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Note

Distributional and Inferential Properties of the Estimated Precision C_p Based on Multiple Samples

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Abstract. Process precision index C_p has been widely used in the manufacturing industry to provide numerical measures on process potential. Pearn et al. (1998) considered an unbiased estimator of C_p for one single sample. They showed that the unbiased estimator is the UMVUE. They also proposed an efficient test for C_p based on one single sample, and showed that the test is the UMP test. In this paper, we consider an unbiased estimator of C_p for multiple samples. We show that the unbiased estimator is the UMVUE of C_p, which is asymptotically efficient. We also consider an efficient test for C_p, and show that the test is the UMP test for multiple samples. The practitioners can use the proposed test on their in-plant applications to obtain reliable decisions.

Key words: process precision index, unbiased estimator, UMVUE, asymptotically efficient, UMP test, p-value, power.

1. Introduction

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications, have been the focus of recent research in quality assurance and process capability analysis. Those capability indices, quantifying process potential and performance, are important for any successful quality improvement activities and quality program implementation. The first process capability index appeared in the literature is the precision index C_p, which is defined by Kane (1986) as:

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma},$$

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the range over which the process measurements are allowable. The denominator gives the range over which the process is actually varying. The index C_p was designed to measure

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the magnitude of the overall process variation relative to the manufacturing tolerance, which is to be used for processes with data that are normal, independent, and in statistical control. Clearly, the index only measures the potential of a process (the potential to reproduce acceptable product), and does not take into account whether the process is centered. The use of the capability indices was first explored within the automotive industry. Ford Motor Company (1986) has used C_p to keep track of the process performance and to reduce process variation. Recently, the manufacturing industries have been making an extensive effort to implement statistical process control (SPC) in their plants and supply bases. Capability indices derived from SPC have received increasing usage not only in capability assessments, but also in the evaluation of purchasing decisions. Capability indices are becoming the standard tools for quality report, particularly, at the management level around the world. Proper understanding and accurate estimating them is essential for the company to maintain a capable supplier.

2. Estimating C_p based on Multiple Samples

For cases where the data are collected as one single sample, Pearn et al. (1998) considered an unbiased estimator of C_p. They showed that the unbiased estimator is the UMVUE (uniformly minimum variance unbiased estimator) of C_p. They also proposed an efficient test for C_p based on one single sample, and showed that the test is the UMP test. For cases where the data are collected as multiple samples, Kirmani et al. (1991) considered m samples each of size n and suggested the following estimator of C_p, where $\bar{X}_i$ is the ith sample mean, and S_i is the ith sample standard deviation:

$$\hat{C}_P^* = \frac{\text{USL} - \text{LSL})d_p}{6},$$

where

$$d_p = \sqrt{\frac{m(n-1)-1}{m(n-1)}}\frac{\epsilon_{m(n-1)-1}}{S_p},$$

$$\begin{aligned}\epsilon_{m(n-1)-1} &= E\left[\frac{S_i}{\sigma}\right] = E\left[\frac{\chi_{m(n-1)-1}}{\sqrt{m(n-1)-1}}\right]\\ &= \sqrt{\frac{2}{m(n-1)-1}}\Gamma\left(\frac{m(n-1)}{2}\right)\left[\Gamma\left(\frac{m(n-1)-1}{2}\right)\right]^{-1},\end{aligned}$$

$$S_p^2 = \frac{1}{m(n-1)}\sum_{i=1}^{m}(n-1)S_i^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2,$$

noting that the statistic S_i/σ is distributed as $\chi_{m(n-1)-1}/[m(n-1)-1]^{1/2}$. Under normality assumption, the estimator $\hat{C}_p^*$ is distributed as:

$$\hat{C}_p^* \sim \frac{\sqrt{m(n-1)-1}\,\epsilon_{m(n-1)-1}}{\sqrt{\chi^2_{m(n-1)}}} C_p.$$

The estimator $\hat{C}_p^*$ is unbiased, and its probability density function (PDF) can be obtained as the following, for $y > 0$, where $k = [m(n-1)-1]C_p^2\epsilon^2_{m(n-1)-1}$, which is a function of C_p.

$$g(y) = \frac{2k^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]} y^{-[m(n-1)+1]} \exp\left[-\frac{k}{2}\left(\frac{1}{y^2}\right)\right].$$

The variance of $\hat{C}_p^*$ can be calculated as the following (Kirmani et al. (1991)). Tables I(a)–I(d) display the values of the variance for $C_p = 1.00$, 1.33, 1.67, 2.00, $m = 10(5)25$, and $n = 2(1)15$. We note that for fixed $m \times n$ sample observations, Var $(\hat{C}_p^*)$ for large m and small n is greater than that for small m and large n. For example, for $C_p = 1.00$ with $m \times n = 60$ Var $(\hat{C}_p^*) = 0.0133$ for $m = 20$, $n = 3$, Var $(\hat{C}_p^*) = 0.0117$ for $m = 15$, $n = 4$, and Var $(\hat{C}_p^*) = 0.0105$ for $m = 10$, $n = 6$. Similarly, for $C_p = 1.00$ with $m \times n = 100$, Var $(\hat{C}_p^*) = 0.0068$ for $m = 25$, $n = 4$, Var $(\hat{C}_p^*) = 0.0064$ for $m = 20$, $n = 5$, and Var $(\hat{C}_p^*) = 0.0056$ for $m = 10$, $n = 10$.

$$\begin{aligned}
\text{Var}\,(\hat{C}_p^*) &= E[(\hat{C}_p^*)^2] - [E(\hat{C}_p^*)]^2 \\
&= (\text{USL} - \text{LSL})^2 \epsilon^2_{m(n-1)-1} \frac{[m(n-1)-1]}{36m(n-1)} E\left(\frac{1}{S_P^2}\right) - C_P^2 \\
&= C_P^2 \left\{\left[\frac{m(n-1)-1}{m(n-1)-2}\right]\epsilon^2_{m(n-1)-1} - 1\right\} \\
&= C_P^2 \left\{\frac{1}{\epsilon^2_{m(n-1)-2}} - 1\right\}
\end{aligned}$$

In the following, we investigate some other statistical properties of $\hat{C}_p^*$. We show that $\hat{C}_p^*$ is the UMVUE of C_p, which is also asymptotically efficient. Under regular conditions, an estimator $\hat{\theta}_n$ is said to be asymptotically efficient if the asymptotic efficiency, $\lim_{n\to\infty} e(\hat{\theta}_n) = \lim_{n\to\infty}[1/I(\theta)\text{Var}\,(\hat{\theta}_n)] = 1$, where $1/I(\theta)$ is the Cramer–Rao lower bound.

THEOREM 1. If the process characteristic follows the normal distribution, then

Table Ia. Variance of $\hat{C}_p^*$ for $C_p = 1.00$, for $m = 10(5)25$, and $n = 2(1)15$.

m n	10	15	20	25
2	0.0643	0.0391	0.0282	0.0220
3	0.0282	0.0180	0.0133	0.0105
4	0.0180	0.0117	0.0087	0.0068
5	0.0133	0.0087	0.0064	0.0050
6	0.0105	0.0068	0.0050	0.0040
7	0.0087	0.0056	0.0042	0.0034
8	0.0074	0.0048	0.0036	0.0028
9	0.0064	0.0042	0.0032	0.0026
10	0.0056	0.0038	0.0028	0.0022
11	0.0050	0.0034	0.0026	0.0020
12	0.0046	0.0030	0.0022	0.0018
13	0.0042	0.0028	0.0020	0.0016
14	0.0040	0.0026	0.0020	0.0016
15	0.0036	00.002	0.0018	0.0014

Table Ib. Variance of $\hat{C}_p^*$ for $C_p = 1.33$, for $m = 10(5)25$, and $n = 2(1)15$.

m n	10	15	20	25
2	0.1138	0.0692	0.0499	0.0388
3	0.0499	0.0319	0.0236	0.0185
4	00.031	0.0207	0.0153	0.0121
5	0.0236	0.0153	0.0114	0.0089
6	0.0185	0.0121	0.0089	0.0071
7	0.0153	0.0099	0.0075	0.0060
8	0.0132	0.0085	0.0064	0.0050
9	0.0114	0.0075	0.0057	0.0046
10	0.0099	0.0067	0.0050	0.0039
11	0.0089	0.0060	0.0046	0.0035
12	0.0082	0.0053	0.0039	0.0032
13	0.0075	0.0050	0.0035	0.0028
14	0.0071	0.0046	0.0035	0.0028
15	0.0064	0.0043	0.0032	0.0025

Table Ic. Variance of $\hat{C}_p^*$ for $C_p = 1.67$, for $m = 10(5)25$, and $n = 2(1)15$.

m n	10	15	20	25
2	0.1795	0.1091	0.0786	0.0612
3	0.0786	0.0503	0.0372	0.0292
4	0.0503	0.0326	0.0241	0.0191
5	0.0372	0.0241	0.0179	0.0140
6	0.0292	0.0191	0.0140	0.0112
7	0.0241	0.0157	0.0118	0.0095
8	0.0208	0.0134	0.0101	0.0078
9	0.0179	0.0118	0.0089	0.0073
10	0.0157	0.0106	0.0078	0.0061
11	0.0140	0.0095	0.0073	0.0056
12	0.0129	0.0084	0.0061	0.0050
13	0.0118	0.0078	0.0056	0.0045
14	0.0112	0.0073	0.0056	0.0045
15	0.0101	0.0067	0.0050	0.0039

Table Id. Variance of $\hat{C}_p^*$ for $C_p = 2.00$, for $m = 10(5)25$, and $n = 2(1)15$.

m n	10	15	20	25
2	0.2574	0.1564	0.1127	0.0878
3	0.1127	0.0722	0.0533	0.0419
4	0.0722	0.0468	0.0346	0.0273
5	0.0533	0.0346	0.0257	0.0201
6	0.0419	0.0273	0.0201	0.0160
7	0.0346	0.0225	0.0169	0.0136
8	0.0298	0.0193	0.0144	0.0112
9	0.0257	0.0169	0.0128	0.0104
10	0.0225	0.0152	0.0112	0.0088
11	0.0201	0.0136	0.0104	0.0080
12	0.0185	0.0120	0.0088	0.0072
13	0.0169	0.0112	0.0080	0.0064
14	0.0160	0.0104	0.0080	0.0064
15	0.0144	0.0096	0.0072	0.0056

(a) $\hat{C}_p^*$ is the UMVUE of C_p.
(b) $(mn)^{1/2}(\hat{C}_p^* - C_p)$ converges to $N(0, [C_p]^2/2)$ in distribution.
(c) $\hat{C}_p^*$ is asymptotically efficient.

Proof: (a) It is easy to show that the statistics S_p^2 is a complete sufficient statistic for C_p since the probability density function of $\hat{C}_p^*$ belongs to the exponential family. Further, since $\hat{C}_p^*$ is an unbiased estimator for C_p, which is also a function of S_p^2 only, then by Lehmann–Scheffe's theorem (see Arnold (1990)), $\hat{C}_p^*$ is an UMVUE of C_p.

(b) If the process characteristic is normally distributed, then the statistic $(mn)^{1/2}(S_p^2 - \sigma^2)$ converges to $N(0, 2\sigma^4)$ in distribution. We define the continuous function, $g(t)$, as

$$g(t) = (\mathrm{USL} - \mathrm{LSL})/(6t^{1/2}),$$

and its derivative is

$$g'(t) = -(\mathrm{USL} - \mathrm{LSL})/(12t^{3/2}).$$

By the Cramer-σ theorem (see Arnold (1990)), we have

$$\sqrt{mn}[g(S_p^2) - g(\sigma^2)] \to N(0, 2\sigma^2[g'(\sigma^2)]^2)$$

in distribution, where $[g'(\sigma^2)]^2 = [C_p]^2/(4\sigma^4)$. The result is obviously equivalent to $\sqrt{mn}(d/3S_P - C_p) \to N(0, C_p^2/2)$ in distribution. Kirmani et al. (1991) proved that $\hat{C}_p^*$ is a consistent estimator of C_p, then $\hat{C}_p^*$ converges to C_p in probability. Thus, by Slutzky's Theorem (see Arnold (1990)) we have

$$(mn)^{1/2}(\hat{C}_p^* - d/3S_P) \to N(0, [C_p]^2/2)$$

in distribution, and so

$$(mn)^{1/2}(\hat{C}_p^* - C_p) \to N[0, [C_p]^2/2)$$

in distribution.

(c) Noted that Var $(\hat{C}_p^*) = [C_P]^2\{1/\epsilon^2_{m(n-1)-2} - 1\}$ by Kirmani (1991). For single sample of size n, the information for C_P is

$$I_1(C_P) = E[-\partial^2 \log f_1(x; C_P)/\partial C_P^2] = 2(n-1)/C_P^2,$$

where

$$f_1(x; C_P) = 2\frac{(\sqrt{(n-1)/2}C_p)^{n-1}}{\Gamma[(n-1)/2]}x^{-n}\exp[-(n-1)(C_p)^2(2x^2)^{-1}], \quad x > 0$$

by Chou and Owen (1989). Therefore, the information for m independent subgroups of each size of n $I(C_P) = mI_1(C_P) = 2m(n-1)/(C_P)^2$. Next, we computed the asymptotic efficiency for $\hat{C}_p^*$.

$$\begin{aligned}
\lim_{m(n-1)\to\infty} e(\hat{C}_P^*) &= \lim_{m(n-1)\to\infty} C_P^2/2m(n-1)C_P^2(\epsilon_{m(n-1)-2}^{-2} - 1, \text{ let } k = m(n-1) \\
&= \lim_{k\to\infty} \epsilon_{k-2}^2/2k(1-\epsilon_{k-2}^2) \\
&= \lim_{k\to\infty} [1 - (1/2k) + (1/8k^2) \\
&\quad + o(1/k^2)]/\{2k[(1/2k) - (1/8k^2) + o(1/k^2)]\} = 1.
\end{aligned}$$

Therefore, $\hat{C}_p^*$ is asymptotically efficient.

3. UMP Test for C_p Based on Multiple Samples

For cases with multiple samples, to determine whether a given process meets the preset requirement and runs under the desired quality condition, we consider the following testing hypotheses with null hypothesis H_0: $C_p \leqq C$ (the process is incapable), versus the alternative H_1: $C_p > C$ (the process is capable). Thus, we may consider the test $\phi^*(x) = 1$ if $\hat{C}_p^* > c^*$, and $\phi^*(x) = 0$, otherwise. The test ϕ^* rejects the null hypothesis if $\hat{C}_p^* > c^*$, with type I error $\alpha(c^*) = \alpha$, the chance of incorrectly judging an incapable process as capable. Kirmani et al. (1991) obtained the critical value c^*, which satisfies the following equation:

$$P[\hat{C}_p^* > c^* \mid H_0 : Cp \leq C] = P\left[\chi_{m(n-1)}^2 < \frac{[m(n-1)-1]\epsilon_{m(n-1)-1}^2}{c^*} C^2\right] = \alpha.$$

$$c^* = C\sqrt{\frac{[m(n-1)-1]\epsilon_{m(n-1)-1}^2}{\chi_{m(n-1),\alpha}^2}},$$

where $\chi_{m(n-1),\alpha}^2$ is the lower α-percentage point on the chi-square distribution with $m(n-1)$ degrees of freedom. The null hypothesis ($C_p \leqq C$) is rejected and the process is declared capable if the value of $\hat{C}_p^*$ is greater than c^*.

A test is said to be the uniformly most powerful test (UMP) against the alternative H_1 (but not against another if H_0 is simple but H_1 is composite) if it is the most powerful against every simple alternative in H_1. As noted by Lindgren (1968), if the process characteristic X has a distribution in the exponential family, with $f(x;\theta) = B(\theta)h(x)\exp[Q(\theta)S(x)]$, and if $Q(\theta)$ is monotone increasing, then the critical region $S(X) > K$ is uniformly most powerful for $\theta \leqq \theta^*$ against $\theta > \theta^*$. Thus, we can show the following:

THEOREM 2. For the testing hypotheses, H_0: $C_p \leqq C$ versus H_1: $C_p > C$, the test defined as $\phi^*(x) = 1$ if $\hat{C}_p^* > c^*$, and $\phi^*(x) = 0$ otherwise, is the UMP test of level α.

Proof: Under the assumption of normality, the density function of $\hat{C}_p^*$ is given below, where $k = [m(n-1)-1]C_p^2 \epsilon_{m(n-1)-1}^2$.

$$f(x) = \frac{2k^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]} x^{-[m(n-1)+1]} \exp\left[-\frac{k}{2}\left(\frac{1}{x^2}\right)\right].$$

We note that the above probability density function belongs to the exponential family with $S(x) = -(x-2)$, $Q(C_P) = (1/2)[m(n-1)-1]C_p^2\epsilon_{m(n-1)-1}^2$, x is real and $Q(C_P)$ is strictly increasing in C_P. Thus, by the theory described in Lindgren (1968) it is clear that the test ϕ^* is the uniformly most powerful. The UMP test rejects the null hypothesis if, and only if, $-x^{-2} \geqq -(c^*)^{-2}$, where $P[-x^{-2} \geqq -(c^*)^{-2}] = \alpha$. Since $\hat{C}_p^* = [m(n-1)-1]^{-1/2}\epsilon_{m(n-1)-1}C_p K^{-1/2}$, where K is distributed as $\chi_{m(n-1)}^2$, and the critical region can be expressed as following:

$$\left\{C \mid \frac{K}{[m(n-1)]\epsilon_{m(n-1)-1}^2 C^2} \leq \frac{1}{[c^*]^2}\right\}.$$

The critical value, c^*, for an α level of significance is derived by satisfying the equation,

$$\frac{[m(n-1)]\epsilon_{m(n-1)-1}^2 C^2}{(c^*)^2} \geq \chi_{m(n-1),\alpha}^2$$

$$(c^*)^2 = \frac{[m(n-1)]\epsilon_{m(n-1)-1}^2 C^2}{\chi_{m(n-1),\alpha}^2}.$$

In the following, we first calculate the p-value (risk for wrongly rejecting the null hypothesis H_0: $C_p \leqq C$) given an observed value of the statistic. Suppose the observed value of the statistic $\hat{C}_p^* = W$, then we can calculate p-value as the following, where K is distributed as $\chi_{m(n-1)}^2$.

$$\begin{aligned}
p\text{-value} &= P\{\hat{C}_P^* \geq W \mid C_P \leq C\} \\
&= P\left\{\frac{\sqrt{m(n-1)-1}\,\epsilon_{m(n-1)-1}C}{\sqrt{K}} \geq W \mid C_P \leq C\right\} \\
&= P\left\{\frac{[m(n-1)-1]\epsilon_{m(n-1)-1}^2 C^2}{K} \geq W^2 \mid C_P \leq C\right\} \\
&= P\left\{\chi_{m(n-1)}^2 \leq \frac{[m(n-1)-1]\epsilon_{m(n-1)-1}^2 C^2}{W^2} \mid C_P \leq C\right\}.
\end{aligned}$$

The power of the UMP test (probability of correctly rejecting the null hypothesis $C_p \leqq C$ when the true $C_p > C$), can also be computed for the given alternative hypothesis, H_1: $C_p = C_I > C$. The power of the test, denoted as $\pi(C_P)$ can be computed as the following:

$$\begin{aligned}
\pi(C_P) &= P\{\hat{C}_P > c^* \mid C_P = C_1\} \\
&= P\left\{\frac{\sqrt{m(n-1)-1}\epsilon_{m(n-1)-1}C_1}{\sqrt{K}} \geq c^* \mid C_P = C_1\right\} \\
&= P\left\{\frac{[m(n-1)-1]\epsilon^2_{m(n-1)-1}C_1^2}{K} \geq c^{*2} \mid C_P = C_1\right\} \\
&= P\left\{\chi^2_{m(n-1)} \leq \frac{[m(n-1)-1]\epsilon^2_{m(n-1)-1}C_1^2}{c^{*2}} \mid C_P = C_1\right\}.
\end{aligned}$$

4. An Application

Consider a forging manufacturing process making a specific type of piston rings for automotive engines. The engineers wish to establish a precision control of the inside diameter of the piston rings to monitor the process performance, for this particular type of piston rings, using the process precision index C_p. The specification limits for the inside diameter of the piston ring are set to the upper specification limit USL = 74.050 mm, and the lower specification limit LSL = 73.950 mm.

Ten samples, each of size five are taken from the process that is demonstrably in control (stable). The inside diameter measurement data for the ten samples are displayed in Table II. The minimal precision requirement of this process is set to $C_p = 1.33$ in the factory, which is continuously used within the automotive industry as a capability benchmark. To test whether the piston ring manufacturing process meets the precision requirement or not, we use the UMP test developed for multiple samples for the hypotheses, H_0: $C_p \leqq 1.33$ versus alternative H_1: $C_p > 1.33$, to obtain a reliable decision making with risk $\alpha = 0.05$. The calculated sample mean and the sample variance for the ten samples are tabulated in Table III. We also run the SAS computer software to obtain the critical value 1.60 for risk $\alpha = 5\%$. Thus, we have

$$\begin{aligned}
S_p^2 &= \frac{1}{m(n-1)}\sum_{i=1}^{m}(n-1)S_i^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2 = 0.000093, \\
\hat{C}_p^* &= 1.69, \quad c^* = 1.60.
\end{aligned}$$

Since calculated C_p from the sample data, 1.69, is greater than the critical value 1.60, then we may conclude, with 95% confidence, that the process meets the

Table II. The collected sample data (10 samples, a total of 50 observations).

Sample 1	73.995	73.992	74.001	74.011	74.004
Sample 2	73.992	74.007	74.015	73.989	74.014
Sample 3	73.985	74.003	73.993	74.015	73.988
Sample 4	73.988	74.000	73.990	74.007	73.995
Sample 5	73.994	73.998	73.994	73.995	73.990
Sample 6	74.012	74.014	73.998	73.999	74.007
Sample 7	74.006	74.010	74.018	74.003	74.000
Sample 8	73.988	74.001	74.009	74.005	73.996
Sample 9	74.015	74.008	73.993	74.000	74.010
Sample 10	73.982	73.984	73.995	74.017	74.013

Table III. The calculated sample mean, and the sample variance for the 10 samples.

Sample 1	74.001	0.000056
Sample 2	74.003	0.000149
Sample 3	73.997	0.000150
Sample 4	73.996	0.000060
Sample 5	73.994	0.000008
Sample 6	74.006	0.000053
Sample 7	74.007	0.000049
Sample 8	74.000	0.000067
Sample 9	74.005	0.000076
Sample 10	73.998	0.000262

precision requirement $C_p > 1.33$. The probability of wrongly judging an incapable process as a capable one is 5%.

5. Conclusions

Process precision index C_p has been widely used in the manufacturing industry to provide numerical measures on process potential. It measures the overall process variation relative to the specification tolerance. Statistical properties of the estimated C_p based on one single sample, have been investigated extensively. But, the properties of the estimated C_p based on multiple samples have been comparatively neglected. In this paper, we considered an estimator of C_p denoted as $\hat{C}_p^*$, based on multiple random samples with each of size n, and investigated its statistical properties. We showed that the estimator $\hat{C}_p^*$ is the UMVUE of C_p, which is also asymptotically efficient. In addition, we showed that the test based on the UMVUE of C_p is the UMP test. Using this test, the practitioners can make reliable decisions on whether their processes meet the precision requirement preset in the factory, with the decision error minimized.

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