# Levinson theorem with the nonlocal Aharonov-Bohm effect

De-Hone Lin*
*Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30043, Taiwan*
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Levinson theorem for a charged particle moving in an arbitrary short-range potential and the field of the Aharonov-Bohm magnetic flux is established. The theorem constructs the relation $\delta_\alpha(0)=n_\alpha\pi$ between the phase shift $\delta_\alpha(k)$ of scattering state at zero momentum and the total number $n_\alpha$ of bound states for the $\alpha$th angular-momentum channel, where $\alpha=|m+\mu_0|$ is a real number ($m=$integer, and $\mu_0=-\Phi/\Phi_0$ with $\Phi$ being the magnetic flux and $\Phi_0=hc/e$ the fundamental flux quantum). The relation means that the phase shift at the threshold of zero momentum can serve as a counter for the bound states in the general angular-momentum channel.

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## I. INTRODUCTION

In 1949, Levinson discovered one of the most beautiful theorems in quantum mechanics [1]. Well known as the Levinson theorem, it clarifies the relation between the phase shifts of a quantum particle scattered by a short-range potential and the number of bound states therein. In three-dimensional space, the theorem can be described as

$$\delta_l(0)=n_l\pi, \quad l=1,2,\ldots,$$

where $\delta_l(0)$ denotes the phase shift of scattered state with a linear momentum $k$ at the threshold of zero momentum, i.e., $k=0$, in the angular-momentum channel $l$, and $n_l$ is the total number of bound states in the angular-momentum channel $l$ allowed by the short-range potential. When the angular momentum $l=0$, the theorem must be modified to

$$\delta_0(0)=(n_0+1/2)\pi$$

due to the existence of a zero-energy resonance (a half-bound state). Later some authors devoted Levinson's verification to discuss the elegant theorem by way of the different manners, or generalized it to the more general cases [2–8]. Slightly different from the three-dimensional case, Levinson theorem in two dimensions can be expressed as [8]

$$\delta_m(0)=n_m\pi, \quad m=0,1,2,\ldots, \tag{1}$$

where $\delta_m(0)$ is the phase shift of scattering state at threshold of angular-momentum channel $m$, and $n_m$ is the total number of the bound states in the same channel. The total number of bound states in the angular-momentum channel $-m$ is the same as that of $m$. This is due to the fact that the phase shift and the number of bound states just relate to the angular momentum $m$ via its absolute value $|m|$ in cylindrically symmetric system. Ten years after Levinson's work, Aharonov and Bohm found that a charged particle can be influenced by the magnetic field even if the particle is nowhere in the region of nonzero field strength [9]. The phenomenon is somewhat counterintuitive and represents a nonlocal and topological effect in quantum mechanics. The term "nonlocal" means that it exists even when the charged particle passes through a field-free region and is only associated with the entire closed curve. It is "topological" in the sense that the phase interference is unaffected when the particle path of closed curve is deformed within the field-free region. Forty years later, Aharonov-Bohm (AB) effect had great impact on our comprehension of the foundation of quantum theory [10], and helped in the understanding of the quantum Hall effect [11,12], superconductivity [12,13], and so forth.

In this paper we shall generalize the Levinson theorem for a charged particle moving in an arbitrary short-range potential to include the field of the nonlocal AB effect. This paper is organized as follows. In the following section we establish the partial-wave method for scattering theory in two dimensions for a short-range potential and the nonlocal AB effect. The asymptotic behavior of phase shifts at threshold is discussed in Sec. III. In Sec. IV, the Levinson theorem is generalized to charged particles moving in the potential $V(\rho)$, which is less singular than $\rho^{-2}$ when $\rho\leqslant a$ and $V(\rho)=0$ when $\rho\geqslant a$, and in the field of the nonlocal AB effect using Green's-function method. The number of bound states $n_\alpha$ for a given general angular momentum $\alpha=|m+\mu_0|$ is related to the phase shifts $\delta_\alpha(0)$ of zero-momentum scattering states as follows:

$$\delta_\alpha(0)=n_\alpha\pi, \quad \alpha=|m+\mu_0|, \tag{2}$$

where $m$ is an integer, and $\mu_0=-\Phi/\Phi_0$ with $\Phi$ being the AB flux and $\Phi_0=hc/e$ the fundamental flux quantum. Our discussions are summarized in Sec. V.

## II. PARTIAL-WAVE METHOD FOR A SHORT-RANGE POTENTIAL AND AN AB MAGNETIC FLUX

The fixed-energy Green's function $G^0(\mathbf{r},\mathbf{r}';E)$ for a charged particle with mass $\mu$ propagating from $\mathbf{r}'$ to $\mathbf{r}$ satisfies the Schrödinger equation

$$\left[E-H_0\left(\mathbf{r},\frac{\hbar}{i}\nabla\right)\right]G^0(\mathbf{r},\mathbf{r}';E)=\delta^2(\mathbf{r}-\mathbf{r}'). \tag{3}$$

*Electronic address: dhlin@mail.nctu.edu.tw

 

Here the Hamiltonian of the system is given by $H_0 = -\hbar^2\nabla^2/2\mu + V(\mathbf{r})$ and $\mathbf{r}$ represents the two-dimensional position vector. In the cylindrically symmetric system, the angular decomposition of the Green's function can be written as

$$G^0(\mathbf{r},\mathbf{r}';E) = \sum_{m=-\infty}^{\infty} G_m^0(\rho,\rho';E)\frac{e^{im(\varphi-\varphi')}}{2\pi}, \tag{4}$$

with $(\rho,\varphi)$ being the polar coordinates in two-dimensional space and $G_m^0(\rho,\rho';E)$ the radial Green's function. As a result, the left-hand side (lhs) of Eq. (3) can be brought to the following form:

$$\sum_{m=-\infty}^{\infty}\left\{E+\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)\right]-V(\rho)\right\} \times G_m^0(\rho,\rho';E)\frac{e^{im(\varphi-\varphi')}}{2\pi}. \tag{5}$$

For a charged particle affected by a magnetic field, Green's function $G(\mathbf{r},\mathbf{r}';E)$ is different from $G^0(\mathbf{r},\mathbf{r}';E)$ by a globally nonintegrable phase factor [14,15]:

$$G(\mathbf{r},\mathbf{r}';E) = G^0(\mathbf{r},\mathbf{r}';E)\exp\left\{\frac{ie}{\hbar c}\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\tilde{\mathbf{r}})\cdot d\tilde{\mathbf{r}}\right\}. \tag{6}$$

Here we have used the vector potential $\mathbf{A}(\tilde{\mathbf{r}})$ to represent the magnetic field. For an infinitely thin tube of finite magnetic flux along the $z$-direction under consideration, the vector potential can be described by

$$\mathbf{A}(\mathbf{r}) = 2g\frac{-y\hat{e}_x + x\hat{e}_y}{x^2+y^2}, \tag{7}$$

where $\hat{e}_x,\hat{e}_y$ stand for the unit vector along the $x,y$ axes, respectively. Introducing the azimuthal angle around the AB tube,

$$\varphi(\mathbf{r}) = \tan^{-1}(y/x), \tag{8}$$

the components of the vector potential can be expressed as

$$A_i = 2g\partial_i\varphi(\mathbf{r}). \tag{9}$$

The associated magnetic-field lines are confined to an infinitely thin tube along the $z$ axis:

$$B_3 = 2g\epsilon_{3ij}\partial_i\partial_j\varphi(\mathbf{r}) = 4\pi g\,\delta(\mathbf{r}_\perp), \tag{10}$$

where $\mathbf{r}_\perp$ stands for the transverse vector $\mathbf{r}_\perp \equiv (x,y)$. Note that the derivatives in front of $\varphi(\mathbf{r})$ commute everywhere, except at the origin where Stokes' theorem yields

$$\int d^2x(\partial_x\partial_y - \partial_y\partial_x)\varphi(\mathbf{r}) = \oint d\varphi = 2\pi. \tag{11}$$

Since the magnetic flux through the tube is defined by the integral

$$\Phi = \int d^2x B_3, \tag{12}$$

the coupling constant $g$ is related to the magnetic flux by $g = \Phi/4\pi$. Inserting $A_i = 2g\partial_i\varphi$ into the nonintegrable phase factor in Eq. (6), the magnetic interaction takes the form $\exp[-i\mu_0\int^\tau d\tau'\dot{\varphi}(\tau')]$, where $\dot{\varphi} = d\varphi/d\tau$ and $\mu_0 = -2eg/\hbar c = -\Phi/\Phi_0$ is a dimensionless number. The minus sign is a matter of convention. According to the discussion in Refs. [14–17], only phase factors with closed-loop contour are considered where the description of the electromagnetic phenomenon is complete [18]. Hence, we have

$$m = \frac{1}{2\pi}\int^\tau d\tau'\dot{\varphi}(\tau'), \tag{13}$$

with integer values $m$ corresponding to the winding number. The magnetic interaction is therefore purely nonlocal, and topological. The nonintegrable phase factor now becomes $\exp\{-i\mu_0(2\pi m + \varphi - \varphi')\}$. It can be included with the help of Poisson's summation formula (e.g., p. 469 of Ref. [19])

$$\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dx \sum_{n=-\infty}^{\infty} e^{2\pi nxi} f(x). \tag{14}$$

So expression (5) can be written as

$$\int dz\sum_{m=-\infty}^{\infty}\left\{E+\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{z^2}{\rho^2}\right)\right]-V(\rho)\right\} \times G_z(\rho,\rho';E)\frac{e^{i(z-\mu_0)(\varphi+2m\pi-\varphi')}}{2\pi}, \tag{15}$$

where the superscript 0 in $G_m^0$ has been suppressed to reflect the inclusion of the AB effect. The summation over all indices $m$ forces $z = \mu_0$ modulo an arbitrary integer number. Thus, we have

$$\sum_{m=-\infty}^{\infty}\left\{E+\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{|m+\mu_0|^2}{\rho^2}\right)\right]-V(\rho)\right\} \times G_{|m+\mu_0|}(\rho,\rho';E)\frac{e^{im\varphi}}{2\pi}. \tag{16}$$

In what follows, we shall denote $|m+\mu_0| = \alpha$ briefly. We see that the influence of the AB effect to the radial Green's function is to replace the integer quantum number $m$ with a real one $\alpha$ which depends on the magnitude of magnetic flux. Applying the Fourier expansion of $\delta$ function,

$$\delta(\varphi-\varphi') = \sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\varphi-\varphi')}, \tag{17}$$

to the right-hand side (rhs) of Eq. (3), one can show that the radial Green's function satisfies

$$\left\{E+\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\alpha^2}{\rho^2}\right)\right]-V(\rho)\right\}G_\alpha(\rho,\rho';E)$$
$$=\delta(\rho-\rho'). \tag{18}$$

As a result, the corresponding radial wave equation reads

$$\left\{E+\left[\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{\alpha^2}{\rho^2}\right)\right]-V(\rho)\right\}R_{\alpha k}(\rho)=0, \tag{19}$$

where the subscript set $(\alpha,k)$ with $k\equiv\sqrt{2\mu E}/\hbar$ denotes the state of scattering particle.

For short-range potential, i.e., $V(\rho)$ vanishes for $\rho>a$, the domain of the variable $\rho$ is divided into an internal region $(\rho<a)$ and an external region $(\rho>a)$. The normalized exterior solution is the linear combination of Bessel functions $J_\alpha(k\rho)$ and $N_\alpha(k\rho)$ of the first and second kind, and may be given by

$$R_{\alpha k}(\rho)=\sqrt{k}[\cos\delta_\alpha(k)J_\alpha(k\rho)-\sin\delta_\alpha(k)N_\alpha(k\rho)], \tag{20}$$

where $\delta_\alpha(k)$ is the phase shift of the scattered radial wave function which is used to measure the interaction in potential. The general solution of a scattering particle $\Psi_k(\mathbf{r})$ is given by superposition of the partial wave $\Psi_{\alpha k}(\mathbf{r})=R_{\alpha k}(\rho)e^{im\varphi}$, and reads

$$\Psi_k(\mathbf{r})=\sum_{m=-\infty}^{\infty}\sqrt{k}[\cos\delta_\alpha(k)J_\alpha(k\rho)$$
$$-\sin\delta_\alpha(k)N_\alpha(k\rho)]e^{im\varphi}. \tag{21}$$

Because it must describe both incident wave and scattered wave at large distance, we naturally expect it to become

$$\Psi_k(\mathbf{r})\xrightarrow{|\mathbf{r}|\to\infty}\mathcal{F}_{\text{asymp}}\left(\exp\{i\mathbf{k}\cdot\mathbf{r}\}\exp\left\{\frac{ie}{\hbar c}\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\tilde{r})\cdot d\tilde{\mathbf{r}}\right\}\right)$$
$$+f(\varphi)\sqrt{\frac{i}{\rho}}\exp\{ik\rho\}, \tag{22}$$

where $\exp(i\mathbf{k}\cdot\mathbf{r})$ describes the incident plane wave of a charged particle with momentum $\mathbf{p}=\mu\mathbf{k}$ and $\mathcal{F}_{\text{asymp}}(\cdot)$ stands for its asymptotic form. The phase modulation of the nonintegrable phase factor comes from the fact that the field $\mathbf{A}(\tilde{\mathbf{r}})$ of AB magnetic flux affects the charged particle globally. To find the amplitude $f(\varphi)$ we note that the plane wave in Eq. (22) can be in terms of the expansion of the partial waves in polar coordinates:

$$e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{m=-\infty}^{\infty}i^mJ_m(k\rho)e^{im\varphi}. \tag{23}$$

Using the same procedure as in Eqs. (14)–(16), the nonlocal flux effect can be combined into the partial-wave expansion, and yields

$$\exp(i\mathbf{k}\cdot\mathbf{r})\exp\left(\frac{ie}{\hbar c}\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\tilde{\mathbf{r}})\cdot d\tilde{\mathbf{r}}\right)=\sum_{m=-\infty}^{\infty}i^\alpha J_\alpha(k\rho)e^{im\varphi}. \tag{24}$$

Inserting the result into Eq. (22), making use of the asymptotic approximations of Bessel functions, and then comparing both asymptotic forms of Eqs. (21) and (22), we find the scattering amplitude in terms of phase shifts:

$$f(\varphi)=\frac{1}{\sqrt{2\pi k}}\sum_{m=-\infty}^{\infty}e^{i(\delta_\alpha-\pi/4)}2i\sin\delta_\alpha e^{im\varphi}. \tag{25}$$

It is worth noting that if the flux is quantized, i.e., $\mu_0$ is an integer, the result reduces to the free of flux case. In most cases, one concern is the total cross section, which is defined by

$$\sigma_t=\int_{-\pi}^{\pi}|f(\varphi)|^2d\varphi. \tag{26}$$

Accordingly, the partial-wave representation of the total cross section for a charged particle scattered by a short-range potential and the nonlocal AB effect is given by

$$\sigma_t=\frac{4}{k}\sum_{m=-\infty}^{\infty}\sin^2\delta_\alpha. \tag{27}$$

We see that the cross section is completely determined by the scattered phase shifts which are concluded by the potential of different types and the magnetic flux. On the other hand, the potential also determines the number of bound states. The relation between the phase shifts and the number of bound states was first clarified by Levinson [1]. Here, due to the nonlocal AB magnetic flux existence, the phase shifts are affected globally, and so are the number of bound states [20]. In next two sections it will be showed that the relation between the phase shift at threshold of the scattered wave function and the number of bound states for the corresponding angular-momentum channel is connected by a general Levinson theorem.

### III. PHASE SHIFTS NEAR $\rho\to 0$ AT THRESHOLD

Since the behavior of phase shifts near $\rho\to 0$ at threshold is useful in the procedure of proof, the asymptotic behavior is discussed in that follows. According to Eq. (19), when a potential $V(\rho)$ is less singular than $\rho^{-2}$, the solution has the power dependence on $\rho$ near $\rho=0$,

$$R_\alpha(\rho,k)\overset{\rho\to 0}{\sim}\rho^\alpha, \tag{28}$$

where we have used $R_\alpha(\rho,k)$ to denote the solution of Eq. (19) which satisfies the boundary condition Eq. (28). On the other hand, the external solution is given by Eq. (20). The boundary conditions at $\rho=a$ require that the logarithmic derivative be continuous,

$$\frac{1}{R_\alpha}\frac{dR_\alpha}{d\rho}\bigg|_{\rho=a^-}=\frac{1}{R_{\alpha k}}\frac{dR_{\alpha k}}{d\rho}\bigg|_{\rho=a^+},\tag{29}$$

and thus yield the formula for the phase shift,

$$\tan\delta_\alpha=\frac{kaJ'_\alpha(ka)-\gamma_\alpha J_\alpha(ka)}{kaN'_\alpha(ka)-\gamma_\alpha N_\alpha(ka)}.\tag{30}$$

Here we define $J'_\alpha(\rho)=dJ_\alpha(\rho)/d\rho$ and $\gamma_\alpha=adR_\alpha/R_\alpha d\rho|_{\rho=a^-}$. Note that Eq. (28) is independent of $k$, and Eq. (18) depends on $k$ only through $k^2$. Therefore either $R_\alpha$ or $dR_\alpha/d\rho$ must be an integral function of $k$, and hence are the even function of $k$. Accordingly, $\gamma_\alpha$ can only have one of the following forms [8]:

$$\gamma_\alpha\to b_\alpha^+(ka)^{2l_\alpha^+},$$

$$\gamma_\alpha\to b_\alpha^-(ka)^{-2l_\alpha^-},$$

$$\gamma_\alpha\to c_\alpha+b_\alpha(ka)^{2l_\alpha},$$

where $b_\alpha^\pm$, and $c_\alpha$, and $b_\alpha$ are nonzero constants, and $l_\alpha^\pm$, $l_\alpha$ are the natural numbers. Using the asymptotic forms of Bessel functions at $\rho\sim 0$, it is easy to find that the leading term of Eq. (30) at threshold in any case is given by

$$\tan\delta_\alpha\to d_\alpha(ka)^{2\tilde{\alpha}},\tag{31}$$

where $d_\alpha\neq 0$ and $\tilde{\alpha}$ is a nonzero positive real number. The result will be useful in the following proof of the Levinson method.

## IV. THE LEVINSON THEOREM WITH THE NONLOCAL AB EFFECT

Using the spectrum representation of the radial Green's function of Eq. (18),

$$G_\alpha(\rho,\rho';E)=\sum_\kappa \frac{u_{\alpha\kappa}(\rho)u^*_{\alpha\kappa}(\rho')}{\sqrt{\rho\rho'}(E-E_{\alpha\kappa}+i\epsilon)},\tag{32}$$

the Green's function $G(\mathbf{r},\mathbf{r}';E)$ is

$$\begin{aligned}G(\mathbf{r},\mathbf{r}';E)&=\sum_{m=-\infty}^{\infty}G_\alpha(\rho,\rho';E)\frac{e^{im(\varphi-\varphi')}}{2\pi}\\&=\sum_{m=-\infty}^{\infty}\sum_\kappa\frac{u_{\alpha\kappa}(\rho)u^*_{\alpha\kappa}(\rho')}{\sqrt{\rho\rho'}(E-E_{\alpha\kappa}+i\epsilon)}\frac{e^{im(\varphi-\varphi')}}{2\pi},\end{aligned}\tag{33}$$

where $\epsilon=0^+$ has been defined for the retarded Green's function and we use $\kappa$ to denote the discretized energy levels of a system moving in an attractive field. It is easy to see that by making use of $R_{\alpha\kappa}=u_{\alpha\kappa}/\sqrt{\rho}$ in Eq. (19) the wave function $u_{\alpha\kappa}(\rho)$ solves

$$\frac{d^2u_{\alpha\kappa}}{d\rho^2}+\left[\frac{2\mu}{\hbar^2}[E_{\alpha\kappa}-V(\rho)]-\frac{\alpha^2-1/4}{\rho^2}\right]u_{\alpha\kappa}=0.\tag{34}$$

The wave function $u_{\alpha\kappa}(\rho)$ satisfies normalization condition

$$\langle u_{\alpha\kappa},u_{\alpha\kappa'}\rangle=\int_0^\infty d\rho u^*_{\alpha\kappa}(\rho)u_{\alpha\kappa'}(\rho)=\delta_{\kappa\kappa'}.\tag{35}$$

So we find the following trace:

$$\int d\rho\rho G_\alpha(\rho,\rho;E)=\sum_\kappa\frac{1}{(E-E_{\alpha\kappa}+i\epsilon)}.\tag{36}$$

With the help of the formula

$$\frac{1}{x+i\epsilon}=\mathrm{P}\frac{1}{x}-i\pi\delta(x),\tag{37}$$

the imaginary part of integral in Eq. (36) reads

$$\mathrm{Im}\int d\rho\rho G_\alpha(\rho,\rho;E)=-\pi\sum_\kappa\delta(E-E_{\alpha\kappa}).\tag{38}$$

Thus the total number of bound states can be read off by integrating the formula over $E$ from $-\infty$ to $0^-$, and yields

$$\mathrm{Im}\int_{-\infty}^{0^-}dE\int d\rho\rho G_\alpha(\rho,\rho;E)=-n_\alpha^-\pi,\tag{39}$$

where $n_\alpha^-$ is the number of bound states with negative energies corresponding to the channel of a general angular momentum $\alpha\hbar$. We have performed the integral over $E$ up to $0^-$ instead of 0 for avoiding ambiguity. The possible existence of a bound state with zero energy will be considered in Sec. V. We point out that the degeneracy between $m$ and $-m$ in Eq. (1) is broken in general due to the existence of magnetic flux of the nonlocal AB effect. A similar discussion as above can be applied to the free charged particle moving in the field of the AB effect and gives

$$\mathrm{Im}\int_{-\infty}^{0^-}dE\int d\rho\rho G^0_\alpha(\rho,\rho;E)=0.\tag{40}$$

Combining both Eqs. (39) and (40), we find

$$\mathrm{Im}\int_{-\infty}^{0^-}dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G^0_\alpha(\rho,\rho;E)]=-n_\alpha^-\pi.\tag{41}$$

It is useful to discuss this result by the Dyson equation

$$\begin{aligned}G(\mathbf{r},\mathbf{r}';E)&=G^0_{\mathrm{AB}}(\mathbf{r},\mathbf{r}';E)\\&\quad+\int d\mathbf{r}''G^0_{\mathrm{AB}}(\mathbf{r},\mathbf{r}'';E)V(\mathbf{r}'')G(\mathbf{r}'',\mathbf{r}';E),\end{aligned}\tag{42}$$

where we have used $G^0_{\mathrm{AB}}(\mathbf{r},\mathbf{r}';E)$ to represent the Green's function $G(\mathbf{r},\mathbf{r}';E)$ in the case of $V(\rho)=0$. With the help of

Eq. (33), the integration of angular part can be carried out and turns the equation into the single-dimensional one with a general quantum number $\alpha$ [16]

$$G_\alpha(\rho,\rho';E)=G_\alpha^0(\rho,\rho';E)+\int d\rho'' G_\alpha^0(\rho,\rho'';E)V(\rho'')G_\alpha(\rho'',\rho';E). \tag{43}$$

Here $G_\alpha^0(\rho,\rho';E)$ is the radial Green's function with $V(\rho)=0$. Its spectrum representation can be in terms of discrete sum

$$G_\alpha^0(\rho,\rho';E)=\sum_\kappa \frac{u_{\alpha\kappa}^0(\rho)u_{\alpha\kappa}^{0*}(\rho')}{\sqrt{\rho\rho'}(E-E_{\alpha\kappa}+i\epsilon)}. \tag{44}$$

This can be achieved by requiring the wave functions to vanish at a sufficiently large radius $R$ when $a\ll R$ for a short-range potential. Taking the same trace with respect to Eq. (43) as Eq. (36), we obtain

$$\int d\rho\rho[G_\alpha(\rho,\rho;E)-G_\alpha^0(\rho,\rho;E)]=\int d\rho\rho\left\{\int d\rho'' G_\alpha^0(\rho,\rho'';E)V(\rho'')G_\alpha(\rho'',\rho;E)\right\}. \tag{45}$$

With the help of Eqs. (32) and (44), the equation becomes

$$\int d\rho\rho[G_\alpha(\rho,\rho;E)-G_\alpha^0(\rho,\rho;E)]=\sum_{\kappa\kappa'}\frac{\langle u_{\alpha\kappa'},u_{\alpha\kappa}^0\rangle\langle u_{\alpha\kappa}^0|V|u_{\alpha\kappa'}\rangle}{(E-E_{\alpha\kappa}^0+i\epsilon)(E-E_{\alpha\kappa'}+i\epsilon)}. \tag{46}$$

The matrix element $\langle u_{\alpha\kappa}^0|V|u_{\alpha\kappa'}\rangle$ can explicitly carry out

$$\begin{aligned}\langle u_{\alpha\kappa}^0|V|u_{\alpha\kappa'}\rangle&=\int_0^\infty d\rho u_{\alpha\kappa}^{0*}(\rho)V(\rho)u_{\alpha\kappa'}(\rho)\\&=\int_0^\infty d\rho u_{\alpha\kappa}^{0*}(\rho)(\tilde{H}-\tilde{H}_0)u_{\alpha\kappa'}(\rho)\\&=(E_{\alpha\kappa'}-E_{\alpha\kappa}^0)\langle u_{\alpha\kappa}^0,u_{\alpha\kappa'}\rangle,\end{aligned} \tag{47}$$

where $\tilde{H}$ is the Hamiltonian in Eq. (34),

$$\tilde{H}u_{\alpha\kappa}\equiv\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{d\rho^2}+\left(V(\rho)+\frac{(\alpha^2-1/4)\hbar^2}{2\mu\rho^2}\right)\right]u_{\alpha\kappa}=E_{\alpha\kappa}u_{\alpha\kappa}, \tag{48}$$

and $\tilde{H}_0$ is the Hamiltonian with $V(\rho)=0$. Substituting Eq. (47) into Eq. (46) and taking the imaginary part, we obtain

$$\mathrm{Im}\int d\rho\rho[G_\alpha(\rho,\rho;E)-G_\alpha^0(\rho,\rho;E)]=\pi\sum_{\kappa\kappa'}[\delta(E-E_{\alpha\kappa}^0)-\delta(E-E_{\alpha\kappa'})]|\langle u_{\alpha\kappa}^0,u_{\alpha\kappa'}\rangle|^2. \tag{49}$$

Integrating this equation over $E$ from $-\infty$ to $\infty$ gives

$$\mathrm{Im}\int_{-\infty}^\infty dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G_\alpha^0(\rho,\rho;E)]=0. \tag{50}$$

The equation indicates that the total number of states in a specific angular-momentum channel is not changed by an attractive field, except that some scattering states are pulled down into the bound-state region. Comparing Eqs. (41) and (50), we obtain the result

$$\mathrm{Im}\int_{0^-}^\infty dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G_\alpha^0(\rho,\rho;E)]=n_\alpha^-\pi. \tag{51}$$

Arriving here we complete the proof of rhs of the Levinson theorem with the nonlocal AB effect in Eq. (2) by discretizing the energy spectrum of continuous part. In the following we shall prove the lhs of the Levinson theorem by directly treating the continuous part of energy spectrum which will gives the phase-shift expression of the total number of bound states at threshold. Including the continuous spectrum, Eq. (32) takes the expression

$$G_\alpha(\rho,\rho';E)=\sum_\kappa\frac{u_{\alpha\kappa}(\rho)u_{\alpha\kappa}^*(\rho')}{\sqrt{\rho\rho'}(E-E_{\alpha\kappa}+i\epsilon)}+\int dk\frac{u_{\alpha k}(\rho)u_{\alpha k}^*(\rho')}{\sqrt{\rho\rho'}(E-E_{\alpha k}+i\epsilon)}, \tag{52}$$

where we have used $\kappa$ and $k$ to denote the discrete and continuous spectrum, respectively. Using Eqs. (35) and (37), we have

$$\begin{aligned}\mathrm{Im}\int d\rho\rho G_\alpha(\rho,\rho;E)&=-\pi\sum_\kappa\delta(E-E_{\alpha\kappa})\\&\quad-\pi\int dk\,\delta(E-E_{\alpha k})\langle u_{\alpha k},u_{\alpha k}\rangle.\end{aligned} \tag{53}$$

Note that $E_{\alpha k}$ may be zero energy, and the wave functions corresponding to the continuous spectrum has the normalization condition

$$\langle u_{\alpha k},u_{\alpha k'}\rangle=\int_0^\infty d\rho u_{\alpha k}^*(\rho)u_{\alpha k'}(\rho)=\delta(k-k'). \tag{54}$$

Integrating Eq. (53) over $E$ from $0^-$ to $\infty$, one finds that

$$\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho G_\alpha(\rho,\rho;E)=-\pi\int dk\langle u_{\alpha k},u_{\alpha k}\rangle, \tag{55}$$

which is divergent due to $\langle u_{\alpha k},u_{\alpha k}\rangle=\delta(0)=\infty$. The same treatment for $G^0_\alpha(\rho,\rho';E)$ gives

$$\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho G^0_\alpha(\rho,\rho;E)=-\pi\int dk\langle u^0_{\alpha k},u^0_{\alpha k}\rangle, \tag{56}$$

which is also infinity due to $\langle u^0_{\alpha k},u^0_{\alpha k}\rangle=\delta(0)$. But both infinities are of different order which leads to the Levinson theory. To see this, let us first evaluate the difference

$$\langle u_{\alpha k},u_{\alpha k'}\rangle_{\rho_0}-\langle u^0_{\alpha k},u^0_{\alpha k'}\rangle_{\rho_0}=\int_0^{\rho_0}d\rho u^*_{\alpha k}(\rho)u_{\alpha k'}(\rho)-\int_0^{\rho_0}d\rho u^{0*}_{\alpha k}(\rho)u^0_{\alpha k'}(\rho), \tag{57}$$

and then take the limit $k'\to k$ and $\rho_0\to\infty$. Here $\rho_0$ is a large but finite radius. Employing Eq. (34) and the boundary conditions

$$u_{\alpha k}(0)=0,\quad u^0_{\alpha k}(0)=0, \tag{58}$$

it is easy to find the expression

$$(k^2-k'^2)\langle u_{\alpha k},u_{\alpha k'}\rangle_{\rho_0}=u^*_{\alpha k}(\rho_0)\frac{du_{\alpha k'}(\rho)}{d\rho}\bigg|_{\rho_0}-u_{\alpha k'}(\rho_0)\frac{du^*_{\alpha k}(\rho)}{d\rho}\bigg|_{\rho_0}. \tag{59}$$

Since $\rho_0$ is a large radius, the asymptotic form of Eq. (20) can be used to evaluate the equality. With the help of asymptotic behavior of the Bessel functions it can be found

$$u_{\alpha k}=\sqrt{\rho}R_{\alpha k}\overset{\rho\to\infty}{\sim}\sqrt{\frac{2}{\pi}}\cos\left[k\rho-\frac{\alpha\pi}{2}-\frac{\pi}{4}+\delta_\alpha(k)\right], \tag{60}$$

which in the limit $k'\to k$ leads to

$$\langle u_{\alpha k},u_{\alpha k'}\rangle_{\rho_0}=\frac{\rho_0}{\pi}+\frac{1}{\pi}\frac{d\delta_\alpha(k)}{dk}-\frac{1}{2\pi k}\times\{\cos\alpha\pi\cos[2k\rho_0+2\delta_\alpha(k)]+\sin\alpha\pi\sin[2k\rho_0+2\delta_\alpha(k)]\}. \tag{61}$$

The same procedure for $u^0_{\alpha k}$ gives

$$u^0_{\alpha k}=\sqrt{\rho}R_{\alpha k}\overset{\rho\to\infty}{\sim}\sqrt{\frac{2}{\pi}}\cos\left[k\rho-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right] \tag{62}$$

and

$$\langle u^0_{\alpha k},u^0_{\alpha k'}\rangle_{\rho_0}=\frac{\rho_0}{\pi}-\frac{1}{2\pi k}\{\cos\alpha\pi\cos2k\rho_0+\sin\alpha\pi\sin2k\rho_0\}. \tag{63}$$

So we obtain

$$\begin{aligned}\langle u_{\alpha k},u_{\alpha k'}\rangle_{\rho_0}-\langle u^0_{\alpha k},u^0_{\alpha k'}\rangle_{\rho_0}&=\frac{1}{\pi}\frac{d\delta_\alpha(k)}{dk}+\frac{1}{2}\delta(k)\cos\alpha\pi\sin2\delta_\alpha(k)\\&+\delta(k)\sin\alpha\pi\sin^2\delta_\alpha(k)\\&+\frac{\cos\alpha\pi}{\pi k}\cos(2k\rho_0)\sin^2\delta_\alpha(k)\\&-\frac{\sin\alpha\pi}{2\pi k}\cos(2k\rho_0)\sin2\delta_\alpha(k),\end{aligned} \tag{64}$$

where we have used the well-known formula

$$\lim_{\rho_0\to\infty}\frac{\sin2k\rho_0}{\pi k}=\delta(k). \tag{65}$$

Since Eq. (31) is valid, two terms containing $\delta(k)$ in Eq. (64) vanish. So from Eqs. (55) and (56) we find

$$\begin{aligned}\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G^0_\alpha(\rho,\rho;E)]&=\delta_\alpha(0)-\delta_\alpha(\infty)-\cos\alpha\pi\lim_{\rho_0\to\infty}\\&\times\int_0^\infty dk\frac{\cos(2k\rho_0)}{k}\sin^2\delta_\alpha(k)+\frac{\sin\alpha\pi}{2}\lim_{\rho_0\to\infty}\\&\times\int_0^\infty dk\frac{\cos(2k\rho_0)}{k}\sin2\delta_\alpha(k).\end{aligned} \tag{66}$$

The integrals can be divided into two regions. The first from 0 to $0^+$ vanishes on account of Eq. (31), while the second from $0^+$ to $\infty$ also vanishes in the limit $\rho_0\to\infty$ because the factor $\cos(2k\rho_0)$ oscillates very rapidly. Thus we have

$$\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G^0_\alpha(\rho,\rho;E)]=\delta_\alpha(0)-\delta_\alpha(\infty). \tag{67}$$

Combining Eqs. (51) and (67), we obtain the Levinson theorem with the nonlocal AB effect:

$$\delta_\alpha(0)-\delta_\alpha(\infty)=n^-_\alpha\pi,\quad \alpha=|m+\mu_0|,\quad m=0,\pm1,\pm2,\ldots. \tag{68}$$

## V. DISCUSSION

### A. On the existence of a zero-energy bound state

As an explanation, let us consider a potential well with radius $a$ and depth $V(\rho)=-V_0$ for $\rho<a$; $V(\rho)=0$ for $\rho>a$. Using Eq. (19), it is not difficult to find that the energy spectrum is determined by

$$\beta\frac{J_{\alpha-1}(\beta a)}{J_\alpha(\beta a)}=i\lambda\frac{H^{(1)}_{\alpha-1}(i\lambda a)}{H^{(1)}_\alpha(i\lambda a)}, \tag{69}$$

where $\beta=\sqrt{2\mu(V_0-|E|)}/\hbar$, $\lambda=\sqrt{2\mu|E|}/\hbar$, and $H^{(1)}_\alpha$ is the Hankel function of the first kind. So a zero-energy bound state in this case is determined by $J_{\alpha-1}(k_0a)=0$ with $k_0=\sqrt{2\mu V_0}/\hbar$ for $\alpha>1$ (see below). The existence of a zero-energy bound state would not change the result in Eqs. (39)–(41), and thus Eq. (51). But Eq. (55) will receive an additional $\pi$ to become

$$\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho G_\alpha(\rho,\rho;E)=-\pi-\pi\int dk\langle u_{\alpha k},u_{\alpha k}\rangle. \tag{70}$$

Hence Eq. (67) gets an additional $\pi$ and turns into

$$\mathrm{Im}\int_{0^-}^{\infty}dE\int d\rho\rho[G_\alpha(\rho,\rho;E)-G^0_\alpha(\rho,\rho;E)]=\delta_\alpha(0)-\delta_\alpha(\infty)-\pi. \tag{71}$$

Therefore when a system contains a zero-energy bound state, the Levinson theorem reads

$$\delta_\alpha(0)-\delta_\alpha(\infty)=(n^-_\alpha+1)\pi=n_\alpha\pi, \tag{72}$$

with $n_\alpha\equiv(n^-_\alpha+1)$. Here only when $\alpha>1$ the bound state is a real zero-energy bound state. To see this recall Eq. (34). When $E_{\alpha0}=0$, the exterior solution ($\rho>0$) satisfies

$$\frac{d^2u_{\alpha0}}{d\rho^2}-\frac{\alpha^2-1/4}{\rho^2}u_{\alpha0}=0. \tag{73}$$

Explicitly, $u_{\alpha0}$ is given by

$$u_{\alpha0}\sim\rho^{-\alpha+1/2}, \tag{74}$$

which leads to the fact that the wave function $\Psi_{\alpha0}\sim u_{\alpha0}/\sqrt{\rho}=1/\rho^\alpha$ cannot be normalized when $\alpha\leqslant1$. On the flip side, as $\alpha>1$, $\delta_\alpha(0)$ obtains an additional $\pi$ if a zero-energy solution actually exists. For this case, Levinson theorem becomes Eq. (72).

### B. The phase shifts at high energies

Let us investigate the behavior of the phase shifts when $\alpha$ is fixed but $k\to\infty$. For this purpose, we consider the scattering by two potentials $V(\rho)$ and $\tilde{V}(\rho)$. The corresponding radial equations read

$$\frac{d^2u_{\alpha k}}{d\rho^2}+\left[\frac{2\mu}{\hbar^2}[E_{\alpha k}-V(\rho)]-\frac{\alpha^2-1/4}{\rho^2}\right]u_{\alpha k}=0, \tag{75}$$

$$\frac{d^2\tilde{u}_{\alpha k}}{d\rho^2}+\left[\frac{2\mu}{\hbar^2}[E_{\alpha k}-\tilde{V}(\rho)]-\frac{\alpha^2-1/4}{\rho^2}\right]\tilde{u}_{\alpha k}=0. \tag{76}$$

With the boundary conditions of Eq. (58) and the asymptotic form of radial function $u_{\alpha k}$ in Eq. (60), it is easy to find that

$$\sin[\delta_\alpha(k)-\tilde{\delta}_\alpha(k)]=-\frac{\pi\mu}{\hbar^2k}\int_0^\infty d\rho[V(\rho)-\tilde{V}(\rho)]u_{\alpha k}(\rho)\tilde{u}_{\alpha k}(\rho). \tag{77}$$

When $\tilde{V}(\rho)=0$ we deduce the integral representation

$$\sin\delta_\alpha(k)=-\frac{\pi\mu}{\hbar^2k}\int_0^\infty d\rho V(\rho)u_{\alpha k}(\rho)u^0_{\alpha k}(\rho). \tag{78}$$

In the case $k=\sqrt{2\mu E_{\alpha k}}/\hbar\to\infty$ we expect that the potential will become vanishingly small since the potentials $V(\rho)$ and $\tilde{V}(\rho)$ should not be more singular than $r^{-2}$ at the origin and well behaved elsewhere as assumed. So the radial function $u_{\alpha k}$ will be very close to the corresponding free wave, i.e., $u_{\alpha k}(\rho)$ can be replaced with $u^0_{\alpha k}$. Thus with the help of the asymptotic expression of Eq. (62) we deduce that

$$\sin\delta_\alpha(k)=-\frac{2\mu}{\hbar^2k}\int_0^\infty d\rho V(\rho)\cos^2\left[k\rho-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right]. \tag{79}$$

The square of cosine function can be replaced with its mean value 1/2 since a very large $k$ value leads to very rapid oscillations. So we have

$$\sin\delta_\alpha(k)\overset{k\to\infty}{\to}-\frac{\mu}{\hbar^2k}\int_0^\infty d\rho V(\rho). \tag{80}$$

Hence we see that the phase shifts $\delta_\alpha(k)$ tend to zero (modulo $\pi$) as $k\to\infty$ provided that the integral exists. This suggests that a reasonable absolute definition of the phase shift may be given by requiring that

$$\lim_{k\to\infty}\delta_\alpha(k)=0. \tag{81}$$

The definition is physically reasonable since we require that $\delta_\alpha(k)=0$ when the particle is effectively free. With this convention the Levinson theorem is given by

$$\delta_\alpha(0)=n_\alpha\pi,\quad \alpha=|m+\mu_0|,\quad m=0,\pm1,\pm2,\ldots, \tag{82}$$

a result given in Eq. (2). It means that the phase shift at the threshold serves as a counter for the bound states in a general angular-momentum channel.

### C. The effects of magnetic flux

Several interesting effects caused by the nonlocal influence of the magnetic flux are concluded as follows.

(a) When the flux is quantized, i.e., $\Phi = m\Phi_0$ the multiple of a fundamental flux quantum $hc/e$, the Levinson theorem will reduce to the free of flux case as in [8]. In this case the total number of bound states for the quantum number $m$ and $-m$ are the same except $m=0$, and thus have the same phase shifts.

(b) When the flux satisfies $\Phi/\Phi_0 =$ half-odd integer, there are two different $m$ corresponding to the same total number of the bound states, so are the phase shifts at threshold. These are such that the number pairs $(m,m)=(1,-2),(2,-3)$ for $\Phi/\Phi_0 = -1/2$.

(c) In general, when $\Phi/\Phi_0 \neq$ integer, and half-odd integer, the total number of bound states for $\pm m$ are no longer identical, and the phase shifts will be different from each other.

### D. Extension of the potential to a more general case

Although in the procedure of our proof we assume that the potential must be less singular than $\rho^{-2}$ in Eq. (28) and $V(\rho)=0$ for $\rho > a$, we do not specify the radius $a$ beyond which $V(\rho)=0$. Hence we expect that the Levinson theorem given in the paper should be valid for a very general potential as long as the potential decreases rapidly enough when $r\to\infty$ such that the total number of bound states in a general angular-momentum channel is finite.

### E. A possible experimental test

In Ref. [21], a general fractional (nonquantized) magnetic flux is observed in the superconducting film. Because of the inevitable pinning of flux in superconductor, the flux finally attaches to the defect or impurity which may carry the charge. A thin film can be viewed as a two-dimensional system and because the screen effect exists in solid, the electric interaction becomes a finite range interaction as mentioned in the preceding paragraph. If a charged particle moves near the impurity which may be captured by the impurity and forms a bound-state system during a period, the system scattered by the other low-energy charged particle can be the test ground of the phase shift and the number of bound states for a general angular-momentum channel.

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