-009-023. We thank Professor P. HOLMES for many helpful comments and S.-F. CHANG for assistance with Appendix II.

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(Accepted December 6, 1993)