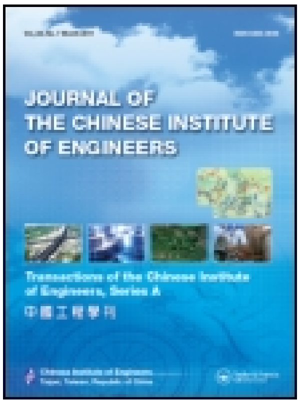

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Numerical time domain BEM experiment for 2-D elastodynamics

Gin-Show Liou [a], Gin-Zen Lai [b] & Chung-Cheng Wang [c]

[a] Department of Civil Engineering, Chiao-Tung University, Hsing-Chu, Taiwan 300, R.O.C. Phone: 886-3-5712121 ext. 54906 E-mail:

[b] Department of Civil Engineering, Chiao-Tung University, Hsing-Chu, Taiwan 300, R.O.C.

[c] Chung Shan Institute of Science and Technology, Taoyuan, Taiwan 325, R.O.C.
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NUMERICAL TIME DOMAIN BEM EXPERIMENT FOR 2-D ELASTODYNAMICS

Gin-Show Liou*, Gin-Zen Lai and Chung-Cheng Wang

ABSTRACT

This paper investigates some numerical aspects of solving 2-D time domain elastodynamic problems by the Boundary Element Method. In the investigation, quadratic spatial elements on the boundaries of domains, and linear temporal variations for displacement and constant temporal variation for traction in one time step are employed. For calculating internal displacement and stress, the traction kernels and internal stress kernels are derived by following the methodology developed by Israil and Banerjee.

Since the non-dimensional time step $\beta = \frac{c_1 \Delta t}{\ell}$ is a major parameter in judging the accuracy and stability of time domain BEM, the effects of β value on numerical results are examined thoroughly in the investigation. Also, how accurate the time domain BEM is for calculating internal stress and displacement is also investigated comprehensively. Some conclusions are drawn from numerical experiments.

Key Words: time domain BEM, elastodynamics, internal stress, β-values, numerical damping.

I. INTRODUCTION

Although both the boundary element method and the finite element method were proposed at almost the same time, more than 40 years ago, the finite element method has gotten more attention from the researchers in the field of computational mechanics. The reasons for this trend in the past 40 years are that the finite element method, in general, is a physical approach with simple mathematical manipulation, which is more favored by practicing engineers, and the boundary element method has problems such as numerical singularities, a complex integration scheme, and an asymmetrically operating matrix. If one wants to obtain internal stress and displacement, one has to solve the boundary values first.

However, after so many resources have been poured into researches on the finite element method, the method has become mature and the finite element method also has some limitations. For example: difficulties in dealing with an infinite domain, and numerical precision problems in dealing with wave propagation problems. The boundary element method has some advantages over the finite element method such as better numerical precision with only boundary meshing necessary, which greatly reduces the number of system equations while solving engineering problems. Therefore, in the past decade, the boundary element method has attracted more and more academic researchers to work on it, and has gradually become the main stream of research in the field of computational mechanics (Banerjee, 1994; Kane, 1994).

Especially in solving elastodynamic problems, the boundary element method has advantages over finite element method, since artificial wave reflection may occur in the finite element method when the element sizes change in the domain and the boundary element method gives much better precision for wave propagation problems.

To solve two-dimensional time-domain problems, Niwa *et al*. (1980) integrated three-dimensional

*Corresponding author. (Tel: 886-3-5712121 ext. 54906; Email: gsliou@mail.nctu.edu.tw)

G. S. Liou and G. Z. Lai are with the Department of Civil Engineering, Chiao-Tung University, Hsing-Chu, Taiwan 300, R.O.C.

C. C. Wang is with the Chung Shan Institute of Science and Technology, Taoyuan, Taiwan 325, R.O.C.

transient kernels at each time step with respect to the third spatial coordinate only in the range of wave propagation distance. Mansur (1983) was the first to formulate a time-stepping algorithm directly using 2*D* time-domain elasto-dynamic kernels. However, the accuracy of the formulation suffers as indicated by Israil and Banerjee (1990) for the following reasons: mathematical complexity resulting from the treatment of Heaviside functions in the kernel functions, simplified assumptions of constant variation of spatial variables, modeling of boundary geometry by using straight line segments, and inadequate treatment of edges and corners, etc. After these formulations, the integral equation solutions of elasto-dynamic problems in the time domain have also been presented by the weighted residual method and the reciprocity method. Spyrakos and Antes (1986) have found that the reciprocity method takes considerably less calculation time than the weighted residual method for elasto-dynamic transient problems with short duration. Israil and Banerjee (1990, 1991) have made certain contributions to the numerical implementation of time-stepping techniques and also presented a number of numerical solutions. In their works, the temporal convolution integrals are evaluated analytically and the spatial integrations are performed numerically at each time step. Wang and Takemiya (1992) also analytically obtained both spatial and temporal integration for scalar wave by the Cagninard-de Hoop method. However, in the aforementioned methods, the temporal solution is assumed to be either the zeroth or first order (i.e. constant or linear variation) with one-time-step piecewise continuity. Wang *et al*. (1996, 1997) followed the methodology of Israil and Banerjee to develop the quadratic temporal solution, *QC* or *QL* method, which is called the second order method with two-time-step piecewise continuity. In the procedure, quadratic temporal variation for displacement and constant or linear variation for traction are adopted, and spatial field variations are assumed to be quadratic. Just like Israil and Banerjee's works, the temporal integrations can be obtained analytically and the spatial integration is obtained using the Gaussian quadrature method. Also, Chen and Hong (1990, 1993) have made some contributions to the dual boundary element method which can be employed to solve elastodynamic problems.

Since one disadvantage of the boundary element method (BEM) is that BEM must obtain boundary solutions first and then calculate internal stress and displacement, this paper is devoted to investigating some numerical schemes and precision of the boundary element method when internal stress and displacement are involved. To calculate the internal stress, Israil and Banerjee (1991) have proposed a pair of first order temporal internal stress kernels, which will be used in the following investigations. In the investigations, the parameter of non-dimensionalized time step $\beta=\frac{c_1\Delta t}{\ell}$, in which c_1 is compressional wave velocity, Δt is time step interval and ℓ is length of quadratic element at boundary, will be examined in great detail for solving elasto-dynamic internal displacements and stresses, and evaluating the precision for calculating internal displacement and stress while using the boundary element method. This precision of internal displacement and stress will depend upon locations where internal displacement and stress are calculated. Besides, the paper will compare the precision of BEM at boundary points with that at internal points, since which result is better is still arguable in the engineering community. In these numerical experiments, the variations of displacement and traction with respect to time will be linear and constant respectively in one-time-step (the so-called *LC* method) and the kernel integrations with respect to time will be calculated analytically, and at boundary the spatial field variation is assumed to be quadratic and Gaussian quadrature is used to do the numerical integrations.

II. OUTLINED FORMULATIONS OF BEM EQUATIONS

To solve the internal stress and displacement in a domain, the boundary values must be obtained first. To obtain the boundary values, the quadratic element for the spatial boundary is assumed, time discretization is assumed to be uniform segments by equal time step, and temporal variations are assumed to be linear for displacement and constant for tractions in one step. Therefore, following Israil and Banerjee's (1990, 1991) methodology, the boundary equations for the N^{th} time step (i.e. time $t=N\Delta t$, Δt is time step) can be deduced to be

$$\sum_{n=1}^{N}([\boldsymbol{G}_{CFij}^{N-n+1}+\boldsymbol{G}_{CBij}^{N-n}]\{T_j^n\} - [\boldsymbol{F}_{LFij}^{N-n+1}+\boldsymbol{F}_{LBij}^{N-n}]\{\boldsymbol{U}_j^n\})=\{0\} \tag{1}$$

In Eq. (1) kernel matrices $[G_{ij}]$ and $[F_{ij}]$ can be found in the work of Wang *et al*. (1996, 1997) in which the integration with respect to time is calculated analytically and the integration with respect to space is calculated by Gaussian quradrature, $\{T_j^n\}$ and $\{U_j^n\}$ are vectors of values at boundary nodes for tractions and displacements respectively, N in superscripts represents the N^{th} time step of boundary solutions, *CF* and *CB* in subscripts represent constant variations in one time step for forward and backward temporal points respectively, *LF* and *LB* in subscripts represent linear variation in one time step for forward and backward temporal points respectively, and subscripts i and j mean spatial dimensions and i, j=1 or 2 for two dimensional problems. In deriving Eq. (1), Wang *et al*. (1996, 1997) have proved that rigid body motion method to remove the singularity problem in the numerical scheme can also be employed for elastodynamic problems.

To solve Eq. (1), half of the number of values in traction vector $\{T_j^n\}$ and displacement vector $\{U_j^n\}$ must be known due to boundary conditions. Eq. (1) can be rearranged as

$$[\boldsymbol{A}]\{\boldsymbol{X}^N\}=[\boldsymbol{B}]\{\boldsymbol{Y}^N\}+\{\boldsymbol{R}^N\} \tag{2}$$

in which $\{X^N\}$ and $\{Y^N\}$ represent unknown boundary quantities and known boundary quantities at time step N respectively, matrices $[A]$ and $[B]$ are the rearranged kernel matrices $[G_{ij}]$ and $[F_{ij}]$ in Eq. (1), and vector $\{R^N\}$ is the influence due to the dynamic effect from time step n=1, 2, ..., N–1. Therefore,

$$\{\boldsymbol{R}^N\} = -\sum_{n=1}^{N-1}([G_{CFij}^{N-n+1}+G_{CBij}^{N-n}]\{T^n\} - [F_{LFij}^{N-n+1}+F_{LBij}^{N-n}]\{U^n\}) \tag{3}$$

After solving Eq. (2) for all the boundary values, one can proceed to obtain the displacement and stress at a specified internal point in the domain. Israil and Banerjee (1991) proposed the internal stress kernels for calculating the stress at internal points in a domain. After some mathematical manipulations, the displacement and stress at an internal point can be obtained as follows:

$$u_i^N = \sum_{n=1}^{N}([G_{CFij}^{N-n+1}+G_{CBij}^{N-n}]\{T_j^n\} - [F_{LFij}^{N-n+1}+F_{LBij}^{N-n}]\{U_j^n\}) \tag{4}$$

$$\sigma_{ij}^N = \sum_{n=1}^{N}(\{G_{CFijk}^{\sigma^{N-n+1}}+G_{CBijk}^{\sigma^{N-n}}\}\{T_k^n\} - \{F_{LFijk}^{\sigma^{N-n+1}}+F_{LBijk}^{\sigma^{N-n}}\}\{U_k^n\}) \tag{5}$$

In Eqs. (4) and (5), kernel matrices $\{G_{CFij}^{N-n+1}+G_{CBij}^{N-n}\}$ and $\{F_{LFij}^{N-n+1}+F_{LBij}^{N-n}\}$ contain the elements after spatial integrations of convoluted kernels with the source at the specified internal point, kernel matrices $\{G_{CFijk}^{\sigma^{N-n+1}}+G_{CBijk}^{\sigma^{N-n}}\}$ and $\{F_{LFijk}^{\sigma^{N-n+1}}+F_{LBijk}^{\sigma^{N-n}}\}$ contain the elements after spatial integrations of internal convoluted stress kernels with the source at the specified internal point. These forms of integration can be found in Israil and Banerjee's work (1990, 1991), and the vectors $\{U_j^n\}$ and $\{T_j^n\}$ are the known and solved

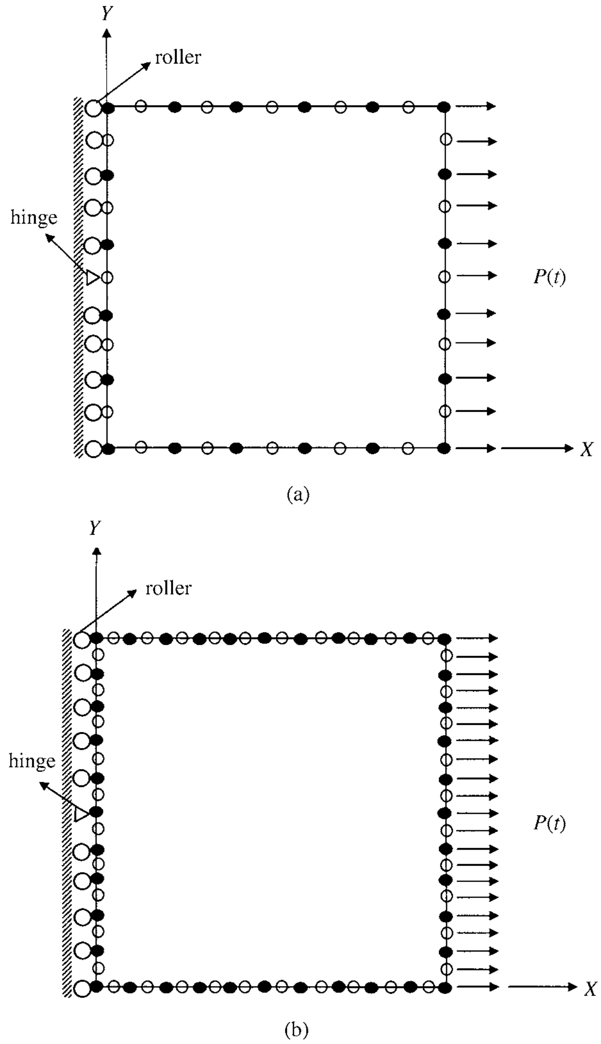

Fig. 1 (a) $L\times L$ square domain with 20 elements; (b) $L\times L$ square domain with 40 elements

boundary values from Eq. (2). To solve Eqs. (4) or (5), one should note that u_i^N and σ_{ij}^N can be obtained consecutively for all time steps simply by matrix dot product operation.

After summarizing the solution process for BEM, we are in the position to proceed to do the investigation of numerical prospects of the outlined method.

III. NUMERICAL INVESTIGATION

To perform the numerical investigation, an $L\times L$ square domain and an $L\times L/10$ rectangular domain in which L is the length of the domian as shown in Figs. 1 and 2 are selected with uniform ramp-step load, as shown in Fig. 3 in which the rising time t_R=0.8L/c_1, is applied at the right-hand side, and hinge and roller supports are at the left-hand side as the figures show. Poisson's ratio (ν) is assumed to be zero. In order to

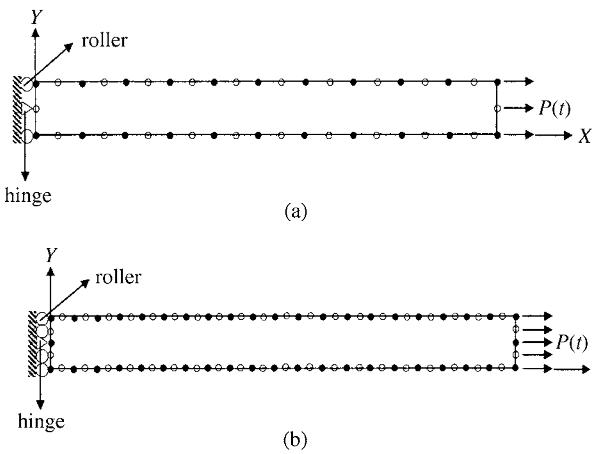

Fig. 2 (a) $L\times L/10$ rectangular domain with 22 elements; (b) $L\times L/10$ rectangular domain with 44 elements

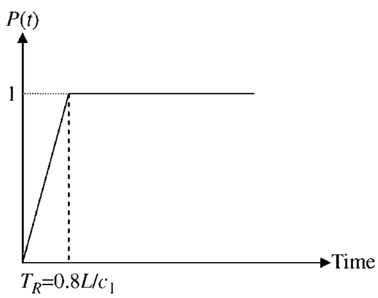

Fig. 3 Ramp-step load (T_R is the rising time)

investigate the effect of the meshing scheme on the accuracy of internal stresses and displacements, two meshes for the boundary are employed. One is 40 boundary nodes with 20 quadratic boundary elements as shown in Fig. 1(a), and the other is 80 boundary nodes with 40 quadratic boundary elements as shown in Fig. 1(b). Similar meshing schemes are used for the rectangular domain as shown in Figs. 2(a) and 2(b). In the calculations, the non-dimensionalized time step $\beta=\frac{c_1\Delta t}{\ell}$ is selected to be in the range of 0.05~1.00 in order to obtain the effect of β-value on the accuracy of results of boundary values and internal values, since β is an important parameter for using BEM to solve time domain elasto-dynamic problems. In order to indicate the locations of interior points and boundary points, the dimensions of the domains and the coordinate systems are shown in Figs 1(a), 1(b), 2(a) and 2(b). Therefore, in the following figures of the numerical results, the coordinates of internal and boundary points will be indicated for reference to the locations of these numerical results.

Figure 4 shows some numerical results for the case of the $L\times L$ domain with 20 boundary elements,

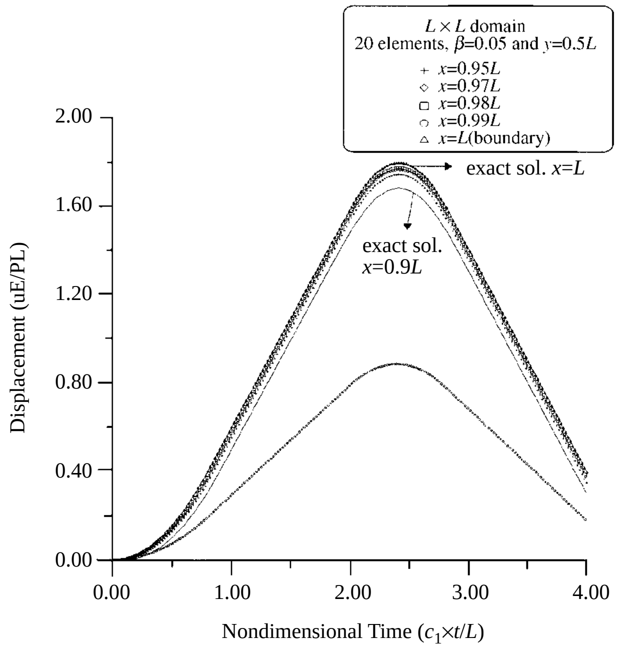

Fig. 4 Displacement for y=0.5L and β=0.05

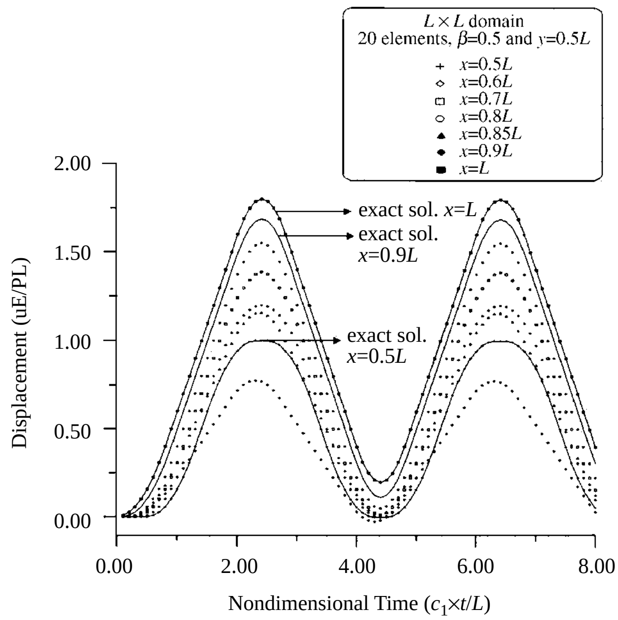

Fig. 5 Displacement for y=0.5L and β=0.5

β=0.05 and y=0.5L. In the figure and the figures thereafter, time step is nondimensionalized by compressional wave velocity c_1 and the length of domain L. From the figure, one can observe that the result for x=0.95L, 0.97L, 0.98L and L (x=L means boundary point) are excellent. However, for x=0.99L the result is not good. Also, from Fig.5, one can see that the results for x=0.5L, 0.6L, 0.7L and 0.8L are very good, but for x=0.85L and 0.9L, the precision of the results is bad. Therefore, a thorough numerical

Table 1 Summary of result precisions for different β-value and location (stress column value is traction at x/L=1.0)

β	β=0.05		β=0.1		β=0.2		β=0.5		β=0.8		β=1	
x/L	disp.	stress	disp.	stress	disp.	stress	disp.	stress	disp.	stress	disp.	stress
0.5	○	○	○	○	○	○	○	○	○	○	○	○
0.55	○	○	○	○	○	○	○	○	○	○	○	○
0.6	○	○	○	○	○	○	○	○	○	○	○	×
0.65	○	○	○	○	○	○	○	○	○	×	×	×
0.7	○	○	○	○	○	○	○	○				
0.75	○	○	○	○	○	○	○	○				
0.8	○	○	○	○	○	○	○	○				
0.85	○	○	○	○	○	○	×					
0.9	○	○	○	○	○	○	×					
0.91	○	○	○	○	○	○	×					
0.92	○	○	○	○	○	○	×					
0.93	○	○	○	○	×	×	×					
0.94	○	○	○	○	×	×	×					
0.95	○	○	○	○	×	×	○					
0.96	○	○	○	○	×	×	×					
0.97	○	○	×	×	×	×	×					
0.98	○	○	×	×	○	×	×					
0.99	×	×	○	×	×	×	×					
1.0(boundary)	○	○	○	○	○	○	○	○	○	○	○	○

experiment investigating the numerical precision of the results for internal points has been done by using different β-values. The results are summarized in Table 1. In the table, ○ means a good result which almost matches the exact solution* and × means a bad result which deviates from the exact solution like the results for x=0.99L in Fig. 4 and x=0.85L and 0.9L in Fig. 5. All the results summarized in Table 1 are for 20 quadratic boundary elements and the same ramp-step load. However, the result for 40 elements as shown in Fig. 1(b) and for the case of $L\times L/10$ domain as shown in Figs. 2(a) for 22 elements and 2(b) for 44 elements have similar patterns to that in Table 1. From the table, one can easily observe the two phenomena as follows: (1) If the internal point is closer to the boundary, the accuracy of the result will be getting worse; (2) If β is getting smaller, one can obtain a good result for an internal point closer to the boundary. Although the table shows the results by changing x-coordinate, a similar pattern is obtained by changing the y-coordinate. Therefore, one can conclude that when the internal point of interest is closer to the boundary, a good result may be difficult to obtain. Although one may use a smaller β, the problem of numerical stability of BEM may arise. The aspect of numerical instability due to small β will be discussed later in the paper. By generalizing all the numerical results, one can also conclude as follows: If the internal point is close to the boundary with a distance less than $\frac{3}{2}\beta\ell$ or $\frac{3}{2}c_1\Delta t$, in which ℓ is the length of quadratic element at boundary, the precision of the numerical results for the internal point will not be good. Although the latter expression, $\frac{3}{2}c_1\Delta t$, seems to tell us that the numerical results of internal stresses and displacements are not dependent upon the element size ℓ, one should notice that the calculation of internal stresses or displacements is based on the results at the boundary where precisions will be affected by the element size ℓ. Therefore, the criterion $\frac{3}{2}\beta\ell$ or $\frac{3}{2}c_1\Delta t$ is valid only under the condition of accurate results at the boundary. The conclusion is sketched in Fig. 6. If the internal point falls in the shaded area of the domain shown in Fig. 6, it is very difficult to obtain good numerical results for the point. The explanation for this phenomenon may be as follows: When the internal point is close to the boundary, the singularity behavior of internal displacement and stress kernels will make the numerical integrations of Gaussian quadrature break down. So just increasing the number of subsegmentation and integration points may not be enough. A new numerical integration scheme, like rigid body motion

*$u=\frac{1}{\rho c_1}\int_0^{t-\frac{x}{c_1}} p(s)ds$, where ρ is mass density, c_1 is compressional wave velocity and $p(s)$ is applied traction.

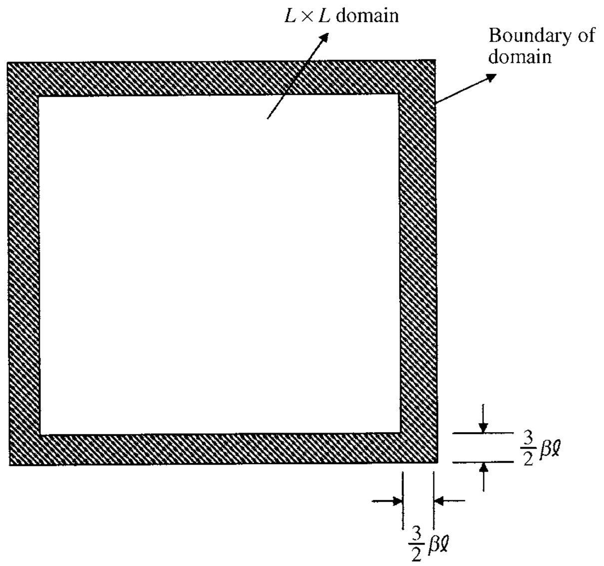

Fig. 6 Shaded area with bad interior results

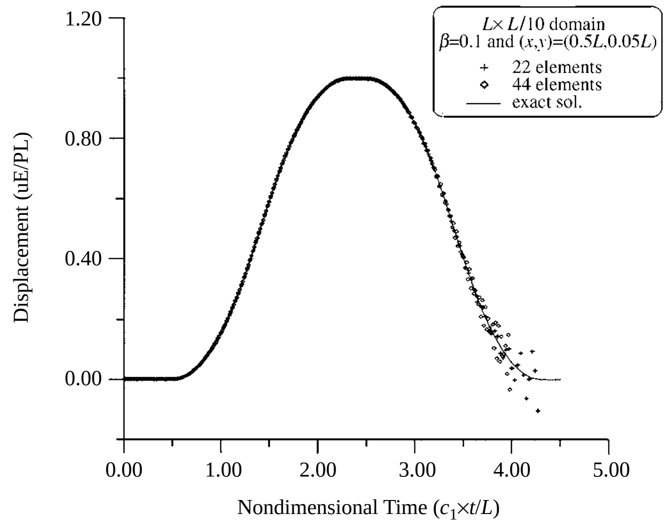

Fig. 7 Comparison of displacements for 22 and 44 elements

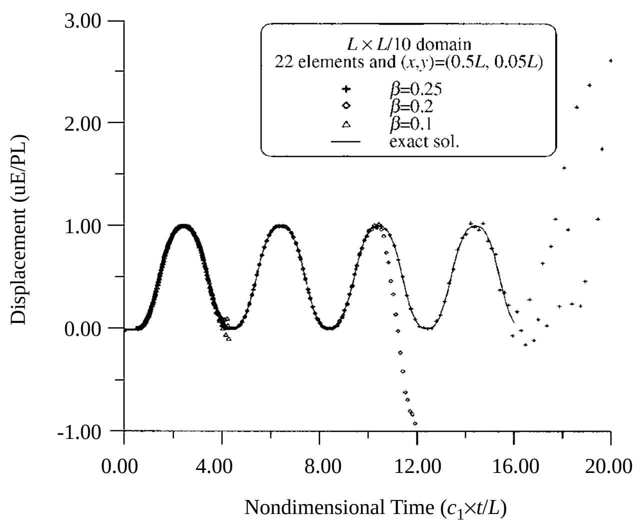

Fig. 8 Comparison of displacements with different β-value

method for integration of kernel for boundary nodes, must be developed to deal with the problem of the strong singularity behavior.

Conventionally, the suggested non-dimensional time step β should be between 0.25 and 0.75 for better results of time domain BEM of first order. However, in this investigation, the β value could be as small as 0.05 for the case of an $L\times L$ domain as shown in Figs. 1(a) and 1(b) without numerical instability. But for the case of an $L\times L/10$ domain shown in Figs. 2(a) and 2(b), small β value may create numerical problem after some time steps. Fig. 7 shows the displacements at coordinates (x, y)=(0.5L, 0.05L) for the cases of 44 elements and 22 elements with β=0.1. From the figure, one can see that both cases have good results at the beginning of the time steps. But, after some time steps, the results start to diverge. Also, one can observe that the result diverges a little earlier for the case of 44 elements (finer mesh). This is because one needs more time steps (twice as many as in the 22 element case) to reach a certain time point, and the accumulation of truncation error in the calculation of each time step will make the numerical divergence occur earlier. Fig. 8 compares the displacement results of β=0.1, 0.2 and 0.25. From the figure, one can conclude that a smaller β will make numerical instability occur earlier. One of the reasons causing this instability could be the truncation error propagation, since using smaller β means that more calculation steps are needed in order to reach the same time point. However, this truncation error propagation is not the only culprit to cause the instability, since for the case of an $L\times L$ domain, the instability never occurs even for β=0.05 in Fig. 4. This may be because the number of time steps is not large enough. In this investigation, the time span for calculation is only up to $t=\frac{16L}{c_1}$ second. Therefore, one can easily say that β is not the only parameter to control the numerical instability and precision. There must be some other factors that will influence the numerical precision and stability of BEM while solving time domain elastodynamic problems. Or, the numerical integration scheme of uniform subsegmentation may not be good enough, and a better integration scheme is needed if one wants to calculate a result for a long time span. Although Figs. 7 and 8 show the results for an internal point, the results for a boundary point also have a similar divergent pattern.

In the engineering community, some still argue that using BEM to calculate internal stresses and

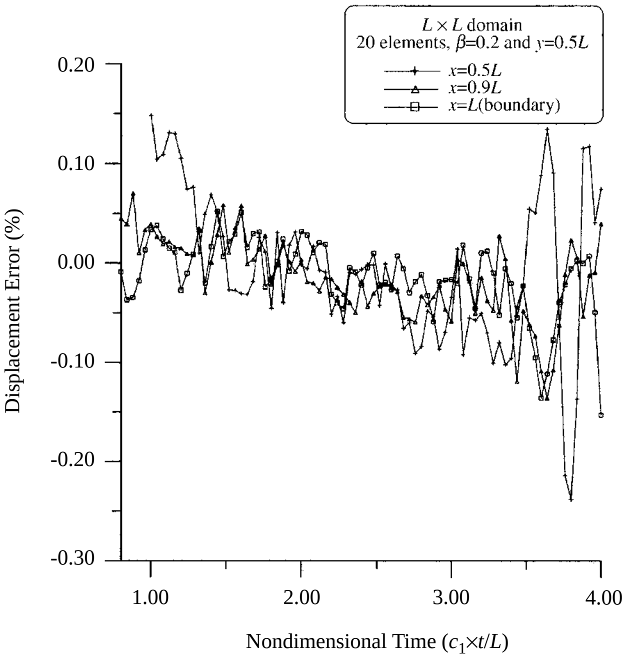

Fig. 9 Comparison of precisions at boundary and interior points

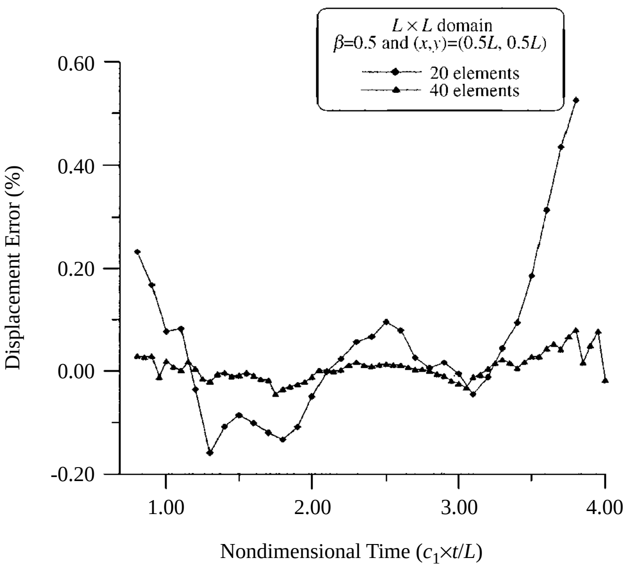

Fig. 10 Comparison of disp. precisions for two boundary meshing schemes

displacements may or may not be as accurate as calculating boundary tractions and displacements. Therefore, some comparisons of the results have been made in order to show some clues. Fig. 9 compares some results for the case of an $L \times L$ domain with β= 0.2. In the figure, the error is defined by subtracting the exact result from the numerical result and then dividing by the exact result. From the figure, one can see that, in general, the result for a boundary point

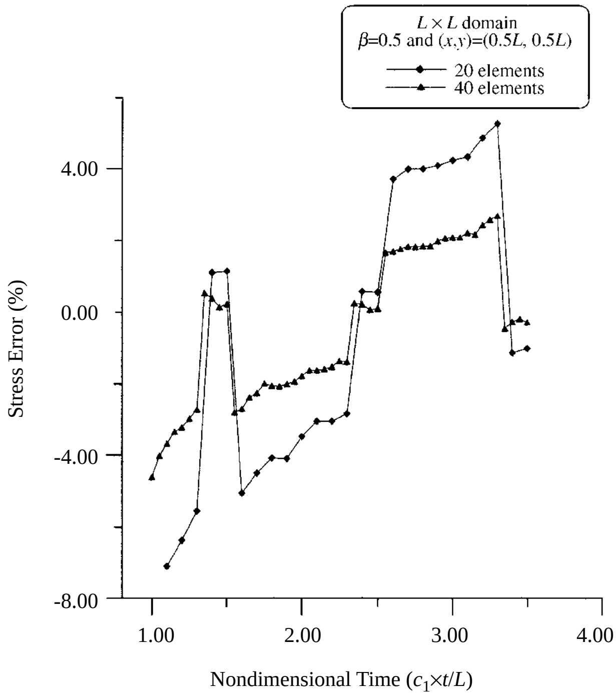

Fig. 11 Comparison of stress precisions for two boundary meshing schemes

is better than that for an interior point although all the results are pretty good. However, at certain time steps, the result for an internal point may be better. Therefore, it doesn't matter which result is better, since both results are good anyway. Generally speaking, that the result for boundary points is good means that the results for internal points will be good too, if the internal point doesn't fall in the region defined in Fig. 6. Fig. 10 shows the comparison of results between two boundary meshing schemes. From the figure, one can observe that finer boundary mesh can give better precision of displacement result for an internal point just like for a boundary point. A similar situation can be observed for the case of calculating internal stress. This is shown in Fig. 11. By comparing Fig. 11 to Fig. 10, one can also conclude that using BEM to calculate internal stress and displacement, one can obtain better precision for displacement. The reason for this phenomenon is that the singularity order for the more complicated traction kernels is stronger than that for the displacement kernels in the numerical investigation.

Also, for a larger β value, the accuracy of results may deteriorate, and numerical damping may become a problem as time steps march on. Fig. 12 shows this numerical damping phenomenon. From Fig. 12, one can see that as β becomes larger, the numerical damping is getting stronger. Therefore, if one wants to use a large β in time domain BEM with

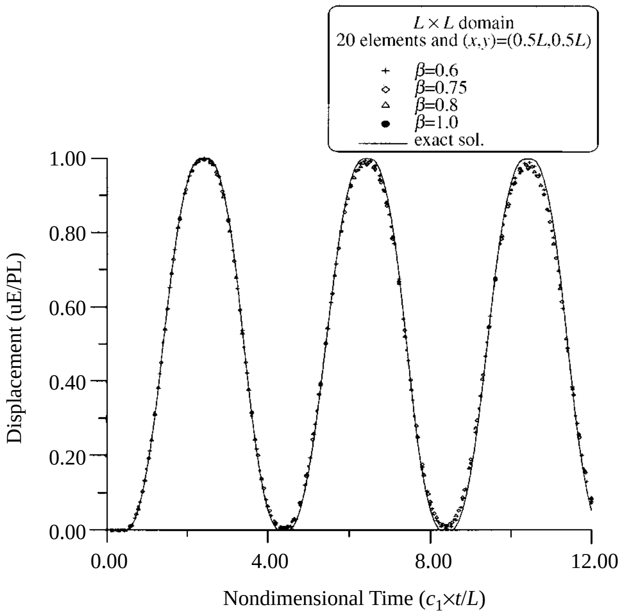

Fig. 12 Numberical damping with larger β-value

a long time span in order to save computational cost, one should be aware that numerical damping may cause inaccuracy as time steps march. If one wants to improve the precision, higher order temporal variation for traction and displacement reported by Wang *et al*. (1996, 1997) should be employed.

IV. CONCLUSIONS

After a thorough numerical experiment, some conclusions can be outlined as follows:

1. To calculate the internal stresses and displacements, the locations of the internal points should not fall in the shaded region defined in Fig. 6. Otherwise, the precision of results will suffer severely, if the traditional integration scheme of uniform subsegmentation is employed. Therefore a more sophisticated integration method may need to be developed.
2. Also, one should notice that finer mesh would make the area of the shaded region, shown in Fig. 6, smaller.
3. For cases of small β value, the instability of numerical results will occur quite early in the time step series. And as β is getting smaller, the occurrence of divergence of numerical result comes earlier in the time step series. However, how early it is will depend upon what kind of domain it is. Therefore, there must be more than one parameter of β to define the accuracy and stability of the numerical results by time domain BEM.
4. For the cases with large β, the accuracy of the numerical results will deteriorate and numerical damping will greatly affect the numerical accuracy, especially for a long time span solution. If one wants to get better numerical results, one may use higher order temporal variations for displacement and traction as reported by Wang *et al*. (1996, 1997).
5. In general, the precision for calculating internal displacement is better than that for calculating internal stress. Although the accuracy of internal stresses and displacements may not be as good as that for boundary tractions and displacements in time domain BEM, the difference between the two accuracies is minor.

NOMENCLATURES

$[A]$	rearranged kernel matrix of $[G_{ij}]$ and $[F_{ij}]$
$[B]$	rearranged kernel matrix of $[G_{ij}]$ and $[F_{ij}]$
c_1	compressional wave velocity
CF	constant variation in one time step for forward temporal point
CB	constant variation in one time step for backward temporal point
$\{F_{LFij}^{N-n+1}+F_{LBij}^{N-n}\}$	corresponding condensed linear traction kernel matrix
$\{F_{LFijk}^{\sigma^{N-n+1}}+F_{LBijk}^{\sigma^{N-n}}\}$	corresponding condensed linear traction kernel matrix related to stress
$\{G_{CFij}^{N-n+1}+G_{CBij}^{N-n}\}$	corresponding condensed constant displacement kernel matrix
$\{G_{CFijk}^{\sigma^{N-n+1}}+G_{CBijk}^{\sigma^{N-n}}\}$	corresponding condensed constant displacement kernel matrix related to stress
ℓ	length of quadratic element
L	length of square domain
LB	linear variation in one time step for backward temporal point
LF	linear variation in one time step for forward temporal point
N	number of total time step
n	n^{th} time step
$p(s)$	applied traction
$\{R^N\}$	vector due to influence of dynamic effect from previous time step
$\{T_j^n\}$	vector of tractions in j^{th} direction at n^{th} time step
t	time duration
t_R	rising time of ramp-step load
$\{U_j^N\}$	vector of displacements in j^{th} direction at n^{th} time step
u_j^N	displacement in i^{th} direction at N^{th} time step
$\{X^N\}$	vector of unknown at N^{th} time step
$\{Y^N\}$	vector of the known at N^{th} time step

β non-dimensionalized time step
ν Poisson's ratio
ρ mass density
σ_{ij}^{N} stress tensor at N^{th} time step
Δt time step interval

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