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Cost benefit analysis of series systems with warm standby components

Kuo-Hsiung Wang[1], **Wen-Lea Pearn**[2]

[1] Department of Applied Mathematics, National Chung-Hsing University, Taichung, TAIWAN, ROC
[2] Department of Industrial Engineering and Management, National Chiao Tung University, Hsinchu, TAIWAN, ROC

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Abstract. This paper deals with the cost benefit analysis of series systems with warm standby components. The time-to-repair and the time-to-failure for each of the primary and warm standby components is assumed to have the negative exponential distribution. We develop the explicit expressions for the mean time-to-failure, *MTTF*, and the steady-state availability, $A_T(\infty)$ for three configurations and perform a comparative analysis. Under the cost/benefit (C/B) criterion, comparisons are made based on assumed numerical values given to the distribution parameters, and to the cost of the components. The configurations are ranked based on: *MTTF*, $A_T(\infty)$, and C/B where B is either *MTTF* or $A_T(\infty)$.

Key words: Availability, Cost/benefit, Reliability, Series system, Warm standbys

1 Introduction

Uncertainty is one of the important issues in management decisions. Two of the most useful uncertainty measures are: system reliability and system availability. Maintaining a high or required level of reliability and/or availability is often an essential requisite.

In this paper, we study the reliability, the availability, and the cost/benefit analysis of three different series system configurations with warm standby components. These three configurations are compared based on their mean time-to-failure, *MTTF*, their steady-state availability, $A_T(\infty)$, and their cost/benefit ratio C/B. Benefit is divided into two categories according to whether the measure utilized is the system reliability given by *MTTF* or the system availability given by $A_T(\infty)$.

Analytical solutions of the Markovian model for the machine repair problem with warm standbys were first developed by Sivazlian and Wang

(1989). Wang and Kuo (2000) investigates the cost and probabilistic analysis of series systems with mixed standby components. The problem considered in this paper is more general than the works of Galikowsky et al. (1996). This paper accomplishes three objectives. The first one is to provide a systematic methodology to develop the explicit expressions for the mean time-to-failure, $MTTF_i$, and the steady-state availability, $A_{T_i}(\infty)$, for configuration i, where $i = 1, 2, 3$. The second one is to perform a parametric investigation which provides numerical results to show the effects of various values of system parameters to the cost/benefit ratios. The third one is to rank three configurations for the *MTTF*, the $A_T(\infty)$, and the C/B, based on specific values of distribution parameters, as well as of the costs of the components.

2 Description of the system

For the sake of discussion, we consider the requirements of a 10 MW power plant. We assume that generators are available in units of both 10 MW and 5 MW. We also assume that standby generators are allowed to fail while inactive before they are put into full operation, and that the standby generators are continuously monitored by a fault detecting device in order to identify if they fail or not. Let us assume that all switch over times are instantaneous and switching is perfect, e.g. never fails and never does any damage. Primary components and standby components can be considered to be repairable. Each of the primary components fails independently of the state of the others and has an exponential time-to-failure distribution with parameter λ. Whenever one of these components fails, it is immediately replaced by a standby component if one is available. We assume that each of the available standby components fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter α $(0 < \alpha < \lambda)$. Whenever a primary component or a standby component fails, it is immediately repaired in the order of breakdowns with a time-to-repair which is exponentially distributed with parameter μ. Once a component is repaired, it is as good as new. Further, failure times and repair times are independently distributed random variables.

The following configurations are considered. The first configuration is a serial system of one primary 10 MW component with two standby 10 MW components (Figure 1). The second configuration is a serial system of two primary 5 MW components and one standby 5 MW component (Figure 2). The standby unit can replace either one of the initially working units in case of failure. The last configuration is a serial system of two primary 5 MW components with two interchangeable standby 5 MW components. (Figure 3). Each standby unit can replace either one of the failed components. If necessary, a standby unit can also replace the first used standby unit in case of failure.

2.1 Cost-benefit factor

We assume that the size-proportional costs for the primary components and warm standby components are given in Table 1. With this, we calculate the costs for each configuration i $(i = 1, 2, 3)$ shown in Table 2. Let C_i be the cost

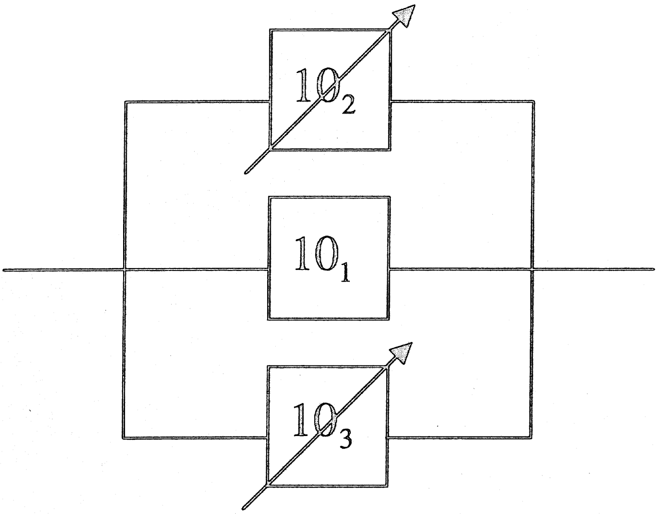

Fig. 1. Configuration 1: one 10 MW component with two standby components

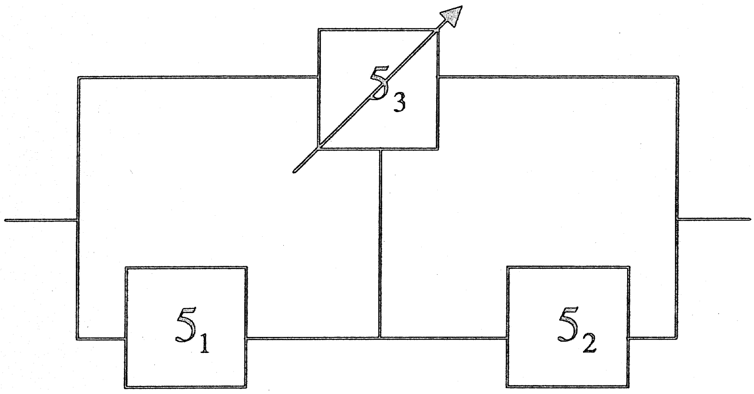

Fig. 2. Configuration 2: two 5 MW components with one standby component

of the configuration i, and B_i be the benefit of the configuration i, where B_i may either be the $MTTF_i$ (for system reliability), or the $A_{T_i}(\infty)$ (for system availability), where $i = 1, 2, 3$.

3 Problem solutions

3.1 Calculations for configuration 1

3.1.1 MTTF as benefit

For configuration 1, let $P_n(t)$ be the probability that exactly n components are working at time t ($t \geq 0$). If we let $\mathbf{P}(t)$ denote the probability row vector at time t, then the initial conditions for this problem are

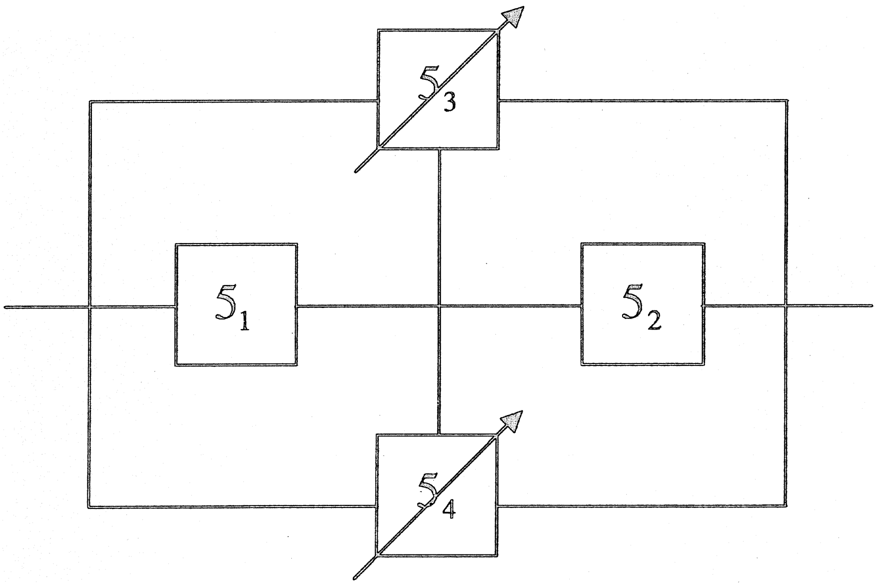

Fig. 3. Configuration 3: two 5 MW components with two standby components

Table 1. The size-proportional cost for the primary and warm standby components

Component	Cost (in $)
Primary 10 MW	10×10^6
Primary 5 MW	5×10^6
Warm standby 10 MW	6×10^6
Warm standby 5 MW	3×10^6

Table 2. The costs for each configuration i, $i = 1, 2, 3$

Configuration	Cost (in $)
Configuration 1	22×10^6
Configuration 2	13×10^6
Configuration 3	16×10^6

$$\mathbf{P}(0) = [P_3(0), P_2(0), P_1(0), P_0(0)] = [1, 0, 0, 0]. \tag{1}$$

Omitting the argument t in $P_n(t)$ so that $P_n(t) \equiv P_n$, we obtain the following differential equations:

$$\frac{dP_3}{dt} = -(\lambda + 2\alpha)P_3 + \mu P_2,$$
$$\frac{dP_2}{dt} = (\lambda + 2\alpha)P_3 - (\lambda + \alpha + \mu)P_2 + 2\mu P_1,$$
$$\frac{dP_1}{dt} = (\lambda + \alpha)P_2 - (\lambda + 2\mu)P_1,$$
$$\frac{dP_0}{dt} = \lambda P_1.$$

This can be written in matrix form as

$$\dot{\mathbf{P}} = \mathbf{QP},$$

where

$$Q = \begin{pmatrix} -\lambda - 2\alpha & \mu & 0 & 0 \\ \lambda + 2\alpha & -\lambda - \alpha - \mu & 2\mu & 0 \\ 0 & \lambda + \alpha & -\lambda - 2\mu & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}.$$

Without deriving the transient solutions, I propose the simple procedure to develop the explicit expression for the *MTTF*. To derive the *MTTF*, we take the transpose matrix of **Q** and delete the rows and columns for the absorbing state (s). The new matrix is called **A**. The expected times to reach an absorbing state is obtained from

$$E[T_{P_{(0)} \to P(absorbing)}] = \mathbf{P}(0)(-\mathbf{A}^{-1}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{2}$$

where

$$A = \begin{pmatrix} -\lambda - 2\alpha & \lambda + 2\alpha & 0 \\ \mu & -\lambda - \alpha - \mu & \lambda + \alpha \\ 0 & 2\mu & -\lambda - 2\mu \end{pmatrix}.$$

This method is successful because of the following relations

$$E[T_{P_{(0)} \to P(absorbing)}] = \mathbf{P}(0) \int_0^\infty e^{\mathbf{A}t} dt. \tag{3}$$

and

$$\int_0^\infty e^{\mathbf{A}t} dt = -\mathbf{A}^{-1}. \tag{4}$$

For configuration 1, we obtain the following explicit expression for the $MTTF_1$:

$$E[T_{P_{(0)} \to P(absorbing)}] = MTTF_1 = \frac{3\lambda(\lambda + 2\alpha + \mu) + 2(\alpha + \mu)^2}{\lambda(\lambda + \alpha)(\lambda + 2\alpha)}. \tag{5}$$

3.1.2 Availability as benefit

For the availability case of configuration 1, the initial conditions for this problem are the same as for the reliability case:

$$\mathbf{P}(0) = [P_3(0), P_2(0), P_1(0), P_0(0)] = [1, 0, 0, 0]. \tag{6}$$

For this case, the following differential equations written in matrix form can be obtained:

$$\begin{pmatrix} \dot{P}_3 \\ \dot{P}_2 \\ \dot{P}_1 \\ \dot{P}_0 \end{pmatrix} = \begin{pmatrix} -\lambda - 2\alpha & \mu & 0 & 0 \\ \lambda + 2\alpha & -\lambda - \alpha - \mu & 2\mu & 0 \\ 0 & \lambda + \alpha & -\lambda - 2\mu & 3\mu \\ 0 & 0 & \lambda & -3\mu \end{pmatrix} \begin{pmatrix} P_3 \\ P_2 \\ P_1 \\ P_0 \end{pmatrix}.$$

Let T_1 denote the time-to-failure of the system for configuration 1. To obtain the steady-state availability, we utilise the following procedure. In the steady-state, the derivatives of the state probabilities become zero. That allows us to calculate the steady-state probabilities with

$$A_{T_1}(\infty) = 1 - P_0(\infty), \tag{7}$$

and

$$\mathbf{QP}(\infty) = 0,$$

or, in matrix form:

$$\begin{pmatrix} -\lambda - 2\alpha & \mu & 0 & 0 \\ \lambda + 2\alpha & -\lambda - \alpha - \mu & 2\mu & 0 \\ 0 & \lambda + \alpha & -\lambda - 2\mu & 3\mu \\ 0 & 0 & \lambda & -3\mu \end{pmatrix} \begin{pmatrix} P_3(\infty) \\ P_2(\infty) \\ P_1(\infty) \\ P_0(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

Using the following normalizing condition

$$\sum_{i=0}^{3} P_i(\infty) = 1, \tag{9}$$

we substitute (9) in any one of the redundant rows in (8) to yield

$$\begin{pmatrix} -\lambda - 2\alpha & \mu & 0 & 0 \\ \lambda + 2\alpha & -\lambda - \alpha - \mu & 2\mu & 0 \\ 0 & \lambda + \alpha & -\lambda - 2\mu & 3\mu \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_3(\infty) \\ P_2(\infty) \\ P_1(\infty) \\ P_0(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

The solution of (10) provides the steady-state probabilities in the availability case. For configuration 1, the explicit expression for the $A_{T_1}(\infty)$ is given by

$$A_{T_1}(\infty) = \frac{3\mu\left[(\lambda + 2\alpha)(\lambda + \alpha + 2\mu) + 2\mu^2\right]}{\lambda(\lambda + \alpha)(\lambda + 2\alpha) + 3\mu\left[(\lambda + 2\alpha)(\lambda + \alpha + 2\mu) + 2\mu^2\right]}. \tag{11}$$

3.2 Calculations for configuration 2

3.2.1 MTTF as benefit

For configuration 2. The initial conditions are

$$\mathbf{P}(0) = [P_3(0), P_2(0), P_1(0)] = [1, 0, 0]. \tag{12}$$

The differential equations written in matrix form are given by

$$\dot{\mathbf{P}} = \mathbf{QP},$$

where

$$Q = \begin{pmatrix} -2\lambda - \alpha & \mu & 0 \\ 2\lambda + \alpha & -2\lambda - \mu & 0 \\ 0 & 2\lambda & 0 \end{pmatrix}.$$

It may be difficult to derive the transient solution. We use the above procedure shown in configuration 1. The expected times to reach an absorbing state is calculated from

$$E[T_{P_{(0)} \to P(absorbing)}] = \mathbf{P}(0)(-\mathbf{A}^{-1})\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2\lambda - \alpha & 2\lambda + \alpha \\ \mu & -2\lambda - \mu \end{pmatrix}.$$

For configuration 2, the explicit expression for the $MTTF_2$ is given by

$$E[T_{P_{(0)} \to P(absorbing)}] = MTTF_2 = \frac{4\lambda + \alpha + \mu}{2\lambda(2\lambda + \alpha)}. \tag{13}$$

3.2.2 Availability as benefit

For the availability case of configuration 2, the initial conditions are the same as for the reliability case:

$$\mathbf{P}(0) = [P_3(0), P_2(0), P_1(0)] = [1, 0, 0]. \tag{14}$$

The differential equations are given by

$$\begin{pmatrix} \dot{P}_3 \\ \dot{P}_2 \\ \dot{P}_1 \end{pmatrix} = \begin{pmatrix} -2\lambda - \alpha & \mu & 0 \\ 2\lambda + \alpha & -2\lambda - \mu & 2\mu \\ 0 & 2\lambda & -2\mu \end{pmatrix} \begin{pmatrix} P_3 \\ P_2 \\ P_1 \end{pmatrix}.$$

Let T_2 represent the time-to-failure of the system for configuration 2. In steady-state, the derivatives of the state probabilities become zero. We calculate the steady-state availability with

$$A_{T_2}(\infty) = 1 - P_1(\infty). \tag{15}$$

and

$$\mathbf{QP}(\infty) = 0,$$

or, in matrix form:

$$\begin{pmatrix} -2\lambda - \alpha & \mu & 0 \\ 2\lambda + \alpha & -2\lambda - \mu & 2\mu \\ 0 & 2\lambda & -2\mu \end{pmatrix} \begin{pmatrix} P_3(\infty) \\ P_2(\infty) \\ P_1(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

To obtain $P_1(\infty)$, we solve (16) and use the following normalizing condition:

$$\sum_{i=1}^{3} P_i(\infty) = 1.$$

For configuration 2, we obtain the explicit expression for the $A_{T_2}(\infty)$

$$A_{T_2}(\infty) = \frac{\mu(2\lambda + \alpha + \mu)}{\lambda(2\lambda + \alpha) + \mu(2\lambda + \alpha + \mu)}. \tag{17}$$

3.3 Calculations for configuration 3

3.3.1 MTTF as benefit

For configuration 3, The initial conditions are

$$\mathbf{P}(0) = [P_4(0), P_3(0), P_2(0), P_1(0)] = [1, 0, 0, 0]. \tag{18}$$

The differential equations written in matrix form are expressed as:

$$\dot{\mathbf{P}} = \mathbf{QP},$$

where

$$Q = \begin{pmatrix} -2\lambda - 2\alpha & \mu & 0 & 0 \\ 2\lambda + 2\alpha & -2\lambda - \alpha - \mu & 2\mu & 0 \\ 0 & 2\lambda + \alpha & -2\lambda - 2\mu & 0 \\ 0 & 0 & 2\lambda & 0 \end{pmatrix}.$$

Again, it is extremely difficult to develop the transient solution. We use the above procedure shown in configuration 1. The expected times to reach an absorbing state is evaluated from

$$E[T_{P_{(0)} \to P(absorbing)}] = \mathbf{P}(0)(-\mathbf{A}^{-1}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2\lambda - 2\alpha & 2\lambda + 2\alpha & 0 \\ \mu & -2\lambda - \alpha - \mu & 2\lambda + \alpha \\ 0 & 2\mu & -2\lambda - 2\mu \end{pmatrix}.$$

For configuration 3, we get the explicit expression for the $MTTF_3$

$$E[T_{P_{(0)} \to P(absorbing)}] = MTTF_3 = \frac{3\lambda(2\lambda + 2\alpha + \mu) + (\alpha + \mu)^2}{2\lambda(\lambda + \alpha)(2\lambda + \alpha)}. \tag{19}$$

3.3.2 Availability as benefit

For the availability case of configuration 3, the initial conditions are

$$\mathbf{P}(0) = [1, 0, 0, 0]. \tag{20}$$

The steady-state equations are given by

$$\begin{pmatrix} -2\lambda-2\alpha & \mu & 0 & 0 \\ 2\lambda+2\alpha & -2\lambda-\alpha-\mu & 2\mu & 0 \\ 0 & 2\lambda+\alpha & -2\lambda-2\mu & 3\mu \\ 0 & 0 & 2\lambda & -3\mu \end{pmatrix} \begin{pmatrix} P_4(\infty) \\ P_3(\infty) \\ P_2(\infty) \\ P_1(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let T_3 represent the time-to-failure of the system for configuration 3. The steady-state availability for configuration 3 is

$$A_{T_3}(\infty) = 1 - P_1(\infty). \tag{21}$$

Solving (21) and using the following normalizing condition:

$$\sum_{i=1}^{4} P_i(\infty) = 1,$$

we obtain $P_1(\infty)$.

For configuration 3, the explicit expression for the $A_{T_3}(\infty)$ yields

$$A_{T_3}(\infty) = \frac{3\mu\left[(\lambda+\alpha)(2\lambda+\alpha+2\mu)+\mu^2\right]}{2\lambda(\lambda+\alpha)(2\lambda+\alpha)+3\mu\left[(\lambda+\alpha)(2\lambda+\alpha+2\mu)+\mu^2\right]}. \tag{22}$$

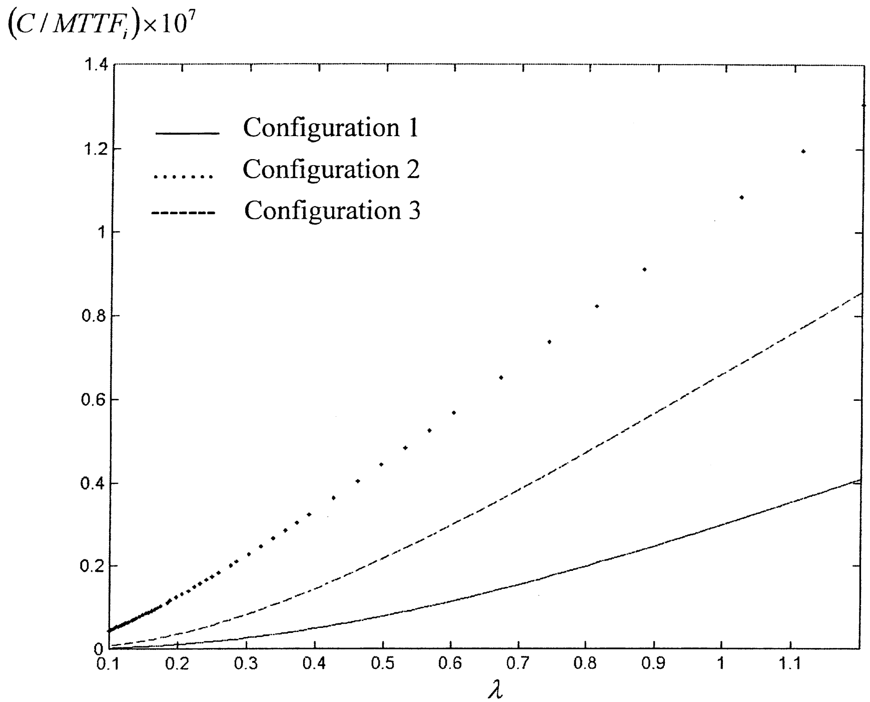

Fig. 4. $C/MTTF_i$ versus failure rate λ

4 Comparison of the three configurations

4.1 Comparison of all configurations

In this section we compare the three configurations in terms of their $MTTF_i$ and their $A_{T_i}(\infty)$, where $i = 1, 2, 3$.

We first perform a comparison of the *MTTF* for the configurations 1, 2, and 3. Using the algebraic manipulations, it can be shown that the following results hold for all choices of λ, α, and μ.

$$MTTF_1 > MTTF_3 > MTTF_2. \tag{23}$$

Next, using the computer software *MALAB*, the comparison of the $A_T(\infty)$ for the configurations 1, 2, and 3, are given by:

$$A_{T_1}(\infty) > A_{T_3}(\infty) > A_{T_2}(\infty). \tag{24}$$

4.2 Comparison of all configurations based on their cost/benefit ratios

We consider that the various configurations may have different costs when comparing all configurations. From Table 2, the cost (C_i) of the configuration i $(i = 1, 2, 3)$ are listed in the following:

$$C_1 = \$22 \times 10^6, \quad C_2 = \$13 \times 10^6, \quad C_3 = \$16 \times 10^6.$$

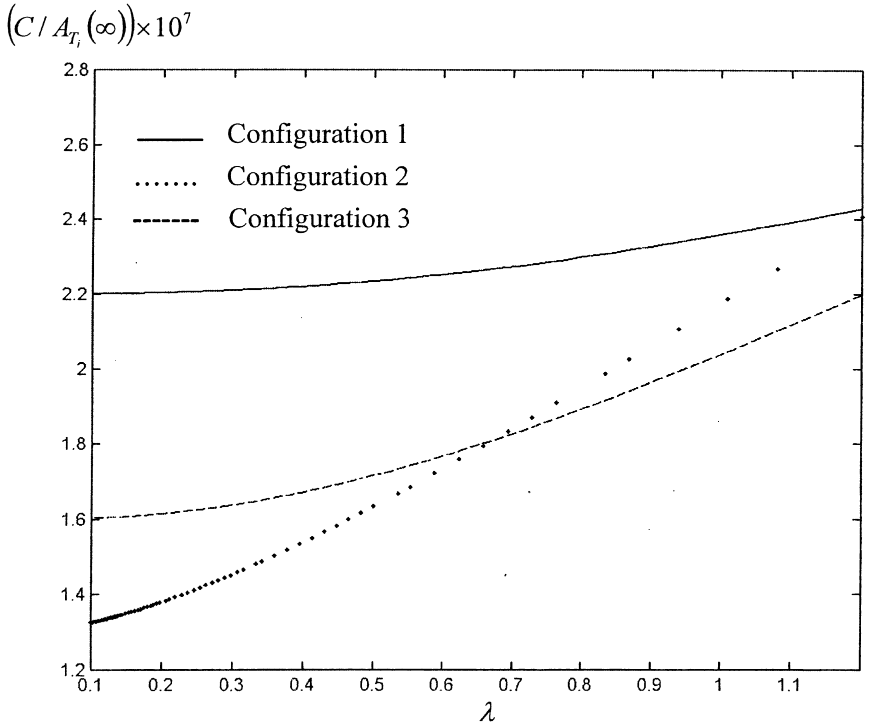

Fig. 5. $C/A_{T_i}(\infty)$ versus failure rate λ

We first fix $\alpha = 0.05$, $\mu = 1.0$, and vary the values of λ from 0.1 to 1.2. The results of the cost/benefit (C_i/B_i) ratios, namely, $C_i/MTTF_i$ and $C_i/A_{T_i}(\infty)$ for each configuration i $(i = 1, 2, 3)$ are depicted in Figure 4 and Figure 5, respectively. Figure 4 and Figure 5 show that the $C_i/MTTF_i$ and the $C_i/A_{T_i}(\infty)$ increase as λ increases for any configuration. One observes from Figure 4 that the optimal configuration using the $C_i/MTTF$ value is configuration 1. We observe from Figure 5 that the optimal configuration using the $C_i/A_T(\infty)$ value depends on the value of λ. When $\lambda < 0.7$, the optimal configuration is configuration 2, but when $\lambda \geq 0.7$, the optimal configuration is configuration 3.

Next, we fix $\alpha = 0.05$, $\lambda = 0.6$, and vary the values of μ from 0.1 to 1.2. The results of the cost/benefit (C_i/B_i) ratios, namely, $C_i/MTTF_i$ and $C_i/A_{T_i}(\infty)$ for each configuration i $(i = 1, 2, 3)$ are depicted in Figure 6 and Figure 7, respectively. We can easily see Figures 6 and 7 that the $C_i/MTTF_i$ and the $C_i/A_{T_i}(\infty)$ decrease as μ decreases for any configuration. Figure 6 shows that the optimal configuration using the $C_i/MTTF$ value is configuration 1. One sees from Figure 7 that the optimal configuration using the $C_i/A_T(\infty)$ value is configuration 2.

5 Conclusions

The primary objectives of this paper have been:

(1) to model three different series system configurations with warm standby components;

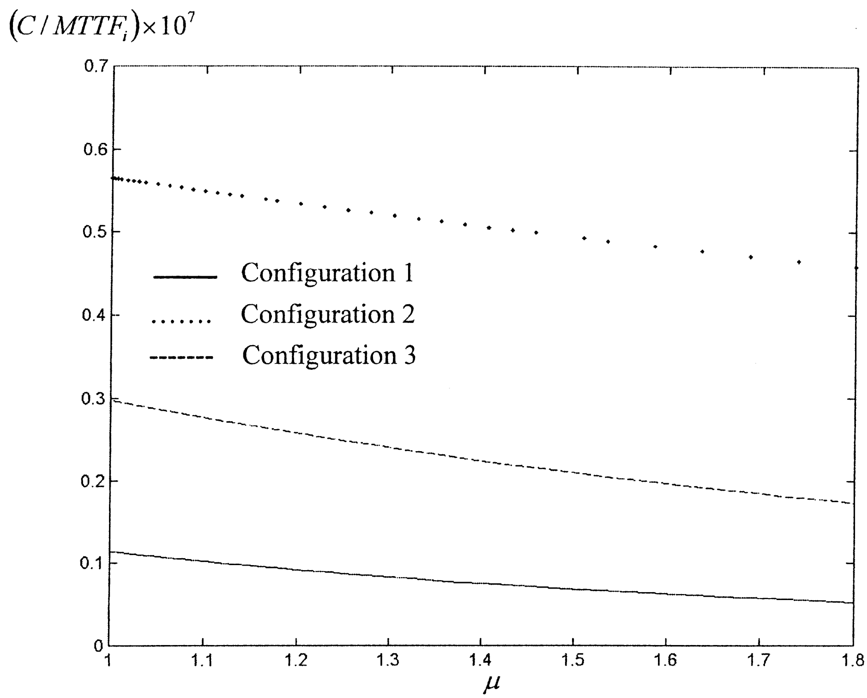

Fig. 6. $C/MTTF_i$ versus failure rate μ

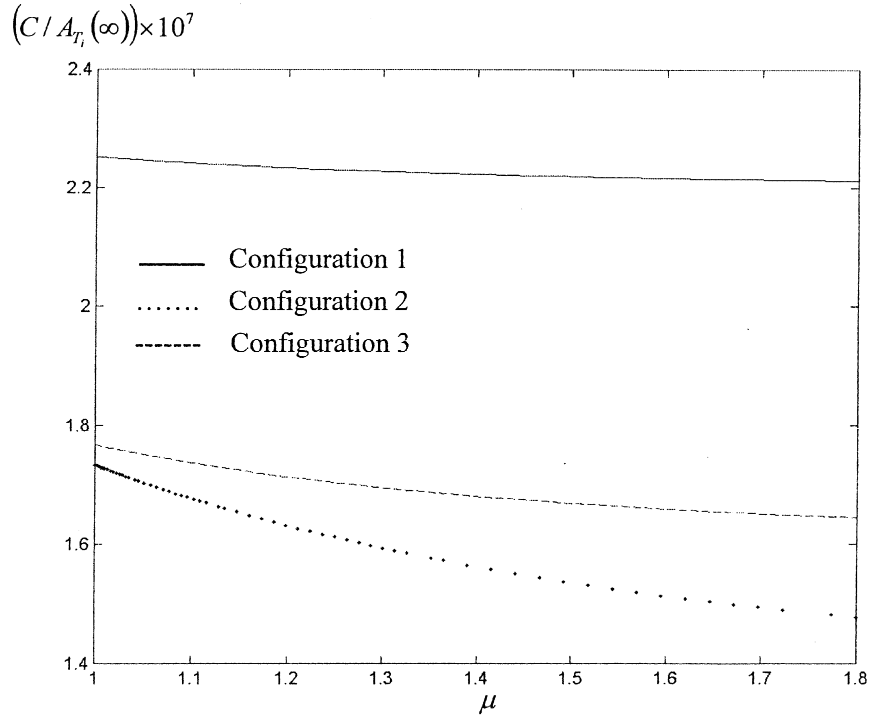

Fig. 7. $C/A_{T_i}(\infty)$ versus failure rate μ

(2) to present cost/benefit analysis of three configurations under uncertainty;
(3) to develop explicit expressions for the *MTTF*, the $A_T(\infty)$, and the C/B, for three configurations and perform comparisons;
(4) to rank three configurations based on the *MTTF*, the $A_T(\infty)$, and the C/B, where B is either *MTTF* or $A_T(\infty)$.

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