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Edge Domatic Numbers of Complete n-Partite Graphs*

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Abstract. An edge dominating set of a graph is a set of edges D such that every edge not in D is adjacent to an edge in D. An edge domatic partition of a graph $G = (V, E)$ is a collection of pairwise disjoint edge dominating sets of G whose union is E. The maximum size of an edge domatic partition of G is called the edge domatic number of G. In this paper we study the edge domatic numbers of complete n-partite graphs. In particular, we give exact values for the edge domatic numbers of complete 3-partite graphs and balanced complete n-partite graphs with odd n.

1. Introduction

In this paper all graphs are simple, i.e., finite, undirected, loopless, and without multiple edges. An *edge dominating set* of a graph is a set of edges D such that every edge not in D is adjacent to an edge in D. An *edge domatic partition* of a graph $G = (V, E)$ is a collection of pairwise disjoint edge dominating sets of G whose union is E. The *edge domatic number problem* is to determine the *edge domatic number* $\mathrm{ed}(G)$ of G, which is the maximum size of an edge domatic partition of G. Zelinka [8] showed that $\delta(G) \le \mathrm{ed}(G) \le \delta_e(G) + 1$ where $\delta(G)$ is the minimum degree of a vertex in G and $\delta_e(G)$ is the minimum degree of an edge in G. He also determined the values of $\mathrm{ed}(G)$ when G are circuits, complete graphs, complete bipartite graphs, and trees. Algorithmic results on domatic numbers are also extensively studied in [1, 2, 3, 5, 6].

The purpose of this paper is to study the edge domatic number of a complete n-partite graph $K_{m_1,\ldots,m_n}$ whose parts are $M_1, \ldots, M_n$ of size $m_1, \ldots, m_n$, respectively. For simplicity, we assume $m_1 \le \cdots \le m_n$. For $1 \le i < j \le t$, we denote by E_{ij} the set of all edges between M_i and M_j, i.e., $E_{ij} = \{(a, b)\colon 1 \le a \le m_i \text{ and } 1 \le b \le m_j\}$.

It is well known that $\mathrm{ed}(K_{m_1,m_2}) = m_2$. In general, the exact value of the edge domatic number of a complete n-partite graph with $n \ge 3$ is not easy to find. This

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paper gives exact values for the edge domatic numbers of complete 3-partite graphs and balanced complete n-partite graphs with odd n.

2. Edge Domatic Numbers of Complete 3-Partite Graphs

This section determines the exact value of the edge domatic number $\mathrm{ed}(K_{m_1,m_2,m_3})$ of a complete 3-patite graph K_{m_1,m_2,m_3} with $m_1 \le m_2 \le m_3$.

In a graph G, an edge set A is said to *cover* a vertex v i times if v is incident to at least i edges in A. An edge set A is said to *cover* a vertex set B if every vertex in B is incident to some edge in A. Note that an edge set D is an edge dominating set of $K_{m_1,\ldots,m_n}$ if and only if D covers $\bigcup_{j\neq i} M_j$ for some i.

First of all, we shall establish the lower bound of $\mathrm{ed}(K_{m_1,m_2,m_3})$. To do this, we want to partition $E(K_{m_1,m_2,m_3})$ into as many edge dominating sets as possible. For convenience, we need the following notation. Suppose r, s, t are integers and $1 \le t \le m_i m_j/d_{ij}$ where d_{ij} is the greatest common divisor of m_i and m_j. Denote by $E_{ij}(r,s,t)$ the set $\{(a \bmod m_i, (a+r) \bmod m_j) \in E_{ij}: a = s+1, s+2, \ldots, s+t\}$ and $E_{ij}(r) = E_{ij}(r, 0, m_i m_j/d_{ij})$, where "$x \bmod y$" always results a positive integer between 1 and y.

Lemma 2.1. $|E_{ij}(r,s,t)| = t$. *Also,* $E_{ij}(r,s,m_i)$ *covers* M_i *and* $E_{ij}(r,s,m_j)$ *covers* $M_i \cup M_j$.

Proof. Suppose $(a \bmod m_i, (a+r) \bmod m_j) = (b \bmod m_i, (b+r) \bmod m_j)$ for some $s+1 \le a \le b \le s+t$. Then $b-a$ is a common multiple of m_i and m_j. However $0 \le b-a \le t-1 < m_i m_j/d_{ij}$ where $m_i m_j/d_{ij}$ is the least common multiple of m_i and m_j. So $a = b$ and hence $|E_{ij}(r,s,t)| = t$.

$E_{ij}(r,s,m_i)$ covers M_i since $a \bmod m_i$ ranges over $\{1, 2, \ldots, m_i\}$ when a ranges over $\{s+1, s+2, \ldots, s+m_i\}$. Similar arguments and the fact that $m_i \le m_j$ imply that $E_{ij}(r,s,m_j)$ covers $M_i \cup M_j$. □

Lemma 2.2. E_{ij} *can be partitioned into* $E_{ij}(r)$, $r = 1, 2, \ldots, d_{ij}$.

Proof. Suppose there exist $1 \le r < s \le d_{ij}$ such that $E_{ij}(r) \cap E_{ij}(s) \neq \varnothing$, i.e., $a \equiv b \pmod{m_i}$ and $a + r \equiv b + s \pmod{m_j}$ for some $1 \le a$, $b \le m_i m_j/d_{ij}$. Since d_{ij} is a common divisor of m_i and m_j, $a \equiv b \pmod{d_{ij}}$ and $a + r \equiv b + s \pmod{d_{ij}}$. These imply that $r \equiv s \pmod{d_{ij}}$, in contradiction to $1 \le r < s \le d_{ij}$. □

Lemma 2.3. $\mathrm{ed}(K_{m_1,m_2,m_3}) \ge \begin{cases} m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor, \\ m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor. \end{cases}$

Proof. By Lemma 2.2, we can partition E_{12} into $E_{12}(r)$, $1 \le r \le d_{12}$. By Lemma 2.1, we can further partition each $E_{12}(r)$ into $E_{12}(r, (i-1)m_1, m_1)$, $1 \le i \le m_2/d_{12}$, each of size m_1 and covering M_1 but covering only m_1 vertices of M_2. For simplicity, we denote these m_2 sets by $A_1, \ldots, A_{m_2}$. The idea is for each j to find a subset B_j of E_{23} such that $|B_j| = m_2 - m_1$ and $A_j \cup B_j$ covers M_2.

Suppose $m_2(m_2 - m_1) = \alpha(m_2 m_3/d_{23}) + \beta$ where $0 \le \beta < m_2 m_3/d_{23}$. Note that

β is a multiple of m_2. Let $B = \bigcup_{r=1}^{\alpha} E_{23}(r) \cup E_{23}(\alpha+1, 0, \beta)$. Then B is of size $m_2(m_2 - m_1)$ and covers each vertex of M_2 exactly $m_2 - m_1$ times. Since $E_{12} = \bigcup_{j=1}^{m_2} A_j$ covers each vertex of M_2 exactly m_1 times and each A_j covers each vertex of M_2 at most once, we can partition B into $B_1, \ldots, B_{m_2}$ each of size $m_2 - m_1$ such that $A_j \cup B_j$ covers M_2 for $1 \le j \le m_2$. Then each $A_j \cup B_j$ covers $M_1 \cup M_2$, i.e., $E_{12} \cup B$ can be partitioned into m_2 edge dominating sets of K_{m_1,m_2,m_3}.

For the first lower bound, we shall partition the remaining edges into dominating sets of size m_3. Note that $E_{23} - B = E_{23}(\alpha+1, \beta, m_2m_3/d_{23} - \beta) \cup \bigcup_{r=\alpha+2}^{d_{23}} E_{23}(r)$ and $E_{13} = \bigcup_{r=1}^{d_{13}} E_{13}(r)$. By Lemma 2.1, $E_{23}(\alpha+1, \beta, m_2m_3/d_{23} - \beta)$ can be partitioned into $\lfloor (m_2m_3/d_{23} - \beta)/m_3 \rfloor = \lfloor m_2/d_{23} - \beta/m_3 \rfloor$ edge dominating sets and each $E_{23}(r)$ (resp. $E_{13}(r)$) can be partitioned into m_2/d_{23} (resp. m_1/d_{13}) edge dominating sets. Hence $E(K_{m_1,m_2,m_3})$ can be partitioned into $m_2 + \lfloor m_2/d_{23} - \beta/m_3 \rfloor + (d_{23} - \alpha - 1)m_2/d_{23} + d_{13}m_1/d_{13} = m_1 + 2m_2 + \lfloor -\alpha m_2/d_{23} - \beta/m_3 \rfloor = m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor$ edge dominating sets. These give the first lower bound in the lemma.

For the second lower bound, by Lemma 2.1, $E_{23} - B = E_{23}(\alpha+1, \beta, m_2m_3/d_{23} - \beta) \cup \bigcup_{r=\alpha+2}^{d_{23}} E_{23}(r)$ can be partitioned into $(m_2m_3/d_{23} - \beta)/m_2 + (d_{23} - \alpha - 1)(m_2m_3/d_{23})/m_2 = m_3 - m_2 + m_1$ sets $C_1, \ldots, C_{m_3-m_2+m_1}$ each of size m_2 and covering M_2. Suppose $m_1(m_3 - m_2 + m_1) = \lambda(m_1m_3/d_{13}) + \mu$ where $0 \le \mu < m_1m_3/d_{13}$. Note that μ is a multiple of m_1. Let $D = \bigcup_{r=1}^{\lambda} E_{13}(r) \cup E_{13}(\lambda+1, 0, \mu)$. Then, by Lemma 2.1, D can be partitioned into $\lambda(m_1m_3/d_{13})/m_1 + \mu/m_1 = m_3 - m_2 + m_1$ sets $D_1, \ldots, D_{m_3-m_2+m_1}$ each of size m_1 and covering M_1. Then each $C_i \cup D_i$, $1 \le i \le m_3 - m_2 + m_1$, covers $M_2 \cup M_1$. So $(E_{23} - B) \cup D$ can be partitioned into $m_3 - m_2 + m_1$ edge dominating sets. Finally, by Lemma 2.1, $E_{13} - D = E_{13}(\lambda+1, \mu, m_1m_3/d_{13} - \mu) \cup \bigcup_{r=\lambda+2}^{d_{13}} E_{13}(r)$ can be partitioned into $\lfloor (m_1m_3/d_{13} - \mu)/m_3 \rfloor + (d_{13} - \lambda - 1)(m_1m_3/d_{13})/m_3 = \lfloor m_1(m_2 - m_1)/m_3 \rfloor$ edge dominating sets. Hence $\mathrm{ed}(K_{m_1,m_2,m_3}) \ge m_2 + (m_3 - m_2 + m_1) + \lfloor m_1(m_2 - m_1)/m_3 \rfloor = m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor$. These give the second lower bound of the theorem. □

Note that for the two lower bounds in Lemma 2.3, $m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor \le m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor$ if and only if or $m_1 + m_2 \le m_3$.

We now give an example to demonstrate Lemma 2.3, as follows. Consider K_{m_1,m_2,m_3} with $m_1 = 3$, $m_2 = 8$ and $m_3 = 12$. Let $M_1 = \{1, 2, 3\}$, $M_2 = \{1', 2', \ldots, 8'\}$ and $M_3 = \{1'', 2'', \ldots, 12''\}$. In Fig. 2.1 and Fig. 2.2, an entry (a, b) represents an edge (a, b) of K_{m_1,m_2,m_3} and entry (a, b) is numbered by i when edge (a, b) is in the i-th edge dominating set X_i.

In Fig. 2.1, E_{12} is partitioned into $A_1, \ldots, A_8$, each of size 3 and covering M_1 but covering only 3 vertices of M_2. Gray entries in E_{23} form the set B of size 40. B is partitioned into $B_1, \ldots, B_8$ such that $X_1 = A_1 \cup B_1, \ldots, X_8 = A_8 \cup B_8$ are edge

	1'	2'	3'	4'	5'	6'	7'	8'	1"	2"	3"	4"	5"	6"	7"	8"	9"	10"	11"	12"
1	1	2	3	4	5	6	7	8	13	14	15	13	14	15	13	14	15	13	14	15
2	8	1	2	3	4	5	6	7	15	13	14	15	13	14	15	13	14	15	13	14
3	7	8	1	2	3	4	5	6	14	15	13	14	15	13	14	15	13	14	15	13
1'									6	2	9	11	5		10	12	3	4	9	11
2'									11	6	7	9	11	5		10	12	3	4	9
3'									9	11	6	7	9	11	5		10	12	8	4
4'									1	9	11	6	7	9	11	5		10	12	8
5'									8	1	10	12	6	7	9	11	2		10	12
6'									12	8	1	10	12	3	7	9	11	2		10
7'									10	12	8	1	10	12	3	4	9	11	2	
8'										10	12	5	1	10	12	3	4	9	11	2

Fig. 2.1. An edge domatic partition of $K_{3,8,12}$, which has 15 edge dominating sets

	1'	2'	3'	4'	5'	6'	7'	8'	1"	2"	3"	4"	5"	6"	7"	8"	9"	10"	11"	12"
1	1	2	3	4	5	6	7	8	9	13	16	10	14	16	11	15	16	12		16
2	8	1	2	3	4	5	6	7	16	9	13	16	10	14	16	11	15	16	12	
3	7	8	1	2	3	4	5	6		16	9	13	16	10	14	16	11	15	16	12
1'									6	2	10	13	5	9	12	15	3	4	11	14
2'									14	6	7	10	13	5	9	12	15	3	4	11
3'									11	14	6	7	10	13	5	9	12	15	8	4
4'									1	11	14	6	7	10	13	5	9	12	15	8
5'									8	1	11	14	6	7	10	13	2	9	12	15
6'									15	8	1	11	14	3	7	10	13	2	9	12
7'									12	15	8	1	11	14	3	4	10	13	2	9
8'									9	12	15	5	1	11	14	3	4	10	13	2

Fig. 2.2. An edge domatic partition of $K_{3,8,12}$, which has 16 edge dominating sets

dominating sets. $E_{23} - B$ is then partitioned into 4 edge dominating sets $X_9, \ldots, X_{12}$ with 8 edges unused. Finally, E_{13} is partitioned into 3 edge dominating sets X_{13}, X_{14} and X_{15}. So, there are a total of $m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor = 15$ edge dominating sets.

In Fig. 2.2, we also partition E_{12} into $A_1, \ldots, A_8$ and B into $B_1, \ldots, B_8$ to form 8 edge dominating sets $X_1 = A_1 \cup B_1, \ldots, X_8 = A_8 \cup B_8$. Then, $E_{23} - B$ is partitioned into 7 sets $C_1, \ldots, C_7$, each of size 8 and covering M_2. White entries in E_{13} form the set D of size 24. D is partitioned into 7 sets $D_1, \ldots, D_7$, each of size 3 and covering M_1, with 3 edges unused. So, $X_9 = C_1 \cup D_1, \ldots, X_{15} = C_7 \cup D_7$ are 7 edge dominating sets. Finally, $E_{13} - D$ is partitioned into 1 edge dominating set X_{16}. So there are a total of $m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor = 16$ edge dominating sets altogether.

For the upper bound of $\operatorname{ed}(K_{m_1,m_2,m_3})$, assume P is an edge domatic partition of K_{m_1,m_2,m_3} and $x_i = |\{D \in P: |D| = i\}|$ for each i. Note that P contains $\sum_i x_i$ edge dominating sets of K_{m_1,m_2,m_3}. Since the size of each edge dominating set of K_{m_1,m_2,m_3} is at least m_2, we have

$$\sum_{i \geq m_2} i x_i = m_1 m_2 + m_1 m_3 + m_2 m_3 \tag{2.1}$$

Lemma 2.4. $\sum_{i=m_2}^{\alpha} (m_1 + m_2 - i)x_i \leq m_1 m_2$ *where* $\alpha = \min\{m_1 + m_2 - 1, m_3 - 1\}$.

Proof. Let D be any edge dominating set of K_{m_1,m_2,m_3} with $m_2 \leq |D| = i \leq \alpha$. It suffices to prove that $D \cap E_{12}$ has at least $m_1 + m_2 - i$ edges. Suppose not, i.e., $D \cap E_{12}$ has at most $m_1 + m_2 - i - 1$ edges and so it covers at most $m_1 + m_2 - i - 1$ vertices of M_1. Since $|D| \leq \alpha \leq m_3 - 1$, D cannot cover M_3. Then, since D is an edge dominating set of K_{m_1,m_2,m_3}, D covers $M_1 \cup M_2$. Therefore, $D \cap E_{13}$ covers at least $i + 1 - m_2$ vertices of M_1 and so $|D \cap E_{13}| \geq i + 1 - m_2$. However, since D covers M_2, $|D \cap (E_{12} \cup E_{23})| \geq m_2$. Hence $|D| = |D \cap E_{13}| + |D \cap (E_{12} \cup E_{23})| \geq i + 1$, in contradicts to $|D| = i$. Hence $\sum_{i=m_2}^{\alpha} (m_1 + m_2 - i)x_i \leq m_1 m_2$. □

Lemma 2.5. $\sum_{m_2 \leq i < m_3} x_i \leq m_1 + m_3$

Proof. Let D be any edge dominating set of K_{m_1,m_2,m_3} with $m_2 \leq |D| = i < m_3$. Since D cannot cover M_3, D covers $M_1 \cup M_2$. By the fact that D covers M_2, $|D \cap (E_{12} \cup E_{23})| \geq m_2$. Therefore, $m_2 \sum_{m_2 \leq i < m_3} x_i \leq m_2(m_1 + m_3)$, i.e., $\sum_{m_2 \leq i < m_3} x_i \leq m_1 + m_3$. □

Lemma 2.6. $\operatorname{ed}(K_{m_1,m_2,m_3}) \leq \begin{cases} m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor \text{ } if\ m_1 + m_2 \geq m_3, \\ m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor \text{ } if\ m_1 + m_2 \leq m_3. \end{cases}$

Proof. In the case of $m_1 + m_2 \geq m_3$, the α in Lemma 2.4 is equal to $m_3 - 1$. By multiplying the inequality of Lemma 2.4 by $(m_3 - m_2)/m_1$ and adding it to (2.1), we have

$$\sum_{m_2 \leq i < m_3} (i + (m_1 + m_2 - i)(m_3 - m_2)/m_1)x_i + \sum_{i \geq m_3} i x_i$$
$$\leq m_3(m_1 + 2m_2) + m_2(m_1 - m_2)$$

For $m_2 \leq i < m_3$, $i + (m_1 + m_2 - i)(m_3 - m_2)/m_1 = m_3 + (i - m_2)(m_1 + m_2 - m_3)/m_1 \geq m_3$. Therefore, $m_3 \sum_{i \geq m_2} x_i \leq m_3(m_1 + 2m_2) + m_2(m_1 - m_2)$ and so $\operatorname{ed}(K_{m_1,m_2,m_3}) \leq \sum_{i \geq m_2} x_i \leq m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor$.

In the case of $m_1 + m_2 \leq m_3$, $\alpha = m_1 + m_2 - 1$. By multiplying the inequality in Lemma 2.5 by $m_3 - m_1 - m_2$ and adding it to (2.1) and the inequality in Lemma

2.4, we have

$$\sum_{m_2 \le i < m_1 + m_2} m_3 x_i + \sum_{m_1 + m_2 \le i < m_3} (i + m_3 - m_2 + m_1) x_i + \sum_{i \ge m_3} i x_i$$
$$\le m_3(m_1 + m_3) + m_1(m_2 - m_1).$$

Note that each coefficient of x_i in the left hand side is greater than or equal to m_3. Then $\mathrm{ed}(K_{m_1,m_2,m_3}) \le \sum_{i \ge m_2} x_i \le m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor$. □

Theorem 2.7. $\mathrm{ed}(K_{m_1,m_2,m_3}) = \begin{cases} m_1 + 2m_2 + \lfloor m_2(m_1 - m_2)/m_3 \rfloor \text{ if } m_1 + m_2 \ge m_3, \\ m_1 + m_3 + \lfloor m_1(m_2 - m_1)/m_3 \rfloor \text{ if } m_1 + m_2 \le m_3. \end{cases}$

Proof. This theorem follows directly from Lemmas 2.3 and 2.6. □

3. Balanced Complete *n*-Partite Graphs

Now we consider the edge domatic number problem for the balanced complete n-partite graph $K(r,n) \equiv K_{r,\ldots,r}$ in which every part has exactly r vertices. Exact values for $\mathrm{ed}(K(r,n))$ with odd n and $\mathrm{ed}(K(r,4))$ with even r are given in this section. Note that $\mathrm{ed}(K(r,2)) = r$. So we only consider $\mathrm{ed}(K(r,n))$ for $n \ge 3$. Also, $K(1,n) = K_n$ and it was showed in [8] that $\mathrm{ed}(K_n) = n$ for odd $n \ge 3$ and $\mathrm{ed}(K_n) = n - 1$ for even n.

Let $G^{(k)} = (V^{(k)}, E^{(k)})$ be the graph obtained from a graph $G = (V, E)$ by duplicating each vertex k times, i.e.,

$$V^{(k)} = \{v_1, \ldots, v_k \colon v \in V\}$$

and

$$E^{(k)} = \{(u_i, v_j) \colon (u, v) \in E \text{ and } 1 \le i, j \le k\}.$$

Lemma 3.1. $\mathrm{ed}(G^{(k)}) \ge k\,\mathrm{ed}(G)$

Proof. Let $P = \{D_1, \ldots, D_{\mathrm{ed}(G)}\}$ be an edge domatic partition of G. It suffices to prove that $G^{(k)}$ has $k\,\mathrm{ed}(G)$ pairwise disjoint edge dominating sets. For every $D_q \in P$, we construct k edge dominating sets of $G^{(k)}$ as follows:

$$D_{q,j} = \{(u_i, v_{i+j}) \colon (u, v) \in D_q, 1 \le i \le k\}, \qquad 1 \le j \le k,$$

where index of v_{i+j} is taken modulo k. It is straightforward to check that each $D_{q,j}$ is an edge dominating set of $G^{(k)}$, and so $G^{(k)}$ has $k\,\mathrm{ed}(G)$ pairwise disjoint edge dominating sets. □

Theorem 3.2. $\mathrm{ed}(K(r,n)) \le r^2 n(n-1)/2\lceil r(n-1)/2 \rceil \le rn$ *for* $n \ge 3$.

Proof. Since every edge dominating set of $K(r,n)$ must cover at least $n - 1$ parts of $K(r,n)$, every edge dominating set of $K(r,n)$ has at least $\lceil r(n-1)/2 \rceil$ edges. The theorem follows from this and that $K(r,n)$ has $r^2 n(n-1)/2$ edges. □

Although we believe that the upper bound $r^2n(n-1)/2\lceil r(n-1)/2\rceil$ in Theorem 3.2 is the exact value of $\mathrm{ed}(K(r,n))$ for $n \geq 3$, only two cases have been settled: those where n is odd and r is even with $n = 4$.

Theorem 3.3. $\mathrm{ed}(K(r,n)) = rn$ *if* $n \geq 3$ *and* n *is odd.*

Proof. $\mathrm{ed}(K(r,n)) \leq rn$ by Theorem 3.2. On the other hand, by [4], $\mathrm{ed}(K_n) = n$ for odd n. By Lemma 3.1, $\mathrm{ed}(K(r,n)) = \mathrm{ed}(K_n^{(r)}) \geq r\,\mathrm{ed}(K_n) = rn$. So $\mathrm{ed}(K(r,n)) = rn$. □

By [4], $\mathrm{ed}(K_n) = n - 1$ when n is even. Again, by Lemma 3.1, $\mathrm{ed}(K(r,n)) = \mathrm{ed}(K_n^{(r)}) \geq r\,\mathrm{ed}(K_n) = r(n-1)$. There is a gap between the lower bound $r(n-1)$ and the upper bound in Theorem 3.2. We now consider $\mathrm{ed}(K(r,4))$ when r is even.

Theorem 3.4. $\mathrm{ed}(K(r,4)) = 4r$ *if* r *is even.*

Proof. For the case of $r = 2$, let the vertex set of $K(2,4)$ be $\{0^{(i)}, 1^{(i)}: i = 1,2,3,4\}$ and the edge set of $K(2,4)$ be $\{(x^{(i)}, y^{(j)}): x, y \in \{0,1\}, 1 \leq i \neq j \leq 4\}$. Then $\mathrm{ed}(K(2,4)) = 8$, as it is straightforward to check that the following sets $A_1, \ldots, A_8$ are pairwise disjoint edge dominating sets of $K(2,4)$:

$$A_1 = \{(0^{(1)},0^{(2)}),(1^{(2)},0^{(3)}),(1^{(3)},1^{(1)})\},$$
$$A_2 = \{(0^{(2)},0^{(3)}),(1^{(3)},0^{(4)}),(1^{(4)},1^{(2)})\},$$
$$A_3 = \{(1^{(3)},1^{(4)}),(0^{(4)},0^{(1)}),(1^{(1)},0^{(3)})\},$$
$$A_4 = \{(0^{(4)},1^{(1)}),(0^{(1)},1^{(2)}),(0^{(2)},1^{(4)})\},$$
$$A_5 = \{(1^{(1)},1^{(2)}),(0^{(2)},1^{(3)}),(0^{(3)},0^{(1)})\},$$
$$A_6 = \{(1^{(2)},1^{(3)}),(0^{(3)},1^{(4)}),(0^{(4)},0^{(2)})\},$$
$$A_7 = \{(0^{(3)},0^{(4)}),(1^{(4)},1^{(1)}),(0^{(1)},1^{(3)})\},$$
$$A_8 = \{(1^{(4)},0^{(1)}),(1^{(1)},0^{(2)}),(1^{(2)},0^{(4)})\}.$$

In general, let $r = 2s$. Then $K(r,4) = K(2s,4) = K(2,4)^{(s)}$. By Lemma 3.1, $\mathrm{ed}(K(r,4)) \geq s\,\mathrm{ed}(K(2,4)) \geq 8s = 4r$. On the other hand, $\mathrm{ed}(K(r,4)) \leq 4r$ by Theorem 3.2. Hence $\mathrm{ed}(K(r,4)) = 4r$. □

We close this paper by the following conjecture: $\mathrm{ed}(K(r,n)) = r^2n(n-1)/2\lceil r(n-1)/2\rceil$ for any $n \geq 3$. The solutions to the domatic numbers of general complete n-partite graphs are also desirable.

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