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Fault-tolerant cycle-embedding of crossed cubes ☆

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Abstract

The crossed cube CQ_n introduced by Efe has many properties similar to those of the popular hypercube. However, the diameter of CQ_n is about one half of that of the hypercube. Failures of links and nodes in an interconnection network are inevitable. Hence, in this paper, we consider the hybrid fault-tolerant capability of the crossed cube. Letting f_e and f_v be the numbers of faulty edges and vertices in CQ_n, we show that a cycle of length l, for any $4 \leqslant l \leqslant |V(CQ_n)| - f_v$, can be embedded into a wounded crossed cube as long as the total number of faults $(f_v + f_e)$ is no more than $n-2$, and we say that CQ_n is $(n-2)$-fault-tolerant pancyclic. This result is optimal in the sense that if there are $n-1$ faults, there is no guarantee of having a cycle of a certain length in it.

Keywords: Cycle embedding; Crossed cube; Pancyclic; Hamiltonian; Fault tolerance

1. Introduction

Network topology is essential for parallel and distributed computation, and many topologies have been proposed, for example, hypercubes, butterfly graphs and star graphs. The hypercube is one of the most popular networks since it has a simple structure and is easy to implement. However, there are still some different points of view to construct new topologies; for example, a new topology having smaller diameter.

To lower the diameter, we may change some links of the hypercube. Some variations of the hypercube have been studied in the literature. In [2], Efe first studied the crossed cube CQ_n, which has a structure similar to that of the hypercube, including recursive structure, the same number of vertices, and the same number of edges. However, the diameter of CQ_n is only about one half of that of the hypercube, and the diameter is an important factor for parallel computing speed. Other studies have been done to explore more properties of the crossed cube CQ_n, such as edge congestion of CQ_n, as studied in [1]. Furthermore, embedding of binary trees, hamiltonian paths, and hamiltonian cycles into CQ_n were discussed in [5,6].

The graph embedding problem asks if a guest graph is a subgraph of a host graph, and an important benefit of graph embedding is that we can apply

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existing algorithms for guest graphs to host graphs. This problem has attracted a burst of studies over the years.

The pancycle problem asks if a cycle of length l is a subgraph of a given graph with a given positive integer l. Hwang [4] and Fan [3] et al. studied this problem on butterfly graphs and Möbius cubes, respectively. But they did not consider the possibilities of failures of nodes and/or links.

Failures are inevitable when a network is put in use. Therefore, the fault-tolerant capacity of an interconnection network is a crucial issue in parallel computing. Furthermore, both nodes and links may simultaneously be faulty in a network. Hence we study the hybrid fault tolerance of CQ_n in this paper. Letting f_v and f_e be the numbers of faulty vertices and edges in CQ_n, respectively, we show that a cycle of length l, for any $4 \leqslant l \leqslant |V(CQ_n)| - f_v$, is a subgraph of a wounded crossed cube with $(f_v + f_e) \leqslant (n-2)$. That is, CQ_n is $(n-2)$-fault-tolerant pancyclic. In addition, this result is optimal, and the reason is explained as follows: The n-dimensional crossed cube is n-regular. As a result, if there are $(n-1)$ faulty edges incident to a single node, a hamiltonian cycle cannot be embedded into a wounded CQ_n.

The rest of this paper is organized as follows: Section 2 includes the definition of the crossed cubes and some basic notation and terminologies. Then, the proof of the pancyclicity of CQ_n is given in Section 3. For the case $n = 4$, the proof is a little tedious, and we leave some parts of it to Appendix A.

2. Definitions and notation

Given a simple graph G, we use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. In order to define the crossed cube CQ_n, as proposed by Efe [2], the pair related set R is introduced. Let $R = \{(00,00), (10,10), (11,01), (01,11)\}$. Two binary strings a_1a_2 and b_1b_2 of length 2 are pair related, denoted by $a_1a_2 \sim b_1b_2$, if $(a_1a_2, b_1b_2) \in R$. The following is the recursive definition of the n-dimensional crossed cube CQ_n. CQ_n has 2^n vertices, each labeled by a binary string of length n. CQ_1 is a complete graph with two vertices labeled 0 and 1, respectively. For $n \geqslant 2$, CQ_n is obtained by taking two copies of CQ_{n-1}, denoted by CQ^0_{n-1} and CQ^1_{n-1}, respectively, and adding 2^{n-1} edges as follows:

Let

$$V(CQ^0_{n-1}) = \{0x_{n-2}\ldots x_1x_0\colon\ x_i = 0 \text{ or } 1\}$$

and

$$V(CQ^1_{n-1}) = \{1y_{n-2}\ldots y_1y_0\colon\ y_i = 0 \text{ or } 1\}.$$

A vertex $0x_{n-2}\ldots x_1x_0 \in V(CQ^0_{n-1})$ and a vertex $1y_{n-2}\ldots y_1y_0 \in V(CQ^1_{n-1})$ are adjacent if

(1) $x_{n-2} = y_{n-2}$ if n is even, and
(2) $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i < \lfloor (n-1)/2 \rfloor$.

We take CQ_3 and CQ_4 as examples and display them in Fig. 1 (a) and (b), respectively. In Fig. 1(c), we use a different way to draw CQ_3 in order to see its vertex-symmetry.

We now introduce some basic terminologies and notation needed for later discussion. A *path* is a sequence of vertices with any two consecutive vertices being adjacent in G. We use $\langle u_1, u_2, \ldots, u_l\rangle$ to denote a path that begins with u_1 and ends with u_l. In addition, $\langle u_1, u_2, \ldots, u_l\rangle$ is a *cycle* if $u_1 = u_l$. A *hamiltonian path* is defined as a path which contains all the vertices of G exactly once. A graph G is *hamiltonian connected* if, for any two vertices of G, there exists a hamiltonian path between them. We say that a graph G is *pancyclic* if G contains a cycle of length l as a subgraph, for every $4 \leqslant l \leqslant |V(G)|$. A cycle is a *hamiltonian cycle* if it traverses all the vertices of G exactly once. A graph G is *hamiltonian* if G contains a hamiltonian cycle.

To consider a wounded graph, we give the following terminologies and notation. Given a graph G, let $F_v \subseteq V(G)$ and $F_e \subseteq E(G)$; and $F = F_v \cup F_e$. Let G' be the graph obtained from G by deleting all the edges in F_e. We use $G - F$ to denote the subgraph of G' induced by $V(G') - F_v$. We call a graph G *k-fault-tolerant hamiltonian connected* (abbreviated as *k-hamiltonian connected*) if $G - F$ is hamiltonian connected for any F with $|F| \leqslant k$. We call a graph G *k-fault-tolerant hamiltonian* (abbreviated as *k-hamiltonian*) if $G - F$ is hamiltonian for any F with $|F| \leqslant k$. A graph G is called *k-fault-tolerant pancyclic* (abbreviated as *k-pancyclic*) if $G - F$ is pancyclic for any F with $|F| \leqslant k$.

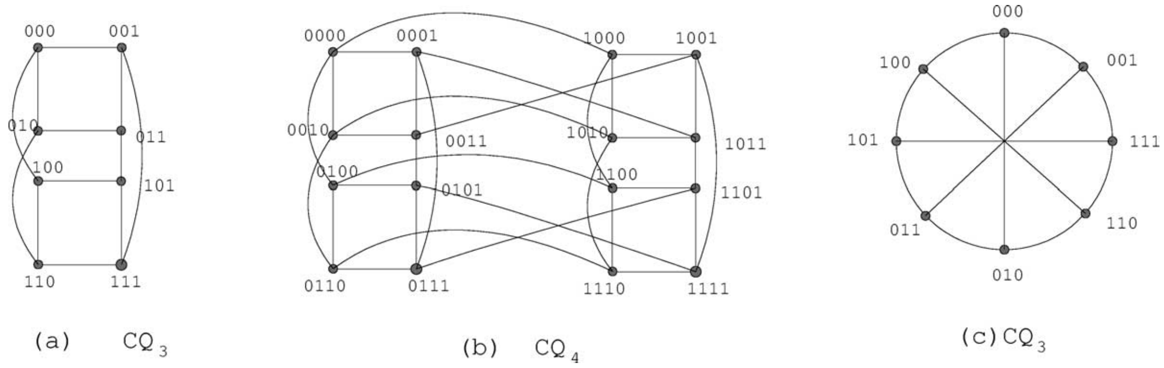

Fig. 1. (a) CQ_3, (b) CQ_4, and (c) CQ_3 drawn in a different way.

3. Main result

We use CQ_{n-2}^{ij} to denote an $(n-2)$-dimensional crossed cube which is a subgraph of CQ_n induced by the vertices labeled $ijx_{n-3}\dots x_0$. We say that an edge is a *critical edge* of CQ_n if it is an edge in CQ_{n-1}^{i} with one *endpoint* in CQ_{n-2}^{i0} and the other in CQ_{n-2}^{i1} *for* $i \in \{0,1\}$.

Lemma 1. *Let* (u_1,u_2) *be a critical edge of* CQ_n *which is in* CQ_{n-1}^{0}*, and* v_1, v_2 *be the neighbors of* u_1 *and* u_2 *in* CQ_{n-1}^{1}*, respectively, for* $n \geqslant 4$*. Then* (v_1,v_2) *is also a critical edge of* CQ_n *in* CQ_{n-1}^{1}.

Proof. We discuss two cases: (1) n is even, and (2) n is odd.

Case 1. *n is even*. Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}\dots x_1x_0$ and $u_2 = 01y_{n-3}y_{n-4}\dots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i \leqslant \lfloor (n-3)/2 \rfloor$. Then $v_1 = 10y_{n-3}y_{n-4}\dots y_1y_0$, and $v_2 = 11x_{n-3}x_{n-4}\dots x_1x_0$. By definition, v_1 and v_2 are adjacent, and (v_1,v_2) is a critical edge in CQ_{n-1}^{1}.

Case 2. *n is odd*. Without loss of generality, we assume that $u_1 = 00x_{n-3}x_{n-4}x_{n-5}\dots x_1x_0$. Suppose that $x_{n-3}=0$. Then $u_1 = 000x_{n-4}x_{n-5}\dots x_1x_0$, $u_2 = 010y_{n-4}y_{n-5}\dots y_1y_0$, where $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \leqslant i \leqslant \lfloor (n-4)/2 \rfloor$, $v_1 = 100y_{n-4}y_{n-5}\dots y_1y_0$, and $v_2 = 110x_{n-4}x_{n-5}\dots x_1x_0$. Thus, v_1 and v_2 are adjacent, and (v_1,v_2) is a critical edge in CQ_{n-1}^{1}. It can be checked that the statement is also true for the case $x_{n-3}=1$. □

It is observed that vertices u_1,u_2,v_1,v_2 in the above lemma form a 4-cycle. We call this cycle a *crossed* 4*-cycle* in CQ_n. It is clear that, for each vertex $00x_{n-3}\cdots x_0$, there is exactly one crossed 4-cycle corresponding to the vertex. Thus, there are 2^{n-2} disjoint crossed 4-cycles in CQ_n. We note that a crossed 4-cycle contains two critical edges.

Huang et al. [5] showed the validity of the following theorem. Based on this theorem, we show the pancyclicity of the crossed cube by induction.

Theorem 1 [5]. *The crossed cube* CQ_n *is* $(n-2)$*-hamiltonian and* $(n-3)$*-hamiltonian connected for* $n \geqslant 3$.

The base case is $n=3$, and the proof is given in the following.

Theorem 2. CQ_3 *is* 1*-pancyclic.*

Proof. Note that CQ_3 can be redrawn as Fig. 1(c), and it is vertex-transitive. We consider two cases (1) one faulty vertex, and (2) one faulty edge as follows:

Case 1. *One faulty vertex*. Without loss of generality, we assume that vertex $x = 000$ is faulty. We list cycles of lengths from 4 to 7 as follows: $\langle 001, 111, 101, 011, 001\rangle$, $\langle 001, 111, 110, 010, 011, 001\rangle$, $\langle 001, 111, 110, 100, 101, 011, 001\rangle$, and $\langle 001, 111, 101, 100, 110, 010, 011, 001\rangle$.

Case 2. *One faulty edge*. Without loss of generality, we assume that the faulty edge e is incident to 000. By case 1, there are cycles of lengths from 4 to 7 in the faulty CQ_3. For a cycle of length 8, suppose that $e = (000, 010)$. Then $\langle 000, 001, 111, 110, 010, 011, 101,$

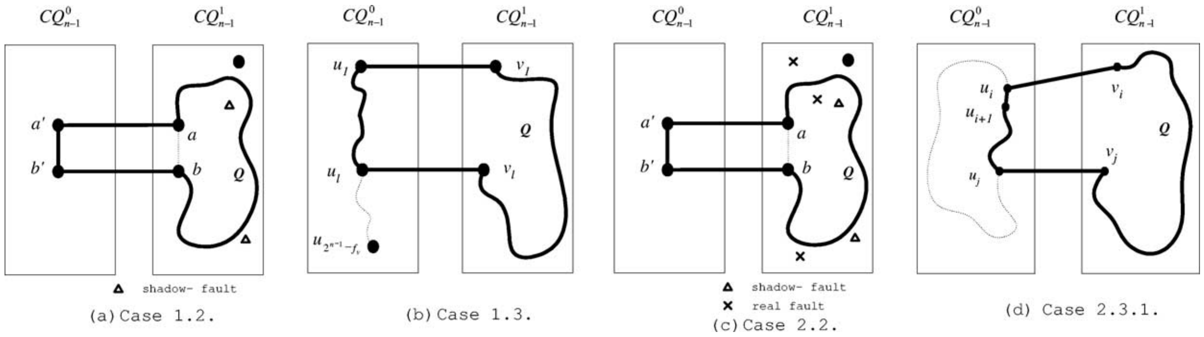

Fig. 2. Cases of Theorem 3.

$100, 000\rangle$ is the desired one. Suppose that $e = (000, 001)$. Then $\langle 000, 010, 110, 111, 001, 011, 101, 100, 000\rangle$ is a cycle of length 8. If $e = (000, 100)$, the case is symmetric to the case $e = (000, 001)$. □

Let F be a set of faults in CQ_n. We say that a vertex u in one subcube of CQ_n is a *safe crossing-point* in $CQ_n - F$ if u still connects to the neighbor in the other subcube in $CQ_n - F$, i.e., the corresponding neighbor u' in the other subcube of u is fault-free, and the edge (u, u') is also fault-free. The main result is as follows.

Theorem 3. *The crossed cube CQ_n is $(n - 2)$-pancyclic for $n \geqslant 3$.*

Proof. We prove this by induction on n. It follows from Theorem 2 that CQ_3 is 1-pancyclic. Now we proceed to the induction step. Suppose that CQ_{n-1} is $(n - 3)$-pancyclic for some $n \geqslant 4$. We will show that CQ_n is $(n-2)$-pancyclic. Let $F \subseteq V(CQ_n) \cup E(CQ_n)$ be the set of faults. We divide F into five disjoint parts:

$$F_v^0 = F \cap V\bigl(CQ^0_{n-1}\bigr), \qquad F_e^0 = F \cap E\bigl(CQ^0_{n-1}\bigr),$$
$$F_v^1 = F \cap V\bigl(CQ^1_{n-1}\bigr), \qquad F_e^1 = F \cap E\bigl(CQ^1_{n-1}\bigr),$$
$$F_e^c = F \cap \bigl\{(u, v) \mid (u, v) \text{ is an edge between } CQ^0_{n-1} \text{ and } CQ^1_{n-1}\bigr\}.$$

Let $f = |F|$, $f_v^0 = |F_v^0|$, $f_e^0 = |F_e^0|$, $f_v^1 = |F_v^1|$, $f_e^1 = |F_e^1|$, and $f_e^c = |F_e^c|$. For convenience of discussion, we define the following subsets of F: $F_v = F \cap V(CQ_n)$, $F_e = F \cap E(CQ_n)$, $F^0 = F_v^0 \cup F_e^0$, and $F^1 = F_v^1 \cup F_e^1$. And let $f_v = |F_v|$, $f_e = |F_e|$, $f^0 = |F^0|$, and $f^1 = |F^1|$. Note that $f^0 + f^1 = f - f_e^c$.

Case 1. There is a subcube containing all the $(n-2)$ faults. Without loss of generality, we assume that $f^0 = n - 2$. Thus, there is no fault outside CQ^0_{n-1}, i.e., $f^1 = f_e^c = 0$. We discuss the existence of cycles of lengths from 4 to $2^n - f_v$ according to the following cases.

Case 1.1. *Cycles of lengths from* 4 *to* 2^{n-1}. Since CQ_{n-1} is $(n-3)$-pancyclic, CQ^1_{n-1} contains cycles of lengths from 4 to 2^{n-1} for $n \geqslant 4$. Clearly, $CQ_n - F$ also contains cycles of these lengths.

Case 1.2. *A cycle of length* $2^{n-1} + 1$. (See Fig. 2(a).) We want to construct a cycle containing $2^{n-1} - 1$ vertices in CQ^1_{n-1} and two vertices in CQ^0_{n-1}. To avoid faults in CQ^0_{n-1}, we introduce a term called the shadows of the faults. Let $\langle u_1, u_2, v_2, v_1, u_1\rangle$ be a crossed 4-cycle with u_1, u_2 in CQ^0_{n-1} and v_1, v_2 in CQ^1_{n-1}, respectively. If there is a fault on this cycle but the fault is not in CQ^1_{n-1}, we call edge (v_1, v_2) a *shadow fault* of F on CQ^1_{n-1}. (Similarly, we may define a shadow fault on CQ^0_{n-1}.) Let $F^s = \{e \mid \text{edge } e$ is a shadow fault of F on $CQ^1_{n-1}\}$. Since all crossed 4-cycles are vertex disjoint, $|F^s| \leqslant n - 2$. If $|F^s| = n - 2$, we arbitrarily pick an edge e_1 in F^s, and let $F' = F^s - e_1$, or else $F' = F^s$. Then $|F'| \leqslant n - 3$ and $CQ^1_{n-1} - F'$ is still pancyclic. So there is a cycle C of length $2^{n-1} - 1$ in $CQ^1_{n-1} - F'$. Clearly, there are two critical edges on C. Let $(a, b) \neq e_1$ be a critical edge on C, so $(a, b) \notin F^s$. Let a', b' be the neighbors of a and b in CQ^0_{n-1}, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n-1} + 1$ in $CQ_n - F$.

Case 1.3. *Cycles of lengths from* $2^{n-1} + 2$ *to* $2^n - f_v$. (See Fig. 2(b).) By Theorem 1, CQ^0_{n-1} is $(n - 3)$-hamiltonian and $f^0 = n-2$, $CQ^0_{n-1} - F^0$ still contains a hamiltonian *path*, say $P = \langle u_1, u_2, \ldots, u_{2^{n-1}-f_v^0}\rangle$, where $f_v^0 = f_v$. Let $2 \leqslant l \leqslant 2^{n-1} - f_v$. We construct a cycle of length $2^{n-1} + l$ as follows: Suppose that the neighbors of u_1 and u_l in CQ_{n-1}^1 are v_1 and v_l, respectively. Since CQ_{n-1} is $(n-4)$-hamiltonian connected and $n \geqslant 4$, there is a hamiltonian path Q in CQ_{n-1}^1 between v_1 and v_l containing 2^{n-1} vertices. So $\langle u_1, \ldots, u_l, v_l, Q, v_1, u_1\rangle$ forms a cycle of length $2^{n-1} + l$.

Case 2. Both f^0 and f^1 are at most $n-3$. By induction hypothesis, $CQ_{n-1}^0 - F^0$ and $CQ_{n-1}^1 - F^1$ are still pancyclic. We discuss the existence of cycles of all lengths from 4 to $2^n - f_v$ in the following cases.

Case 2.1. *Cycles of lengths from* 4 *to* $2^{n-1} - f_v^1$. By induction hypothesis, CQ_{n-1}^1 is $(n-3)$-pancyclic. Thus, we have cycles of lengths from 4 to $2^{n-1} - f_v^1$ in $CQ_{n-1}^1 - F^1$.

Case 2.2. *A cycle of length* $2^{n-1} - f_v^1 + 1$. (See Fig. 2(c).) We construct the cycle using a similar way used in Case 1.2. Let $F^s = \{e \mid \text{edge } e \text{ is a shadow fault of } F \text{ on } CQ_{n-1}^1\}$. Then $|F^s \cup F^1| \leqslant n-2$. If $|F^s \cup F^1| = n-2$, we arbitrarily choose an edge e_1 in F^s, and let $F' = F^s \cup F^1 - e_1$, or else $F' = F^s \cup F^1$. Then $|F'| \leqslant n-3$ and $CQ_{n-1}^1 - F'$ is still pancyclic. Since $F' \cap V(CQ_{n-1}^1) = F_v^1$, there is a cycle C of length $2^{n-1} - f_v^1 - 1$ in $CQ_{n-1}^1 - F'$. Since $2^{n-1} - f_v^1 - 1 > 2^{n-2}$ for $n \geqslant 4$, C contains two critical edges. Let $(a,b) \neq e_1$ be a critical edge on C, so $(a,b) \notin F^s$. Let a', b' be the neighbors of a and b in CQ_{n-1}^0, respectively. Then $\langle a, a', b', b, a\rangle$ is a fault-free crossed 4-cycle. Suppose that $C = \langle a, Q, b, a\rangle$. Then $\langle a', a, Q, b, b', a'\rangle$ forms a cycle of length $2^{n-1} - f_v^1 + 1$ in $CQ_n - F$.

Case 2.3. *Cycles of lengths from* $2^{n-1} - f_v^1 + 2$ *to* $2^n - f_v$. (See Fig. 2(d).) Without loss of generality, we assume that $n-3 \geqslant f^0 \geqslant f^1$. If $f^1 = n-3$, then $f^0 = n-3$, and $2n-6 \leqslant n-2 = f$, which implies $n \leqslant 4$. Thus, we need to discuss the case $f^1 = n-3$ just for $n = 4$. We leave this particular case to Appendix A, and assume that $f^1 \leqslant n-4$ in the following discussion.

By Theorem 1, $CQ_{n-1}^1 - F^1$ is still hamiltonian connected, i.e., there is a path of length $2^{n-1} - f_v^1 - 1$ between any two vertices in $CQ_{n-1}^1 - F^1$. As a result, if we can find a path of length l in $CQ_{n-1}^0 - F^0$ with the two endpoints being safe crossing-points, then we find a cycle of length $l + 2 + (2^{n-1} - f_v^1 - 1)$. Since we want to construct cycles of lengths from $2^{n-1} - f_v^1 + 2$ to $2^n - f_v$, $1 \leqslant l \leqslant 2^{n-1} - (f_v - f_v^1) - 1 = 2^{n-1} - f_v^0 - 1$. Now we construct a path of length l in CQ_{n-1}^0 for each l, $1 \leqslant l \leqslant 2^{n-1} - f_v^0 - 1$. By Theorem 1, CQ_{n-1}^0 is $(n-3)$-hamiltonian. Thus we have a hamiltonian cycle $C = \langle u_0, u_1, \ldots, u_{2^{n-1}-f_v^0-1}, u_0\rangle$ of length $2^{n-1} - f_v^0$ in $CQ_{n-1}^0 - F^0$. We claim that there exist two safe crossing-points u_i and u_j on C such that $(j-i)_{(\mathrm{mod}\ 2^{n-1}-f_v^0)} = l$. Suppose on the contrary that there do not exist such u_i and u_j. Then there are at least $\lceil (2^{n-1} - f_v^0)/2\rceil$ faults outside CQ_{n-1}^0. However, $\lceil (2^{n-1} - f_v^0)/2\rceil + f_v^0 \geqslant 2^{n-2} > n-2$ for $n \geqslant 2$. We obtain a contradiction. Thus, there exist such two vertices u_i and u_j. And then we find a path of length l on C.

Hence, the theorem follows. □

Appendix A

In the following, we construct cycles of lengths from $2^{n-1} - f_v^1 + 2$ to $2^n - f_v$ for the case $f^0 = f^1 = n-3$. Since $f^0 + f^1 = 2n-6 \leqslant n-2$, $n \leqslant 4$. Thus, we need only to discuss the case $f^0 = f^1 = 1$ for $n = 4$ here. We shall use some symmetric properties of CQ_3 to reduce the cases.

For convenience of discussion (see Fig. 3(a)), we call $(000, 010)$, $(001, 011)$, $(111, 101)$, $(110, 100)$ as *inner edges* of CQ_3 and $(000, 001)$, $(001, 111)$, $(111, 110)$, $(110, 010)$, $(010, 011)$, $(011, 101)$, $(101, 100)$, $(100, 000)$ as *outer edges* of CQ_3, respectively. Let x be a vertex of CQ_3. An inner edge e is said to be an *N-edge* of x if x connects to one of the endpoints of e. Hence CQ_3 has two N-edges of x. An inner edge e is said to be an *H-edge* of x if x is not incident to e, and e is not an N-edge of x. Therefore, CQ_3 has one H-edge of x.

To explore the pancyclicity of $CQ_4 - F$, we need an observation, and it is stated in the following lemma.

Lemma 2. *Let x be a faulty vertex in CQ_3. Then the two N-edges of x, e_1 and e_2, are on cycles of lengths from* 4 *to* 7 *in $CQ_3 - x$. And the H-edge of x is on cycles of lengths* 4, 5, *and* 7 *in $CQ_3 - x$.*

Proof. Without loss of generality, we may assume that $x = 000$. Then the two N-edges of x are $(001, 011)$ and $(110, 100)$, and the H-edge of x is $(111, 101)$. We

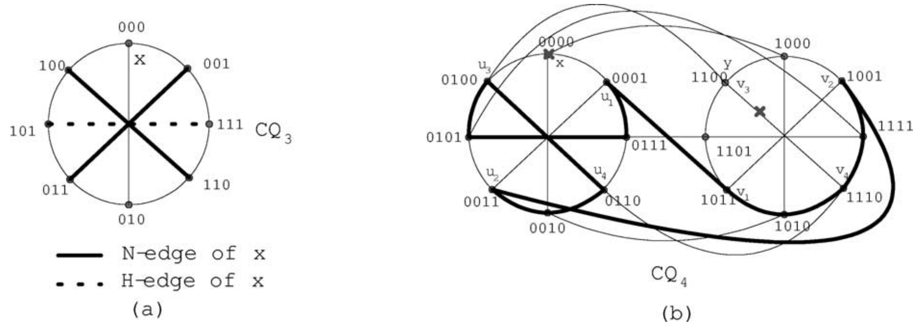

Fig. 3. (a) N-edge and H-edge of x and (b) a cycle of length 12 with one faulty vertex 0000 and one faulty edge $(1100, 1110)$.

list all the cycles as follows: $\langle 001, 111, 101, 011, 001\rangle$, $\langle 111, 110, 100, 101, 111\rangle$, $\langle 001, 111, 110, 010, 011, 001\rangle$, $\langle 111, 110, 010, 011, 101, 111\rangle$, $\langle 110, 010, 011, 101, 100, 110\rangle$, $\langle 001, 111, 110, 100, 101, 011, 001\rangle$, and $\langle 001, 111, 101, 100, 110, 010, 011, 001\rangle$. □

We continue to discuss cycles in $CQ_4 - F$, and consider two situations: (1) cycles of lengths from 8 to 14 and (2) cycles of lengths 15 and 16.

Case 1. *Cycles of lengths from* 8 *to* 14. Suppose that there is a faulty vertex x in CQ_3^0 or a faulty edge e_1 which is incident to x. Let (u_1, u_2) and (u_3, u_4) be the two N-edges of x. By Lemma 2, (u_1, u_2) and (u_3, u_4) are on cycles of lengths from 4 to 7 in $CQ_3^0 - F^0$. Let v_1, v_2, v_3, and v_4 be the neighbors of u_1, u_2, u_3, and u_4 in CQ_3^1, respectively. It is not difficult to check that both (v_1, v_2) and (v_3, v_4) are inner edges of CQ_3^1. (See Fig. 3(b).) And (v_1, v_2) can not reach (v_3, v_4) via exactly one edge of CQ_3^1. Suppose that the other fault is a faulty vertex y in CQ_3^1, or a faulty edge e_2 which is incident to y. Then (v_1, v_2) or (v_3, v_4), say (v_1, v_2) is an N-edge or H-edge of y in $CQ_3^1 - F^1$. By Lemma 2, (v_1, v_2) is on cycles of lengths 4, 5, and 7 in $CQ_3^1 - F^1$. We use C_i to denote a cycle of length i. Let C_i and C_j be cycles containing (u_1, u_2) and (v_1, v_2), respectively, $4 \leqslant i \leqslant 7$, $j = 4$, 5, or 7. Then we can construct a cycle C_l from C_i and C_j by adding (u_1, v_1) and (u_2, v_2), and deleting (u_1, u_2) and (v_1, v_2) for $8 \leqslant l \leqslant 14$. For example, Fig. 3(b) shows a cycle of length 12 in CQ_4 with one faulty vertex 0000 and one faulty edge $(1100, 1110)$.

Case 2. *Cycles of lengths from* 15 *to* 16. If $f_v = 2$, we need only to find cycles of lengths from 4 to 14 which we did in the previous cases. If $f_v = 1$ and $f_e = 1$, we have to find a cycle of length 15. By Theorem 1, CQ_4 is 2-hamiltonian, so there is a cycle of length 15 in $CQ_4 - F$. If $f_e = 2$, since CQ_4 is 2-hamiltonian, there is also a cycle of length 16. Suppose that $F = \{(x_1, y_1), (x_2, y_2)\}$. Let $F' = \{(x_1, y_1), x_2\}$. Then there is a cycle of length 15 in $CQ_4 - F'$. This cycle is also fault-free in $CQ_4 - F$.

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