Wide-Sense Nonblocking Multicast $\mathrm{Log}_2(N, m, p)$ Networks

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***Abstract*—Recently, Tscha and Lee proposed a fixed-size window algorithm for the multicast $\mathrm{Log}_2(N, 0, p)$ network and expressed a desire to see its extension to the $\mathrm{Log}_2(N, m, p)$ network. Later, Kabacinski and Danilewicz generalized the fixed-size window to variable size to improve the results. In this paper, we further extend the variable-size results from the $\mathrm{Log}_2(N, 0, p)$ network to $\mathrm{Log}_2(N, m, p)$. Note that this extension is difficult since each link in the channel graph of $\mathrm{Log}_2(N, 0, p)$ has the same blocking effect, but not so in $\mathrm{Log}_2(N, m, p)$. We also determine the optimal window size and optimal m.**

***Index Terms*—Channel graph, $\mathrm{Log}_2(N, m, p)$ networks, multicast, wide-sense nonblocking (WSNB) network, window algorithm.**

I. INTRODUCTION

FIG. 1 shows an inverse banyan network $\mathrm{BY}^{-1}(n)$ with $n(=4)$ stages and $N = 2^n$ inputs (outputs), and also a $\mathrm{BY}^{-1}(n, m)$ with $m(=2)$ extra stages, which are mirror images of the first m stages.

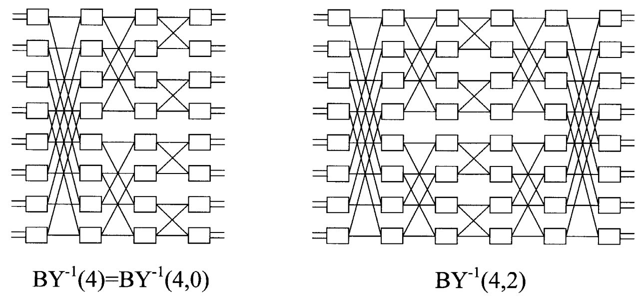

Fig. 1. (Extra-stage) inverse banyan networks.

Lea and Shyy [5] first proposed the $\mathrm{Log}_2(N, m, p)$ network, which consists of a vertical stacking of p copies of $\mathrm{BY}^{-1}(n, m)$, $0 \leq m \leq n-1$, sandwiched between and connected to an input stage and an output stage, each with $N1 \times p$ (or $p \times 1$) crossbars. As shown in Fig. 2, there are three copies of $\mathrm{BY}^{-1}(3, 1)$ sandwiched between the input and output stages.

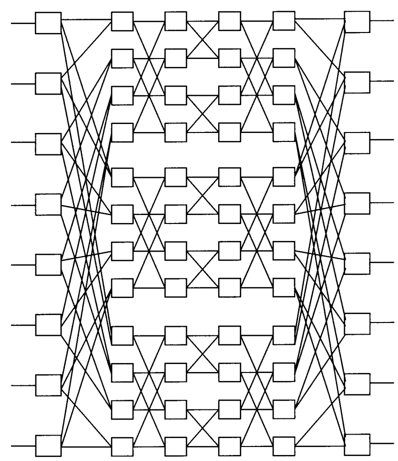

Fig. 2. $\mathrm{Log}_2(8, 1, 3)$.

A multicast network is *strictly nonblocking* if the current request can always be connected regardless of how previous connections were routed, it is wide-sense nonblocking (WSNB) if the connection of the current request is assured only when all connections are routed according to a given algorithm.

Tscha and Lee [7] proved that $\mathrm{Log}_2(N, 0, p)$ is multicast strictly nonblocking if

$$p \geq \begin{cases} n2^{n/2-2} + 1, & \text{for } n \text{ even} \\ (n-1)2^{(n-1)/2-1} + 1, & \text{for } n \text{ odd} \end{cases}.$$

However, Kabacinski and Danilewicz [4] pointed out that their proof using "windows" to split a multicast call implies a routing algorithm, hence, their result is WSNB instead of strictly nonblocking. Recently, Kabacinski and Danilewicz [4] extended the fixed window-size algorithm in [6] to variable window size. Tscha and Lee [7] stated in conclusion that whether their approach could be extended to $\mathrm{Log}_2(N, m, p)$ was unclear. Danilewicz and Kabacinski [2], [3] made such an attempt but encountered some difficulties. In this paper, we give such an extension for the variable window-size algorithm by adopting a channel graph blockage analysis first used by Shyy and Lea [6] on a single-cast network. The $\mathrm{Log}_2(N, m, p)$ network is much more difficult to analyze because of multipaths in the channel graph and each link having a different impact on blockage. We also determine the optimal window size for given m, and then compare the performance among different m.

II. GENERAL APPROACH

Define $\delta = 2^{\lfloor n/2 \rfloor}$. Tscha and Lee [7] partitioned the N outputs of $\mathrm{BY}^{-1}(n, m)$ into N/δ windows, each containing the δ outputs reachable from the same crossbar at stage $n + m - \lfloor n/2 \rfloor + 1$. Kabacinski and Danilewicz [4] extended the notion of window to θ-window, $\theta \geq 1$, which consists of the 2^θ outputs reachable from the same crossbar at stage $n + m - \theta + 1$. In other words, if the outputs are labeled by binary n sequences, then a θ-window consists of those outputs, which have the same $n-\theta$ most significant bits. Although an output can be reached by

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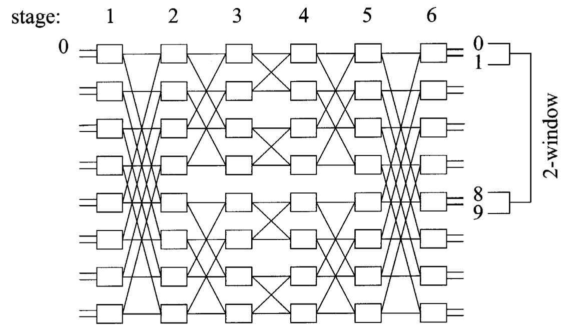


Fig. 3. A 2-window of $\mathrm{BY}^{-1}(4, 2)$.

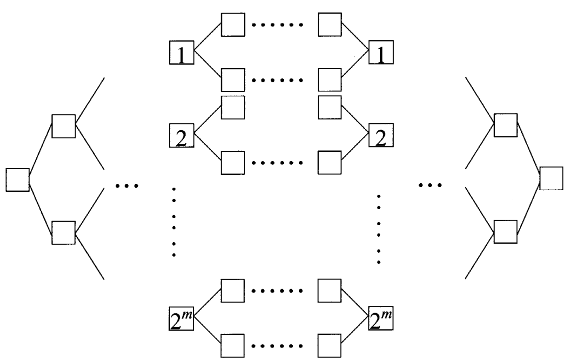


Fig. 4. A channel graph of $\mathrm{BY}^{-1}(n, m)$.

$2^{\theta-1}$ crossbars at stage $n+m-\theta+1$, each such crossbar reaches the same window due to the well-known "buddy" property of banyan type networks. Fig. 3 shows that the outputs {0,1,8,9}, reachable from the first crossbar at stage five, form a 2-window of $\mathrm{BY}^{-1}(4, 2)$. We assume $\theta < n$ to avoid trivial cases.

A *channel graph* between an input crossbar and an output crossbar is the union of all paths between them (see Fig. 4). In $\mathrm{BY}^{-1}(n, m)$, all channel graphs are isomorphic with the following double-tree form (two binary trees with their 2^m leaves linked by paths in a one-to-one fashion).

The channel graph of a multicast call is simply the union of its point-to-point channel graphs.

Following Tscha and Lee [7], we split a multicast request into w multicast requests if the involved outputs spread into w windows, while each request must be routed through the same copy of $\mathrm{BY}^{-1}(n, m)$. When we are discussing a multicast request with respect to a given θ-window, we refer to it as the *designated θ-window*. Further, a *θ'-window* is *designated* if it contains the designated θ-window. As Tscha and Lee [7] dealt only with $\mathrm{BY}^{-1}(n)$, the connection from an input to an output is unique, and whether two connections intersect is determined. Therefore, an intersection graph among the connections within a designated $\lfloor n/2 \rfloor$-window can be defined, and its maximum degree plus one becomes the number of copies of $\mathrm{BY}^{-1}(n)$ sufficient for nonblocking.

For $\mathrm{BY}^{-1}(n, m)$, the analysis is much more complicated as the connection between an input and an output is not unique. First of all, we have to be more specific about the window algorithm. We propose the delayed-splitting θ-window algorithm, which prescribes that a multicast connection to outputs in the same θ-window cannot be split before stage $(n + m - \theta + 1)$. Note that further delay is not always possible, since stage $n + m - \theta + 1$ is the last stage where all outputs in the same window have common reachable crossbars. Also note that such an algorithm fixes only the relative routing of two outputs in the same θ'-window, $\theta' \leq \theta$, but not the absolute routing to an output. Thus, whether two connections intersect is uncertain and the notion of an intersection graph used by Tscha and Lee [7] is not applicable. Instead, we adopt the method of channel graph blockage analysis, first proposed by Shyy and Lea [6] for single cast.

A link connecting stage i and stage $(i + 1)$ is called a *stage-i link*. Consider a k-cast request in a θ-window. An *intersecting connection* is one which contains a link in the channel graph of the request. We can count an intersecting connection either from its input end or its output end. An intersecting connection is an i-intersecting connection if it first (last) intersects the channel graph in a stage-i link when counted from the input (output) side.

We count all i-intersecting connections, $n+m-\theta \leq i \leq n+m-1$, from the output side. Note that the outputs of these connections must all be in the designated θ-window. Thus, there are, at most, $2^\theta - k$ of such connections. Further, they have different impacts in blocking the paths in the channel graph, depending on i. For example, for $m \geq 2$, an $(n + m - 1)$-intersecting connection blocks a proportion of 1/2, since the channel graph has only two stage-$(n + m - 1)$ links, while an $(n + m - 2)$-intersecting connection blocks a proportion of 1/4, since the channel graph has four stage-$(n + m - 2)$ links.

On the other hand, we will count all i-intersecting connections, $1 \leq i \leq n + m - \theta - 1$, from the input side. Again, an i-intersecting connection has a greater (or equality permitted) blocking impact than an $(i + 1)$-intersecting call for $i \leq \lfloor (n + m)/2 \rfloor$. We will show that we never need to count from the input side over the stage $\lfloor (n + m)/2 \rfloor$. Therefore, we adopt the method used in [4] to count from small i to large i to maximize the blocking impact.

III. A Necessary and Sufficient Condition for Nonblocking

For any two stages $j < k$ in a multistage network, let v and v' denote two crossbars at stage j, and V and V' be two sets of crossbars at stage k that v and v' can reach, respectively. Then the network is said to have the buddy property if either $V = V'$ or $V \cap V' = \phi$. It is well known [1] that $\mathrm{BY}^{-1}(n)$ and many other networks have the buddy property. Note that in a buddy network, the set of inputs which can generate an intersecting connection to a multicast request is independent of the size of that request. To see this, consider a 2-cast call from input i to two outputs o and o'. Then an input $i' \neq i$ can generate a k-intersecting connection (at a crossbar u') to the path from i to o' if and only if it can generate a k-intersecting connection (at a crossbar u) to the path from i to o, since the buddy property assures that if i' can reach u', it can reach u. Hence, increasing the size of the request does not increase the number of inputs which can generate intersecting connections, but the fact that these outputs are in the request makes them unavailable as outputs to generate intersecting connections (see Fig. 5,

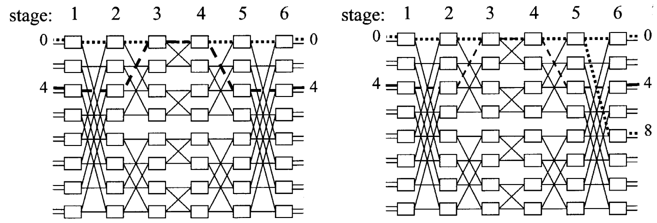


Fig. 5. Input 4 generates a 3-intersecting connection (4, 4) to (a) a 1-cast request (0, 0) and (b) a 2-cast request (0, {0, 8}).

for example). Further, each intersecting connection blocks one copy, so it is the number of intersecting connections that counts. Obviously, a 1-cast request maximizes that number.

For $\mathrm{BY}^{-1}(n,m)$, although the same analysis on the number of intersecting connections applies, the i-intersecting connections block different fractions of a copy, depending on i. Since more outputs in a multicast request induce more i-intersecting calls for larger i, the worst case is not necessarily a 1-cast request.

We consider two cases.

A. $0 \le m \le 1$

The number of stage-i links, $1 \le i \le n+m-1$, in the channel graph is constant, one for $m=0$, and two for $m=1$. Therefore, each intersecting connection has the same impact, regardless of which stage it intersects. The worst case occurs when there is a maximum number of intersecting connections, i.e., $2^\theta - 1$ from the designated window, which cause a blocking of $(2^\theta-1)/2^m$ copies.

B. $2 \le m$

Let R denote the part of the new request which goes to a designated θ-window. Suppose R is k-cast and a 1-window contains r outputs in R. Then it can block, at most

$$\begin{aligned}2\times\frac{1}{4} =& \frac{1}{2} \text{ if } r=0\\ &\text{(only for the } 1-\text{window which}\\ &\text{is in the designated } 2-\text{window)},\\ 1\times\frac{1}{2} =& \frac{1}{2}, \quad \text{if } r=1\\ =& 0, \quad \text{if } r=2.\end{aligned}$$

For instance, in Fig. 6, the first output crossbar corresponds to the case $r=1$, and the third output crossbar corresponds to the case $r=0$.

Therefore, a 1-window can block, at most, 1/2 copy of the channel graph. Consequently, a θ-window can block, at most, $2^{\theta-2}$ copies, which is achieved by having either $k=2^{\theta-1}$ (each 1-window has $r=1$) or $k=2^{\theta-2}$ (half of the 1-window has $r=1$ and half has $r=0$).

To count i-intersecting connections for $1 \le i \le n+m-\theta-1$ we consider two cases.

A. $\theta \le \lfloor n+m/2 \rfloor - 1$

The argument for this part is a straightforward extension of the argument in [4] for $m=0$.

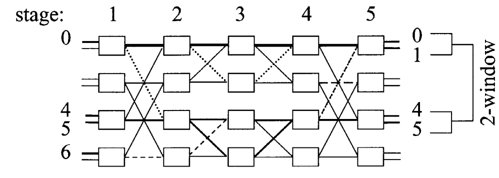


Fig. 6. Assume $\theta = 2$ and (0, 0) is the request. $r=1$ in the first output crossbar and connection (6, 1) blocks 1/2 copy, while $r=0$ in the third output crossbar and connections (4, 4) and (5, 5) each blocks 1/4 copy. Dotted lines indicate channel graph between the first input and the first output crossbar.

There are 2^{i-1} inputs which can generate an i-intersecting connection. Further, an i-intersecting connection can reach all windows for $i \le m$, and $2^{n-\theta-i+m}$ windows for $i \ge m$. In the worst-case scenario, an i-intersecting connection is a multicast connection going to one output in each window it can reach, except the designated window for $1 \le i \le \theta$. The reason for the exception is that all outputs in the designated window are already counted in the part concerning $n+m-\theta \le i \le n+m-1$. Since an i-intersecting connection blocks 2^{-i} copies for $i \le m$ and 2^{-m} copies for $m \le i \le \lfloor (n+m)/2 \rfloor$, the total blocking of up to stage θ is

$$\begin{aligned}&\sum_{i=1}^{\theta} 2^{i-1}\left(2^{n-\theta}-1\right)2^{-i}\\ &=\sum_{i=1}^{\theta} 2^{n-\theta-1} - \sum_{i=1}^{\theta} 2^{-1}\\ &=\theta\left(2^{n-\theta-1}-\frac{1}{2}\right) \text{ for } \theta \le m\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{m} 2^{i-1}\left(2^{n-\theta}-1\right)2^{-i} + \sum_{i=m+1}^{\theta} 2^{i-1}\left(2^{n-\theta-i+m}-1\right)2^{-m}\\ &=\sum_{i=1}^{m} 2^{n-\theta-1} - \sum_{i=1}^{m} 2^{-1} + \sum_{i=m+1}^{\theta} 2^{n-\theta-1} - \sum_{i=m+1}^{\theta} 2^{i-m-1}\\ &=\theta 2^{n-\theta-1} - \frac{m}{2} - 2^{\theta-m} + 1 \qquad \text{for } \theta \ge m.\end{aligned}$$

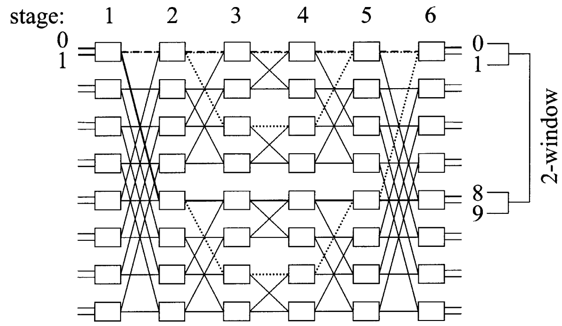
Note that these i-intersecting connections, $1 \le i \le \theta$, use up a maximum of $\sum_{i=1}^{\theta} 2^{i-1} = 2^\theta - 1$ outputs in a window. Therefore, one $(\theta+1)$-intersecting connection can still fit in if $\theta+1 < n+m-\theta$, or $\theta \le \lfloor (n+m)/2 \rfloor - 1$, which is the case here. This $(\theta+1)$-intersecting connection reaches $2^{n-\theta}-1$ windows for $\theta < m$, and $2^{n-2\theta-1+m}-1$ windows for $\theta \ge m$, while each path to a window blocks 2^{-m} copy.

To summarize, the number of blockings from the input side is

$$\begin{aligned}&\theta\left(2^{n-\theta-1}-\frac{1}{2}\right) + 2^{n-\theta-m} - 2^{-m} \text{ for } \theta < m\\ &\theta 2^{n-\theta-1} - \frac{m}{2} - 2^{\theta-m} + 1 + 2^{n-2\theta-1} - 2^{-m} \text{ for } \theta \ge m.\end{aligned}$$

B. $\theta \ge \lfloor n+m/2 \rfloor$

Then $\theta \ge m$. Note that i-intersecting connections for $n+m-\theta \le i \le n+m-1$ are counted from the output side. So the input side counts only up to stage $n+m-\theta-1$ (which is



BY^{-1}(4,2)

Fig. 7. Connection (1, 8) blocks 1/2 copy if counted from the input side, but only 1/4 copy from the output side. Dotted lines indicate channel graph between the first input and the first output crossbar.

upper bounded by θ). Thus, the number of blockings from the input side is

$$\sum_{i=1}^{m} 2^{i-1}\left(2^{n-\theta}-1\right)2^{-i} + \sum_{i=m+1}^{n+m-\theta-1} 2^{i-1}\left(2^{n-\theta-i+m}-1\right)2^{-m}$$
$$= (n+m-\theta-1)\,2^{n-\theta-1} - \frac{m}{2} - 2^{n-\theta-1} + 1$$
$$= (n+m-\theta-2)\,2^{n-\theta-1} - \frac{m}{2} + 1.$$

Since each intersecting connection counted from the output side blocks in the worst-case scenario, i.e., $k = 2^{\theta-1}$ or $2^{\theta-2}$, at least 1/4 copy, there is no reason for the counting from input side to go over stage $n+m-\theta$, with one exception.

For $\theta \ge 2$, we can increase the blocking by allowing the unique 1-intersecting connection from the input side to also go to the designated window to reach an output blocking 1/4 copy (such an output exists when $k = 2^{\theta-2}$). Then this intersecting connection blocks 1/2 copy if counted from the input side, greater than its original value 1/4, as counted from the output side (see Fig. 7, for example). Note that no other such reversal of counting will bring any further increase, since the 1-intersecting connection is the only one which blocks more than 1/4 copy when counted from the input side. On the other hand, since all intersecting connections counted from the input side are before the middle stage, reversing them to the output side will only decrease their impact on blocking.

Combining the above, we have:

Theorem 1: $\mathrm{Log}_2(N, m, p)$ is WSNB for broadcast under the θ-window algorithm if and only if p is as shown in the equation at the bottom of the page.

Results for $m = 0$ correspond to the results in [4]; results for $m = 1$, 2 correspond to the results in [2] and [3].

Note that $\mathrm{Log}_2(N, n-1, p)$ is the Cantor network.

Corollary 2: The Cantor network is WSNB for broadcast under the θ-window algorithm if and only if $p > 2^{\theta-2} + \theta \cdot 2^{n-\theta-1} - \theta/2 + 2^{1-\theta} - 2^{1-n} + 1/4$ (0 if $\theta = 1$), for $n \ge 3$.

IV. Optimization

Let $f(\theta, m)$ denote the maximum number of blockings required in *Theorem 1* for given θ and m. In this section, we determine optimal θ^0 for given n and m, and also compare the optimal solutions among different m.

$f(\theta, 0)$ is decreasing in θ for $\theta \le \lfloor n/2 \rfloor - 1$. Hence, $\theta^0 = \lfloor n/2 \rfloor - 1$ in that range. Since

$$f\left(\left\lfloor \frac{n}{2} \right\rfloor - 1, 0\right) - f\left(\left\lfloor \frac{n}{2} \right\rfloor, 0\right) = \left[\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\cdot 2^{\lceil n/2 \rceil} + 2^{n-2\cdot\lfloor n/2 \rfloor+1} - 1\right] - \left[2^{\lfloor n/2 \rfloor} + \left(\left\lceil \frac{n}{2} \right\rceil - 2\right)\cdot 2^{\lceil n/2 \rceil - 1}\right] > 0 \text{ for } n \ge 3$$

we conclude for $m = 0$ and $n \ge 3$, $\theta^0 \ge \lfloor n/2 \rfloor$. It was shown in [4] that $\lceil n/2 \rceil$ is a better choice than $\lfloor n/2 \rfloor$. Since $f(\theta, 0)$ for $\theta \ge \lfloor n/2 \rfloor$ has a unique minimum, we can start with $\lceil n/2 \rceil$ and increase the window size until $f(\theta, 0)$ increases. In general, θ^0 grows slowly with rate $\log_2 n$ and can be quickly found.

$f(\theta, 1)$ is decreasing in θ for $\theta \le \lfloor n/2 \rfloor - 1$.

Since

$$f\left(\left\lfloor \frac{(n+1)}{2} \right\rfloor - 1, 1\right) - f\left(\left\lfloor \frac{(n+1)}{2} \right\rfloor, 1\right) = \left[\left(\left\lfloor \frac{(n+1)}{2} \right\rfloor - 1\right)\cdot 2^{\lceil (n-1)/2 \rceil} + 2^{n-2\cdot\lfloor (n+1)/2 \rfloor+1} - \frac{1}{2}\right] - \left[2^{\lfloor (n-1)/2 \rfloor} + \left(\left\lceil \frac{(n-1)}{2} \right\rceil - 1\right)\cdot 2^{\lceil (n-1)/2 \rceil - 1}\right] > 0 \text{ for } n \ge 3$$

$\theta^0 \ge \lfloor (n+1)/2 \rfloor$. Again, $f(\theta, 1)$ has a unique minimum, and $\lfloor n/2 \rfloor + 1$ is a good value to start the upward searching.

Finally, for $m \ge 2$, we note that $f(\theta, m)$ is increasing in m for all $\theta \ge m$. Since a larger m implies more stages and larger cost, there is no reason to consider $m > 2$ when it costs more but performs worse. For $\theta \ge m = 2$

$$f(\theta, 2) = \begin{cases} \theta \cdot 2^{n-\theta-1} + 2^{n-2\theta-1}, & \text{for } \theta \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2^{\theta-2} + (n-\theta)\cdot 2^{n-\theta-1} + \frac{1}{4}, & \text{for } \theta \ge \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$p > \begin{cases} \theta \cdot 2^{n-\theta-1} + 2^{n-2\theta-1} - 1, & \text{for } m = 0,\ \theta \le \lfloor \frac{n}{2} \rfloor - 1 \\ \theta \cdot 2^{n-\theta-1} + 2^{n-2\theta-1} - \frac{1}{2}, & \text{for } m = 1,\ \theta \le \lfloor \frac{(n+1)}{2} \rfloor - 1 \\ 2^{\theta} + (n-\theta-2)\cdot 2^{n-\theta-1}, & \text{for } m = 0,\ \theta \ge \lfloor \frac{n}{2} \rfloor \\ 2^{\theta-1} + (n-\theta-1)\cdot 2^{n-\theta-1}, & \text{for } m = 1,\ \theta \ge \lfloor \frac{(n+1)}{2} \rfloor \\ 2^{\theta-2} + \theta \cdot 2^{n-\theta-1} - \frac{m}{2} - 2^{\theta-m} + 2^{n-2\theta-1} - 2^{-m} + \frac{5}{4}, & \text{for } 2 \le m \le \theta \le \lfloor \frac{(n+m)}{2} \rfloor - 1 \\ 2^{\theta-2} + \theta \cdot 2^{n-\theta-1} - \frac{\theta}{2} + 2^{n-\theta-m} - 2^{-m} + \frac{1}{4}(0 \text{ if } \theta = 1), & \text{for } m > \max\{\theta, 1\}, \theta \le \lfloor \frac{(n+m)}{2} \rfloor - 1 \\ 2^{\theta-2} + (n+m-\theta-2)\cdot 2^{n-\theta-1} - \frac{m}{2} + \frac{5}{4}, & \text{for } \theta \ge \lfloor \frac{(n+m)}{2} \rfloor \ge m \ge 2 \end{cases}$$

TABLE I
BEST CHOICE OF θ AND CORRESPONDING VALUE OF p FOR $m = 2$ AND SOME n

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
θ	2	3	4	4	5	5,6	6	7	7	8	8	9	9	10
P	3	4	6	9	13	21	29	45	65	97	145	209	321	449

n	17	18	19	20	21	22	23	24	25	26	27	28	29	30
θ	10,11	11	12	12	13	13	14	14	15	15	16	16	17	17
p	705	961	1473	2049	3073	4353	6401	9217	13313	19457	27649	40961	57345	86017

n	31	32	33	34	35	36	37	38	39	40	41
θ	18	18	19	19,20	20	21	21	22	22	23	23
p	118785	180225	245761	376833	507905	770049	1048577	1572865	2162689	3211265	4456449

n	42	43	44	45	46	47	48	49	50
θ	24	24	25	25	26	26	27	27	28
p	6553601	9175041	13369345	18874369	27262977	38797313	55574529	79691777	113246209

The first equation is decreasing in θ in its range. Hence, $\theta^0 = \lfloor n/2 \rfloor$.

Since

$$f\left(\left\lfloor \frac{n}{2} \right\rfloor, 2\right) - f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\right) = \left\lfloor \frac{n}{2} \right\rfloor \cdot 2^{\lceil n/2 \rceil - 1} + 2^{n - 2\lfloor n/2 \rfloor - 1} - 2^{\lfloor n/2 \rfloor - 1} - \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \cdot 2^{\lceil n/2 \rceil - 2} - \frac{1}{4} > 0 \text{ for } n \geq 4$$

$\theta^0 \geq \lfloor n/2 \rfloor + 1$. $f(\theta, 2)$ has a unique minimum and $\lfloor n/2 \rfloor + 1$ is a good value to start the upward searching.

We next compare the optimal solutions for $m = 0$, 1, 2. We will only compare the starting values in the search process.

$$f\left(\left\lceil \frac{n}{2} \right\rceil, 0\right) = 2^{\lceil n/2 \rceil} + \left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right) \cdot 2^{\lfloor n/2 \rfloor - 1}$$
$$f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 1\right) = 2^{\lfloor n/2 \rfloor} + \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) \cdot 2^{\lceil n/2 \rceil - 2}$$
$$f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\right) = 2^{\lfloor n/2 \rfloor - 1} + \left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \cdot 2^{\lceil n/2 \rceil - 2} + \frac{1}{4}.$$

Clearly, $f(\lfloor n/2 \rfloor + 1, 1) < f(\lceil n/2 \rceil, 0)$.

Furthermore

$$f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 1\right) - f\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 2\right) = 2^{\lfloor n/2 \rfloor - 1} - 2^{\lceil n/2 \rceil - 2} - \frac{1}{4} \geq 0.$$

So $m = 2$ does better in minimizing the number of copies required. However, we have to recall that a copy with $m = 0$ or $m = 1$ costs less. For all three m values, the number of crosspoints is about $O\left(N^{3/2} \log^2 N\right)$.

According to the above result, we choose $m = 2$, and compute the best choice of θ and the corresponding value of p for each n in Table I.

Note that for $n = 17$, two θ's yield the same m-value. For larger n in the table, we show the p-values mainly for mathematical interest, not for practical use.

V. CONCLUSION

We extended the study of a multicast $\text{Log}_2(N, 0, p)$ network in [2] and [5] to a multicast $\text{Log}_2(N, m, p)$ network by refining their window algorithm. We obtain necessary and sufficient conditions on m such that the network is WSNB. We also estimate the optimal window size.

Intuitively, one would expect the larger m is, the more connecting power the $\text{Log}_2(N, m, p)$ is, and hence, the fewer copies are needed for nonblocking. One would also expect the optimal m grows with N. We obtain the surprising result that $m = 2$ is optimal universally. But this is a technical result, for which we have no insight into why it is so. Nonetheless, it is a very valuable result, since regardless of how large is N, we need only to use moderate-size $\text{Log}_2(N, m, p)$, i.e., $\text{Log}_2(N, 2, p)$, which are relatively inexpensive to construct.

Like all routing algorithms, the delayed splitting algorithm restricts the scope of ways in connecting a multicast call. But it also restricts the scope of interference a multicast connection has on other requests. It is a tradeoff whose net value we do not know for sure. However, the delayed splitting algorithm simplifies routing to a degree that an analysis of the nonblocking condition becomes tractable.

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