Cycle Embedding in Faulty Wrapped Butterfly Graphs

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In this paper, we study the maximal length of cycle embedding in a faulty wrapped butterfly graph BF_n with at most two faults in vertices and/or edges. When there is one vertex fault and one edge fault, we prove that the maximum cycle length is $n2^n - 2$ if n is even and $n2^n - 1$ if n is odd. When there are two faulty vertices, the maximum cycle length is $n2^n - 2$ for odd n. All these results are optimal because the wrapped butterfly graph is bipartite if and only if n is even.

Keywords: Hamiltonian cycle; wrapped butterfly; fault-tolerant; cycle embedding; Cayley graph

1. INTRODUCTION

The performance of a distributed system is significantly determined by its network topology. The hypercube (binary n-cube) is one of the most popular interconnection networks. It has been used to design various commercial multiprocessor machines. One basic drawback with hypercubes is that the vertex degree increases with the number of vertices. Among all networks with fixed degrees, the wrapped butterfly network is one of the most promising networks due to its nice topological properties. On the other hand, the cycle network contains several attractive properties such as simplicity, extensibility, and feasible implementation. Hence, embedding a cycle into a wrapped butterfly network has received many researchers' efforts in recent years [1, 3, 6, 8].

To embed a cycle into a faulty butterfly network BF_n with $n2^n$ vertices, it is desirable to isolate those faulty components from the remaining ones so that a maximal-length cycle can be still embedded. Vadapalli and Srimani [6] proved that there exists a cycle of length $n2^n - 2$ when there is one vertex fault and there is a cycle of length $n2^n - 4$ when two vertex faults occur. In [3], Hwang and Chen showed that the maximal cycle of length $n2^n$ can be embedded in a faulty wrapped butterfly graph which has two edge faults. For all integer $n \geq 3$, these results are optimal because the wrapped butterfly graph is bipartite if and only if n is even.

In the previous two studies, faults are limited to either vertex faults or edge faults. However, faults in both vertices and edges may occur. Consequently, we are motivated to explore the embedding feasibility in the faulty wrapped butterfly graph. In this paper, when there is one vertex fault and one edge fault, we prove that the maximum cycle length is $n2^n - 2$ if n is even and $n2^n - 1$ if n is odd. When there are two faulty vertices, the maximum cycle length is $n2^n - 2$ for odd n. All these results are optimal because the wrapped butterfly graph is bipartite if and only if n is even. In Table 1, we summarize all the results about faulty edges and/or vertices in BF_n.

In the following section, we discuss some properties of wrapped butterfly graphs. In Section 3, we prove that the faulty wrapped butterfly graph contains a cycle of length $n2^n - 2$ if it has one vertex fault and one edge fault. Finally, when n is an odd integer, we prove that the wrapped butterfly graph contains a Hamiltonian cycle if it has at most two faults and at least one of them is a vertex fault.

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TABLE 1. Summarizing all the results about faulty edges and/or vertices in BF_n where the star (*) symbol denotes that the result is optimal.

	Faulty set	Hwang and Chen [3]	Vadapalli and Srimani [6]	Our result
n is odd	1 edge and 1 vertex			$n2^n - 1$*
	2 edges	$n2^n$*		
	2 vertices		$n2^n - 4$	$n2^n - 2$*
n is even	1 edge and 1 vertex			$n2^n - 2$*
	2 edges	$n2^n$*		
	2 vertices		$n2^n - 4$*	

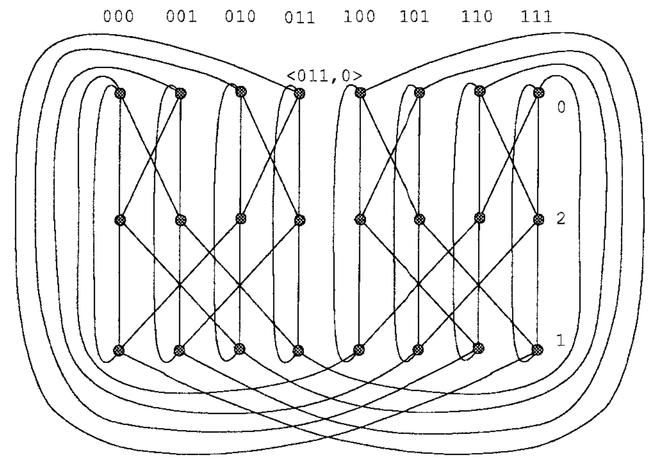

FIG. 1. The structure of BF_3.

2. WRAPPED BUTTERFLY GRAPHS AND THEIR PROPERTIES

An interconnection network can be modeled by an undirected graph $G = (V, E)$ where the set of vertices $V(G)$ represents the processing elements of the network and the set of edges $E(G)$ represents the communication links. Throughout this paper, the graph theoretic definitions and notations in [4] are followed. Let $F = V_1 \cup E_1$ for $E_1 \subseteq E$ and $V_1 \subseteq V$. We use $G - F$ to denote the graph $G' = (V - V_1, (E - E_1) \cap ((V - V_1) \times (V - V_1)))$. A *simple path* (or *path* for short) is a sequence of adjacent edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{m-1}, v_m)$, written as $\langle v_0 \to v_1 \to v_2 \to \ldots \to v_m \rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. We also write the path $\langle v_0 \to P_1 \to v_i \to v_{i+1} \to \ldots \to v_j \to P_2 \to v_k \to v_{k+1} \to \ldots \to v_m \rangle$, where $P_1 = \langle v_0 \to v_1 \to \ldots \to v_i \rangle$ and $P_2 = \langle v_j \to v_{j+1} \to \ldots \to v_k \rangle$. A *cycle* is a path $\langle v_0 \to v_1 \to v_2 \to \ldots \to v_m \to v_0 \rangle$, where $m \geq 2$. A cycle is a *Hamiltonian cycle* if it traverses every vertex of G exactly once. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

2.1. Wrapped Butterfly Graphs

The *wrapped butterfly* (*butterfly* for short) BF_n is a graph with $n2^n$ vertices such that each vertex, at level i, is labeled by $\langle a_0 a_1 \ldots a_{n-1}, i \rangle$ with $0 \leq i \leq n - 1$ and $a_j \in \{0, 1\}$ for all $0 \leq j \leq n - 1$. Edges of BF_n are described as follows: Vertex $\langle a_0 a_1 \ldots a_i \ldots a_{n-1}, i \rangle$ is adjacent to vertex $\langle a_0 a_1 \ldots a_i \ldots a_{n-1}, (i + 1) \textbf{mod } n \rangle$ by a *straight edge* and adjacent to vertex $\langle a_0 a_1 \ldots \bar{a}_i \ldots a_{n-1}, (i + 1) \textbf{mod } n \rangle$ by a *cross edge*. Figure 1 illustrates BF_3.

In [5], Vadapalli and Srimani proposed a family of degree four Cayley graphs, G_n. Later, Chen and Lau [2] pointed out that G_n is isomorphic to BF_n. Thus, we can combine all the results of G_n and BF_n. To prove our main result, we will describe some properties of BF_n proposed by [5]. Throughout the paper, the edges of BF_n are defined by the following four generators g, g^{-1}, f, and f^{-1} in the graph:

$$g(\langle a_0 a_1 \ldots a_{n-1}, k \rangle) = \langle a_0 a_1 \ldots a_{n-1}, (k + 1) \textbf{mod } n \rangle,$$

$$f(\langle a_0 a_1 \ldots a_{n-1}, k \rangle) = \langle a_0 a_1 \ldots a_{k-1} \bar{a}_k a_{k+1} \ldots a_{n-1}, (k + 1) \textbf{mod } n \rangle,$$

$$g^{-1}(\langle a_0 a_1 \ldots a_{n-1}, k \rangle) = \langle a_0 a_1 \ldots a_{n-1}, (k - 1) \textbf{mod } n \rangle, \text{ and}$$

$$f^{-1}(\langle a_0 a_1 \ldots a_{n-1}, k \rangle) = \langle a_0 a_1 \ldots a_{k-2} \bar{a}_{k-1} a_k \ldots a_{n-1}, (k - 1) \textbf{mod } n \rangle.$$

Hence, the *g-edges,* $(u, g(u))$ or $(u, g^{-1}(u))$, and the *f-edges,* $(u, f(u))$ or $(u, f^{-1}(u))$, for some $u \in V(BF_n)$, represent the straight edges and cross edges, respectively. Consequently, we have Lemma 1:

Lemma 1. $f^{-1}(g(u)) = g^{-1}(f(u))$ *for any vertex* u *in* BF_n.

2.2. g-Cycles and g-Edges

Let u be any vertex of BF_n. We observe that $g^n(u) = u$. Moreover, $\langle u \to g(u) \to g^2(u) \to \ldots \to g^n(u) \rangle$ forms a simple cycle C_g^u of length n. We call such a cycle of BF_n a *g-cycle* at u. Hence, $C_g^v \simeq C_g^u$ if and only if $v \in V(C_g^u)$. As a result, all g-cycles form a partition of the straight edges of BF_n. Meanwhile, any f-edge joins vertices from two different g-cycles. It can be seen that $(u, f(u))$ joins vertices from C_g^u and $C_g^{f(u)}$. Lemmas 2 and 3 were proved in [5].

Lemma 2 [5]. $(g(u), g^{-1}(f(u)))$ *is an f-edge joining vertices of* C_g^u *and* $C_g^{f(u)}$. *Moreover, the path* $\langle u \to f(u) \to g^{-1}(f(u)) \to g(u) \to u \rangle$ *forms a cycle of length* 4.

Any C_g^u contains exactly one vertex at each level. In particular, C_g^u contains exactly one vertex at level 0, say $\langle a_0 a_1 \ldots a_{n-1}, 0 \rangle$. We use $C_g^{(a_0 a_1 \ldots a_{n-1})}$ as the name for C_g^u. Now, we form a new graph BF_n^G with all the g-cycles of BF_n as vertices, where two different g-cycles are joined with an edge if and only if there exists at least one f-edge

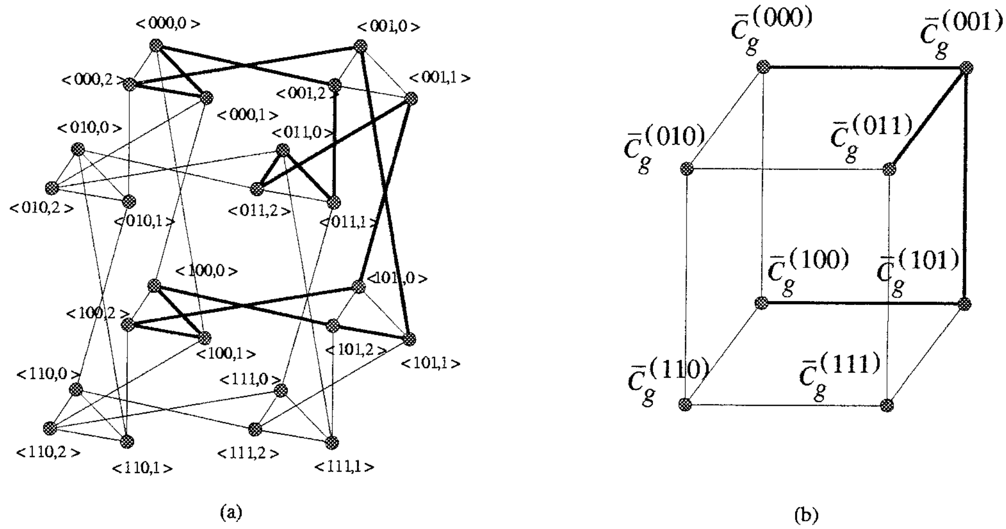

FIG. 2. (a) Another layout of BF_3; (b) the graph BF_3^G.

joining them. The vertex of BF_n^G corresponding to C_g^u is denoted by $\bar{C}_g^u$. We recall the definition of the hypercube as follows: An n-dimensional hypercube (abbreviated to an n-cube) consists of 2^n vertices which are labeled with the 2^n binary numbers from 0 to $2^n - 1$. Two vertices are connected by an edge if and only if their labels differ by exactly one bit.

Lemma 3 [5]. *BF_n^G is isomorphic to an n-dimensional hypercube. Moreover, the set of vertices which are adjacent to $\bar{C}_g^{(a_0a_1\ldots a_{n-1})}$ is $\{\bar{C}_g^{(\bar{a}_0a_1\ldots a_{n-1})}, \ldots, \bar{C}_g^{(a_0a_1\ldots \bar{a}_{n-1})}\}$.*

Let $h = (\bar{C}_g^u, \bar{C}_g^v)$ be any edge of BF_n^G. We use $X(h)$ to denote the set of edges in BF_n joining vertices from C_g^u and C_g^v. Using standard counting techniques, we have the following two corollaries:

Corollary 1 [3]. *If $(u, f(u)) \in X(h)$, then $(g(u), f^{-1}(g(u))) \in X(h)$ and $|X(h)| = 2$.*

Corollary 2 [5]. *There is a unique cycle C such that edges of BF_n joining vertices between C_g^u and C are exactly $(u, f(u))$ and $(g(u), f^{-1}(g(u)))$, and in that case, C is isomorphic to $C_g^{f(u)}$.*

According to Corollaries 1 and 2, any edge $h = (\bar{C}_g^u, \bar{C}_g^v)$ in BF_n^G induces a unique 4-cycle in BF_n, with two f-edges and two g-edges. We use $X_f(C_g^u, C_g^v)$ to denote the set of f-edges in this 4-cycle and $X_g(C_g^u, C_g^v)$ to denote the set of g-edges in this cycle.

Lemma 4. *Let T be any subtree of BF_n^G and let C_g^T be the subgraph of BF_n generated by*

$$\left(\bigcup_{\bar{C}_g^u \in V(T)} E(C_g^u) \cup \bigcup_{\bar{C}_g^u, \bar{C}_g^v \in E(T)} X_f(C_g^u, C_g^v)\right) - \bigcup_{(\bar{C}_g^u, \bar{C}_g^v) \in E(T)} X_g(C_g^u, C_g^v).$$

Then, C_g^T is a cycle of length $n \times |V(T)|$.

Proof. Let T be a subtree of BF_n^G and let $(\bar{C}_g^u, \bar{C}_g^v)$ be an edge of T. Hence, two cycles C_g^u and C_g^v in BF_n are joined by two f-edges $X_f(C_g^u, C_g^v)$. It is easy to see that $E(C_g^u) \cup E(C_g^v) \cup X_f(C_g^u, C_g^v) - X_g(C_g^u, C_g^v)$ forms a cycle of length $2n$ in BF_n. Therefore, the cycle C_g^T of BF_n can be generated by

$$\left(\bigcup_{\bar{C}_g^u \in V(T)} E(C_g^u) \cup \bigcup_{(\bar{C}_g^u, \bar{C}_g^v) \in E(T)} X_f(C_g^u, C_g^v)\right) - \bigcup_{(\bar{C}_g^u, \bar{C}_g^v) \in E(T)} X_g(C_g^u, C_g^v).$$

At the same time, the length of C_g^T is $n \times |V(T)|$. ∎

In Figure 2(a), we have another layout of BF_3. The graph BF_3^G is shown in Figure 2(b). For example, $X_f(C_g^{(000)}, C_g^{(001)}) = \{(\langle 000, 0\rangle, \langle 001, 2\rangle), (\langle 000, 2\rangle, \langle 001, 0\rangle)\}$ and $X_g(C_g^{(000)}, C_g^{(001)}) = \{(\langle 000, 0\rangle, \langle 000, 2\rangle), (\langle 001, 0\rangle, \langle 001, 2\rangle)\}$. Let T be the tree indicated by bold lines in Figure 2(b). The corresponding C_g^T is indicated by bold lines in Figure 2(a).

2.3. *f*-Cycles and *f*-Edges

Let $u = \langle a_0a_1 \ldots a_{n-1}, k\rangle$ be any vertex of BF_n and let $\tilde{u}$ denote the vertex $\langle \bar{a}_0\bar{a}_1 \ldots \bar{a}_{n-1}, k\rangle$. One can see that $f^n(u) = \tilde{u}$ and $f^{2n}(u) = u$. Moreover, $\langle u \to f(u) \to f^2(u) \to \ldots \to f^{2n}(u)\rangle$ forms a simple cycle of length $2n$, denoted by C_f^u. So, all f-cycles form a partition of the cross edges of BF_n. Meanwhile, any g-edge joins vertices from two different f-cycles. Also, then, $(u, g(u))$ joins vertices from C_f^u and $C_f^{g(u)}$. Lemmas 5 and 6 were proved in [5].

Lemma 5 [5]. *$(f(u), g^{-1}(f(u)))$, $(\tilde{u}, g(\tilde{u}))$, $(f(\tilde{u}), g^{-1}(f(\tilde{u})))$ are g-edges joining vertices of C_f^u and $C_f^{g(u)}$. Moreover, the paths $\langle u \to f(u) \to g^{-1}(f(u)) \to g(u) \to u\rangle$ and $\langle \tilde{u} \to f(\tilde{u}) \to g^{-1}(f(\tilde{u})) \to g(\tilde{u}) \to \tilde{u}\rangle$ form two 4-cycles in BF_n.*

Any C_f^u contains exactly two vertices at each level. Suppose that u is one of the vertices in C_f^u at level i. Then, the other vertex in C_f^u at level i is $\tilde{u}$. Thus, C_f^u contains exactly one vertex at level 0, say $\langle a_0a_1 \ldots a_{n-1}, 0\rangle$, with $a_{n-1} = 0$. We use $C_f^{(a_0a_1\ldots a_{n-2}0)}$ as the name for C_f^u. Now, we form a new graph BF_n^F with all the f-cycles of BF_n as vertices, where two different f-cycles are joined with an edge if and only if there exists a g-edge joining them. The vertex of BF_n^F corresponding to C_f^u is denoted by $\bar{C}_f^u$. We recall the definition of the folded hypercube as follows: An n-dimensional folded hypercube is basically an n-cube augmented with 2^{n-1} complement edges. Each complement edge connects two vertices whose labels are complements to each other.

Lemma 6 [5]. *BF_n^F is isomorphic to the $(n-1)$-dimensional folded hypercube. Moreover, the set of vertices which are adjacent to $\bar{C}_f^{(a_0a_1\ldots a_{n-2}0)}$ is $\{\bar{C}_f^{(\bar{a}_0a_1\ldots a_{n-2}0)}, \bar{C}_f^{(a_0\bar{a}_1\ldots a_{n-2}0)}, \ldots, \bar{C}_f^{(a_0a_1\ldots\bar{a}_{n-2}0)}\} \cup \{\bar{C}_f^{(\bar{a}_0\bar{a}_1\ldots\bar{a}_{n-2}0)}\}$.*

Let $h = (\bar{C}_f^u, \bar{C}_f^v)$ be any edge of BF_n^F. We use $Y(h)$ to denote the set of edges of BF_n joining vertices from C_f^u and C_f^v. Using standard counting techniques, we have the following two corollaries:

Corollary 3 [3]. *If $(u, g(u)) \in Y(h)$, then $(f(u), g^{-1}(f(u)))$, $(\tilde{u}, g(\tilde{u}))$, and $(f(\tilde{u}), g^{-1}(f(\tilde{u})))$ are in $Y(h)$ and $|Y(h)| = 4$. Moreover, $\{u, f(u), g(u), g^{-1}(f(u))\}$ and $\{\tilde{u}, f(\tilde{u}), g(\tilde{u}), g^{-1}(f(\tilde{u}))\}$ induce two 4-cycles in BF_n.*

Corollary 4 [5]. *There is a unique cycle C such that edges of BF_n joining vertices between C_f^u and C are exactly $(u, g(u))$, $(f(u), g^{-1}(f(u)))$, $(\tilde{u}, g(\tilde{u}))$, and $(f(\tilde{u}), g^{-1}(f(\tilde{u})))$, and in that case, C is isomorphic to $C_f^{g(u)}$.*

According to Corollaries 3 and 4, any edge $h = (\bar{C}_f^u, \bar{C}_f^v)$ induces two 4-cycles in BF_n. Let α be an assignment of $(\bar{C}_f^u, \bar{C}_f^v) \in E(BF_n^F)$ such that $\alpha(h)$ is the subset of $Y(h)$ induced by the 4-cycles of BF_n. We use $Y_f^\alpha(C_f^u, C_f^v)$ to denote the set of f-edges induced by $\alpha(h)$ and $Y_g^\alpha(C_f^u, C_f^v)$ to denote the set of g-edges induced by $\alpha(h)$. Hence, $|Y_f^\alpha(C_f^u, C_f^v)| = |Y_g^\alpha(C_f^u, C_f^v)| = 2$.

Lemma 7. *Let T be any subtree of BF_n^F and let $C_f^{T,\alpha}$ be the subgraph of BF_n generated by*

$$\left(\bigcup_{\bar{C}_f^u \in V(T)} E(C_f^u) \cup \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_g^\alpha(C_f^u, C_f^v)\right) - \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_f^\alpha(C_f^u, C_f^v).$$

Then, $C_f^{T,\alpha}$ is a cycle of BF_n of length $2n \times |V(T)|$.

Proof. Let T be a subtree of BF_n^F and let $(\bar{C}_f^u, \bar{C}_f^v)$ be an edge of T. Hence, $(\bar{C}_f^u, \bar{C}_f^v)$ induces two 4-cycles in BF_n. Let α be an assignment of $(\bar{C}_f^u, \bar{C}_f^v) \in E(BF_n^F)$ such that $\alpha((\bar{C}_f^u, \bar{C}_f^v))$ induced a 4-cycles in BF_n. Consequently, two cycles C_f^u and C_f^v are joined by two g-edges in $Y_g^\alpha(C_f^u, C_f^v)$. It is easy to see that $E(C_f^u) \cup E(C_f^v) \cup Y_g^\alpha(C_f^u, C_f^v) - Y_f^\alpha(C_f^u, C_f^v)$ forms a cycle of length $4n$ in BF_n. Therefore, the subgraph $C_f^{T,\alpha}$ of BF_n generated by

$$\left(\bigcup_{\bar{C}_f^u \in V(T)} E(C_f^u) \cup \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_g^\alpha(C_f^u, C_f^v)\right) - \bigcup_{(\bar{C}_f^u, \bar{C}_f^v) \in E(T)} Y_f^\alpha(C_f^u, C_f^v)$$

is a cycle. Moreover, the length of $C_f^{T,\alpha}$ is $2n \times |V(T)|$. ∎

In Figure 3(a), we have another layout of BF_3. The graph BF_3^F is shown in Figure 3(b). For example, $Y_g^\alpha(C_f^{(000)}, C_f^{(100)}) = \{(\langle 000, 0\rangle, \langle 000, 1\rangle), (\langle 100, 0\rangle, \langle 100, 1\rangle)\}$ and $Y_f^\alpha(C_f^{(000)}, C_f^{(100)}) = \{(\langle 000, 0\rangle, \langle 100, 1\rangle), (\langle 100, 0\rangle, \langle 000, 1\rangle)\}$. Let T be the tree indicated by bold lines in Figure 3(b). The corresponding $C_f^{T,\alpha}$ is indicated by bold lines in Figure 3(a).

3. CYCLE EMBEDDING IN A FAULTY WRAPPED BUTTERFLY GRAPH

Let F denote a faulty set in a wrapped butterfly graph BF_n and assume that F has at most two elements. Then, we will determine the maximum length of a cycle in the graph $BF_n - F$. In [3], Hwang and Chen showed that the length of the cycle is $n2^n$ when F consists of two edges.

Lemma 8 [3]. *For any integer $n \geq 3$, $BF_n - F$ is Hamiltonian if F consists of two edges.*

On the other hand, Vadapalli and Srimani [6] proposed the following lemma when a wrapped butterfly graph does not have any edge fault but has at most two vertex faults:

Lemma 9 [6]. *For any integer $n \geq 3$, $BF_n - F$ contains a cycle of length $n2^n - 2$ if F consists of one vertex and a cycle of length $n2^n - 4$ if F consists of two vertices.*

In the remaining part of this section, we propose three lemmas: Lemma 10 proves that the cycle of length $n2^n - 2$

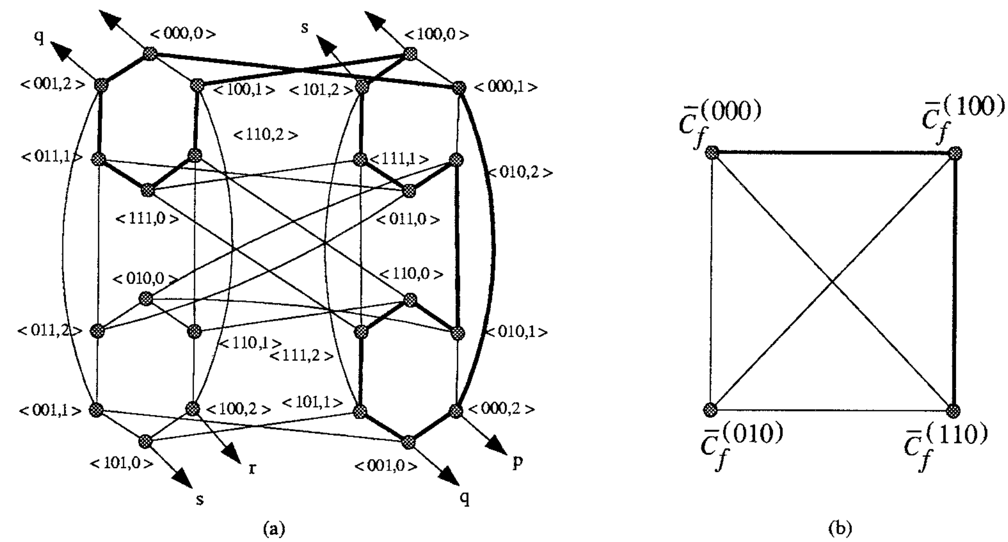

FIG. 3. (a) Another layout of BF_3; (b) the graph BF_3^F.

can be embedded in $BF_n - F$ for any $n \geq 3$ when F consists of one vertex and one edge. Lemma 11 shows that the maximum cycle length is $n2^n - 1$ if n is odd when F consists of one vertex and one edge. Lemma 12 verifies that the maximum cycle length is $n2^n - 2$ for odd $n \geq 3$ when F consists of two vertices. To prove these three lemmas, we use three fundamental cycles, denoted by $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3(j)$, respectively, to construct a larger cycle in a faulty wrapped butterfly graph.

The cycle $\mathcal{B}_1$ shown in Figure 4(a) is constructed as follows: Let $a_1 = \langle \underbrace{00...0}_{n}, 1\rangle$ and P_1 be the path $\langle a_1 \rightarrow g(a_1) \rightarrow g^2(a_1) \rightarrow \ldots \rightarrow g^{n-2}(a_1)\rangle$. Hence, $g^{n-2}(a_1) = a_2 = \langle \underbrace{00...0}_{n}, n-1\rangle$, $f(a_2) = \langle \underbrace{00...0}_{n-1}\ 1, 0\rangle = a_3$, and $f(a_3) = \langle 1\ \underbrace{00...0}_{n-2}\ 1, 1\rangle = a_4$. Let P_2 be the path $\langle a_4 \rightarrow g(a_4) \rightarrow g^2(a_4) \rightarrow \ldots \rightarrow g^{n-1}(a_4)\rangle$. Consequently, $g^{n-1}(a_4) = a_5 = \langle 1\ \underbrace{00...01}_{n-2}, 0\rangle$ and $f^{-1}(a_5) = \langle 1\ \underbrace{00...0}_{n-1}, n-1\rangle = a_6$. Let P_3 be the path $\langle a_6 \rightarrow g^{-1}(a_6) \rightarrow g^{-2}(a_6) \rightarrow \ldots \rightarrow g^{-(n-1)}(a_6)\rangle$. Thus, $g^{-(n-1)}(a_6) = a_7 = \langle 1\ \underbrace{00...0}_{n-1}, 0\rangle$ and $f(a_7) = a_1$. Let $\mathcal{B}_1$ be $\langle a_1 \rightarrow P_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4 \rightarrow P_2 \rightarrow a_5 \rightarrow a_6 \rightarrow P_3 \rightarrow a_7 \rightarrow a_1\rangle$. Obviously, P_1 is a path in $C_g^{a_1}$, P_2 is a path in $C_g^{a_4}$, and P_3 is a path in $C_g^{a_6}$. Since $V(C_g^u) \cap V(C_g^v) = \emptyset$ for $u \neq v \in \{a_1, a_4, a_6\}$ and $n \geq 3$, P_1, P_2, and P_3 are disjoint paths. Therefore, $\mathcal{B}_1$ is a cycle of length $3n$.

Similarly, we can construct the basic cycle $\mathcal{B}_2$ shown in Figure 4(b) as follows:

Let $\mathcal{B}_2 = \langle b_1 \rightarrow Q_1 \rightarrow b_2 \rightarrow b_3 \rightarrow Q_2 \rightarrow b_4 \rightarrow b_5 \rightarrow Q_3 \rightarrow b_6 \rightarrow b_7 \rightarrow Q_4 \rightarrow b_8 \rightarrow b_1\rangle$, where

$$b_1 = \langle \underbrace{00...0}_{n}, 1\rangle, \qquad b_2 = \langle \underbrace{00...0}_{n}, n-1\rangle,$$

$$b_3 = \langle \underbrace{00...0}_{n-2}, 10, n-2\rangle,$$

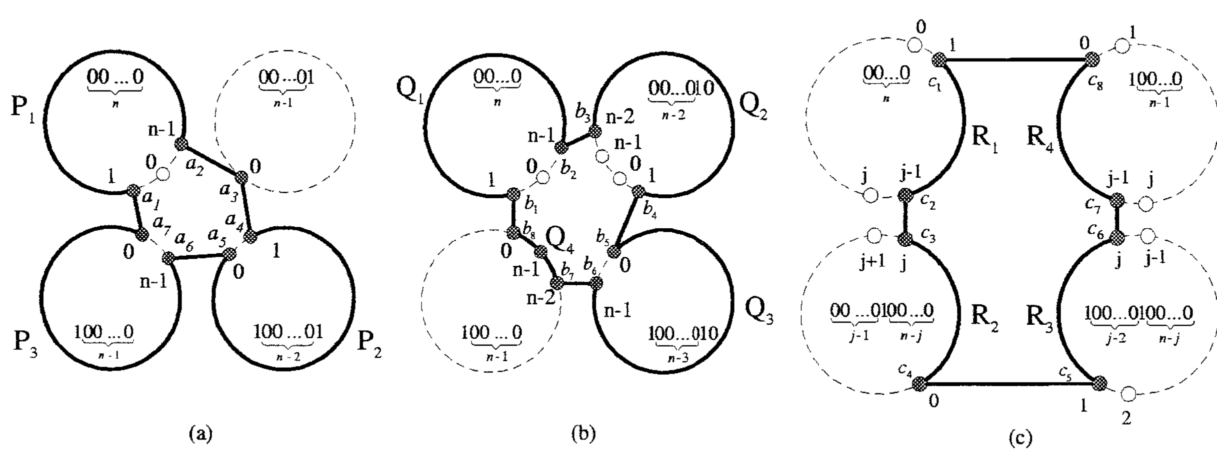

FIG. 4. (a) A cycle $\mathcal{B}_1$ of length $3n$ if $n \geq 3$; (b) a cycle $\mathcal{B}_2$ of length $3n$ if $n \geq 3$; (c) a cycle $\mathcal{B}_3(j)$ of length $2n + 4$ if $n \geq 3$.

$b_4 = \underbrace{1\ 00.\ .\ .0}_{n-2},\ 10,\ 1\rangle, \qquad b_5 = \langle\underbrace{1\ 00.\ .\ .0}_{n-3},\ 10,\ 0\rangle,$

$b_6 = \langle\underbrace{1\ 00.\ .\ .0}_{n-3},\ 10,\ n-1\rangle,$

$b_7 = \langle 1\ \underbrace{00.\ .\ .0}_{n-1},\ n-2\rangle, \qquad b_8 = \langle 1\ \underbrace{00.\ .\ .0}_{n-1}\ 0\rangle,$

$Q_1 = \langle b_1 \rightarrow g(b_1) \rightarrow g^2(b_1) \rightarrow \ldots \rightarrow g^{n-2}(b_1)\rangle,$

$Q_2 = \langle b_3 \rightarrow g^{-1}(b_3) \rightarrow g^{-2}(b_3) \rightarrow \ldots \rightarrow g^{-(n-3)}(b_3)\rangle,$

$Q_3 = \langle b_5 \rightarrow g(b_5) \rightarrow g^2(b_5) \rightarrow \ldots \rightarrow g^{n-1}(b_5)\rangle,$ and

$Q_4 = \langle b_7 \rightarrow g(b_7) \rightarrow g^2(b_7)\rangle.$

Q_1 is a path in $C_g^{b_1}$, Q_2 is a path in $C_g^{b_3}$, Q_3 is a path in $C_g^{b_5}$, and Q_4 is a path in $C_g^{b_7}$. Since $V(C_g^u) \cap V(C_g^v) = \emptyset$ for $u \neq v \in \{b_1, b_3, b_5, b_7\}$ and $n \geq 3$, Q_1, Q_2, Q_3, and Q_4 are disjoint paths. Consequently, $\mathcal{B}_2$ is a cycle of length $3n$.

Fix some j with $2 \leq j \leq n-1$. The cycle $\mathcal{B}_3(j)$ shown in Figure 4(c) is constructed as follows:

Let $\mathcal{B}_3(j) = \langle c_1 \rightarrow R_1 \rightarrow c_2 \rightarrow c_3 \rightarrow R_2 \rightarrow c_4 \rightarrow c_5 \rightarrow R_3 \rightarrow c_6 \rightarrow c_7 \rightarrow R_4 \rightarrow c_8 \rightarrow c_1\rangle$, where

$c_1 = \langle\underbrace{00.\ .\ .0}_{n},\ 1\rangle, \qquad c_2 = \langle\underbrace{00.\ .\ .0}_{n},\ j-1\rangle,$

$c_3 = \langle\underbrace{00.\ .\ .0}_{j-1}\ ,\ 1\ \underbrace{00.\ .\ .0}_{n-j},\ j\rangle,$

$c_4 = \langle\underbrace{00.\ .\ .0}_{j-1}\ 1\ \underbrace{00.\ .\ .0}_{n-j},\ 0\rangle,$

$c_5 = \langle 1\ \underbrace{00.\ .\ .0}_{j-2}\ 1\ \underbrace{00.\ .\ .0}_{n-j},\ 1\rangle,$

$c_6 = \langle 1\ \underbrace{00.\ .\ .0}_{j-2}\ 1\ \underbrace{00.\ .\ .0}_{n-j},\ j\rangle,$

$c_7 = \langle 1\ \underbrace{00.\ .\ .0}_{n-1},\ j-1\rangle, \qquad c_8 = \langle 1\ \underbrace{00.\ .\ .0}_{n-1},\ 0\rangle,$

$R_1 = \langle c_1 \rightarrow g(c_1) \rightarrow g^2(c_1) \rightarrow \ldots \rightarrow g^{j-2}(c_1)\rangle,$

$R_2 = \langle c_3 \rightarrow g^{-1}(c_3) \rightarrow g^{-2}(c_3) \rightarrow \ldots \rightarrow g^{-j}(c_3)\rangle,$

$R_3 = \langle c_5 \rightarrow g^{-1}(c_5) \rightarrow g^{-2}(c_5) \rightarrow \ldots \rightarrow g^{-(n-j+1)}(c_5)\rangle,$

and

$R_4 = \langle c_7 \rightarrow g^{-1}(c_7) \rightarrow g^{-2}(c_7) \rightarrow \ldots \rightarrow g^{-(n-j+1)}(c_7)\rangle.$

R_1 is a path in $C_g^{c_1}$, R_2 is a path in $C_g^{c_3}$, R_3 is a path in $C_g^{c_5}$, and C_4 is a path in $C_g^{c_7}$. Since $V(C_g^u) \cap V(C_g^v) = \emptyset$ for $u \neq v \in \{c_1, c_3, c_5, c_7\}$ and $n \geq 3$, R_1, R_2, R_3, and R_4 are disjoint paths. Consequently, $\mathcal{B}_3(j)$ is a cycle of length $2n + 4$. We even have $b_3 = b_4$ and $c_1 = c_2$ if and only if $n = 3$. We have $V(\mathcal{B}_l) \cap V(\mathcal{B}_k) \neq \emptyset$ for $1 \leq l \neq k \leq 3$ and $n \geq 4$ because $P_1 = Q_1$ and $P_1 \cap R_1 \neq \emptyset$.

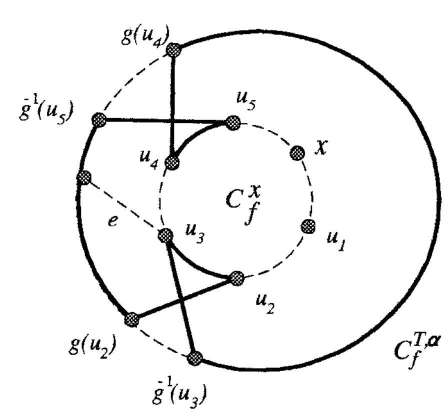

FIG. 5. An illustration for Case 1.1 with $n = 3$, $S_g = \{(u_2, g(u_2)), (u_4, g(u_4))\}$, $S'_g = \{(u_3, g^{-1}(u_3)), (u_5, g^{-1}(u_5))\}$, $S_f = \{(u_2, u_3), (u_4, u_5)\}$, and $S'_f = \{(g(u_2), g^{-1}(u_3)), (g(u_4), g^{-1}(u_5))\}$.

Lemma 10. *For any integer $n \geq 3$, $BF_n - F$ contains a cycle of length $n2^n - 2$ if F consists of one vertex and one edge.*

Proof. Since BF_n is vertex transitive, we assume that the faulty vertex is $x = \langle 00 \ldots 0, 0\rangle$ and the faulty edge is e.

CASE 1. e is a g-edge: Let $e = (u, g(u))$ for some $u \in V(BF_n)$. Since $n \geq 3$ and BF_n^F is isomorphic to an $(n-1)$-dimensional folded hypercube, $BF_n^F - \{\bar{C}_f^x, (\bar{C}_f^u, \bar{C}_f^{g(u)})\}$ is connected. Let T be any spanning tree of $BF_n^F - \{\bar{C}_f^x, (\bar{C}_f^u, \bar{C}_f^{g(u)})\}$. By Lemma 7, $C_f^{T,\alpha}$ for any α forms an $n2^n - 2n$ cycle. Let $S_g = \{(y, g(y)) | y = f^{2k}(x);\ 1 \leq k \leq n-1\}$, $S'_g = \{(f(y), g^{-1}(f(y))) | y = f^{2k}(x);\ 1 \leq k \leq n-1\}$, $S_f = \{(y, f(y)) | y = f^{2k}(x);\ 1 \leq k \leq n-1\}$, and $S'_f = \{(g(y), f^{-1}(g(y))) | y = f^{2k}(x);\ 1 \leq k \leq n-1\}$. Accordingly, $S_f \subset E(C_f^x)$ and $S'_f \subset E(C_f^{g(x)})$.

CASE 1.1. $e \notin S_g \cup S'_g$ (see Fig. 5): One can observe that $S_g \cup S'_g \cup S_f \cup S'_f$ forms $n-1$ disjoint 4-cycles in BF_n. Meanwhile, $S'_f \subset E(C_f^{T,\alpha})$ and $S_f \cap E(C_f^{T,\alpha}) = \emptyset$. Hence,

$$(E(C_f^{T,\alpha}) \cup S_g \cup S'_g \cup S_f) - S'_f$$

forms a cycle in $BF_n - F$ and the cycle length is $n2^n - 2$.

CASE 1.2. $e \in S_g \cup S'_g$: Constructing the cycle of length $n2^n - 2$ is similar to Case 1.1 except that $S_g = \{(y, g(y)) | y = f^{2k-1}(x);\ 1 \leq k \leq n-1\}$, $S'_g = \{(f(y), g^{-1}(f(y))) | y = f^{2k-1}(x);\ 1 \leq k \leq n-1\}$, $S_f = \{(y, f(y)) | y = f^{2k-1}(x);\ 1 \leq k \leq n-1\}$, and $S'_f = \{(g(y), f^{-1}(g(y))) | y = f^{2k-1}(x);\ 1 \leq k \leq n-1\}$.

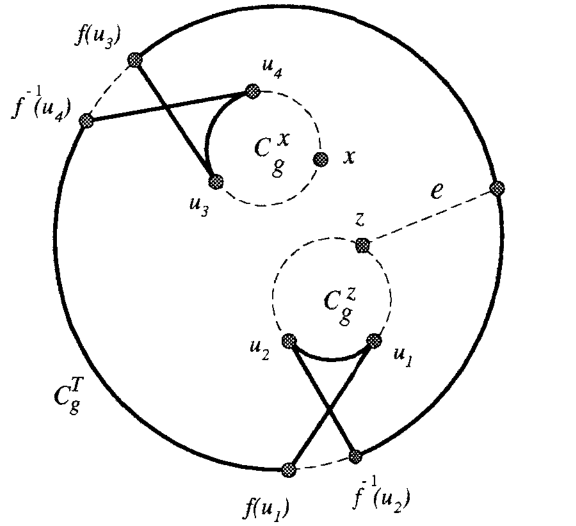

FIG. 6. An illustration for the Case 2.2.1 with $n = 3$, $S_1 = \{(u_3, f(u_3)), (u_4, f^{-1}(u_4))\}$ and $S_2 = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$.

CASE 2. e is an f-edge: Let $e = (u, f(u))$ for some $u \in V(BF_n)$.

CASE 2.1. n is an even integer: Since $n \geq 3$ and BF_n^G is isomorphic to the n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x, (\bar{C}_g^u, \bar{C}_g^{f(u)})\}$ is connected. Let T be any spanning tree of $BF_n^G - \{\bar{C}_g^x, (\bar{C}_g^u, \bar{C}_g^{f(u)})\}$. By Lemma 4, C_g^T forms a cycle spanning $V(BF_n) - V(C_g^x)$. Since T does not contain the edge $(\bar{C}_g^u, \bar{C}_g^{f(u)})$ and $e \in X_f(C_g^u, C_g^{f(u)})$, C_g^T does not contain the faulty edge e. Let $S = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k}(x); 1 \leq k < n/2\}$.

CASE 2.1.1. $e \notin S$: One can observe that $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S$ forms $(n/2) - 1$ disjoint 4-cycles in $BF_n - F$. Let $S_1 = \{(f(y), g^{-1}(f(y))) | y = g^{2k}(x); 1 \leq k < n/2\}$. Hence, $S_1 \subset E(C_g^T)$ and $S_1 \subset \cup_{(u,v)\in S} X_g(C_g^u, C_g^v)$. Therefore,

$$\bigcup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S \cup E(C_g^T) - S_1$$

forms a cycle of length $n2^n - 2$ in $BF_n - F$.

CASE 2.1.2. $e \in S$: The cycle can be constructed using the method of Case 2.1.1 except that $S = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(x); 1 \leq k < n/2\}$.

CASE 2.2. n is an odd integer: Let $S_1 = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2\}$.

CASE 2.2.1. $e \notin S_1$ (see Fig. 6): Since $\bar{C}_g^u \neq \bar{C}_g^{f(u)}$, we can choose $z \in \{u, f(u)\}$ according to the following rules:

(1) If either $\bar{C}_g^u = \bar{C}_g^x$ or $\bar{C}_g^{f(u)} = \bar{C}_g^x$, then we choose z such that $\bar{C}_g^z \neq \bar{C}_g^x$.
(2) Otherwise, we choose z such that $(\bar{C}_g^z, \bar{C}_g^x) \notin E(BF_n^G)$.

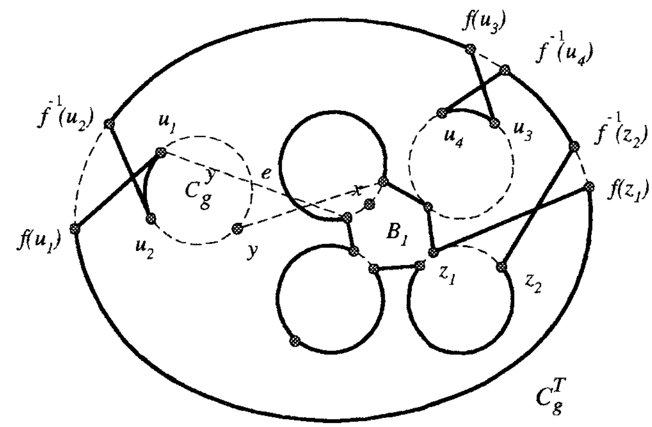

FIG. 7. An illustration for the Case 2.2.2 with $n = 3$, $S_3 = \{(u_3, f(u_3)), (u_4, f^{-1}(u_4))\}$, and $S_4 = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$.

Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^z\}$ is connected. Let T be any spanning tree of $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^z\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - V(C_g^x) - V(C_g^z)$ and it does not contain e. Let $S_2 = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(z); 1 \leq k \leq (n - 1)/2\}$. Hence, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2$ is fault free. We observe that $\cup_{(u,v)\in S_1} X_g(C_g^u, C_g^v) \cup S_1$ forms $(n - 1)/2$ disjoint 4-cycles and $\cup_{(u,v)\in S_2} X_g(C_g^u, C_g^v) \cup S_2$ forms $(n - 1)/2$ disjoint 4-cycles. Let $S_3 = \{(f(y), g^{-1}(f(y))) | y = g^{2k-1}(x); 1 \leq k \leq (n - 1)/2\}$ and $S_4 = \{(f(y), g^{-1}(f(y))) | y = g^{2k-1}(z); 1 \leq k \leq (n - 1)/2\}$. Hence, $S_3 \cap S_4 = \emptyset$ and $S_3 \cup S_4 \subset E(C_g^T)$. Meanwhile, $S_3 \cup S_4 \subset \cup_{(u,v)\in(S_1\cup S_2)} X_g(C_g^u, C_g^v)$. Therefore,

$$\bigcup_{(u,v)\in(S_1\cup S_2)} X_g(C_g^u, C_g^v) \cup S_1 \cup S_2 \cup E(C_g^T) - S_3 - S_4$$

forms a cycle in $BF_n - F$ and this cycle length is $n2^n - 2$.

CASE 2.2.2. $e \in S_1$ (see Fig. 7): Since $n \geq 3$, there exists $y \in V(BF_n)$ such that $f(y) \in V(C_g^x)$. So, both $(y, f(y))$ and $(g(y), f^{-1}(g(y)))$ join vertices of C_g^y and C_g^x. Let $W_1 = V(C_g^{a_1}) \cup V(C_g^{a_3}) \cup V(C_g^{a_5}) \cup V(C_g^{a_7})$ and $\bar{W}_1 = \{\bar{C}_g^{a_1}, \bar{C}_g^{a_3}, \bar{C}_g^{a_5}, \bar{C}_g^{a_7}\}$. Hence, $\bar{C}_g^y$ is not adjacent to any vertex in $\bar{W}_1 - \{\bar{C}_g^x\}$. Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1 - \{\bar{C}_g^y\}$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1 - \{\bar{C}_g^y\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - W_1 - V(C_g^y)$. There exists $w \in V(\mathcal{B}_1)$ such that $X_f(C_g^w, C_g^{f(w)})$ is fault free and $(w, f(w))$ joins some vertex in both $\mathcal{B}_1$ and C_g^T. Then,

$$C_e = (E(\mathcal{B}_1) \cup E(C_g^T) \cup X_f(C_g^w, C_g^{f(w)})) - X_g(C_g^w, C_g^{f(w)})$$

forms a cycle of length $n2^n - 2n$ spanning $(V(BF_n) - W_1 - V(C_g^y)) \cup V(\mathcal{B}_1)$.

Let $S_3 = \{X_f(C_g^s, C_g^{f(s)}) | s = g^{2k-1}(a_3); 1 \leq k \leq (n - 1)/2\}$ and $S_4 = \{X_f(C_g^t, C_g^{f(t)}) | t = g^{2k-1}(y); 1 \leq k$

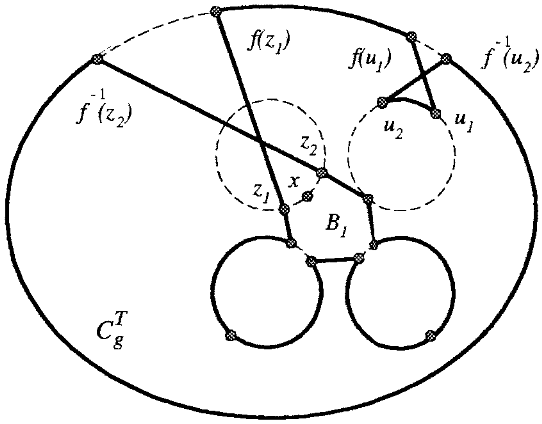

FIG. 8. An illustration for the Case 1.1 with $n = 3$ and $S = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$.

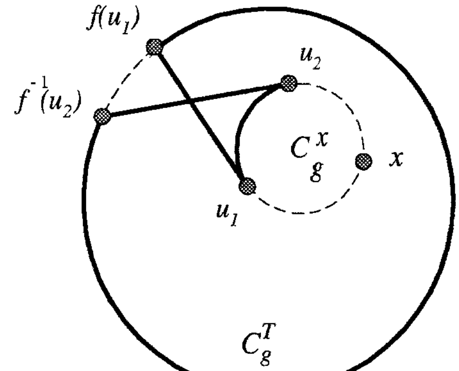

FIG. 9. An illustration for the Case 1.2 with $n = 3$ and $S = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$.

$\leq (n - 1)/2\}$. Hence, $S_3 \cap S_4 = \emptyset$ and $S_3 \cup S_4$ is fault free. We observe that $\cup_{(u,v)\in(S_3\cup S_4)} X_g(C_g^u, C_g^v)) \cup S_3 \cup S_4$ forms $n - 1$ disjoint 4-cycles. Let $S_5 = \{(f(s), g^{-1}(f(s)))|s = g^{2k-1}(a_3);\ 1 \leq k \leq (n - 1)/2\}$ and $S_6 = \{(f(t), g^{-1}(f(t)))|t = g^{2k-1}(y);\ 1 \leq k \leq (n - 1)/2\}$. Hence, $S_5 \cap S_6 = \emptyset$ and $S_5 \cup S_6$ is a subset of both $E(C_e)$ and $\cup_{(u,v)\in(S_3\cup S_4)} X_g(C_g^u, C_g^v))$. Therefore,

$$\bigcup_{(u,v)\in(S_3\cup S_4)} X_g(C_g^u, C_g^v) \cup S_3 \cup S_4 \cup E(C_e) - S_5 - S_6$$

forms a cycle of length $n2^n - 2$ in $BF_n - F$. ∎

Lemma 11. *For any odd integer $n \geq 3$, $BF_n - F$ is Hamiltonian if F consists of one vertex and one edge.*

Proof. Since BF_n is vertex transitive, we may assume that the faulty vertex is $x = \langle 00\ldots 0, 0\rangle$ and the faulty edge is e. Let $W_1 = V(C_g^{a_1}) \cup V(C_g^{a_3}) \cup V(C_g^{a_5}) \cup V(C_g^{a_7})$ and $\bar{W}_1 = \{\bar{C}_g^{a_1}, \bar{C}_g^{a_3}, \bar{C}_g^{a_5}, \bar{C}_g^{a_7}\}$.

CASE 1. e is an f-edge: Let $e = (u, f(u))$ for some $u \in V(BF_n)$.

CASE 1.1. $\{u, f(u)\} \cap (V(C_g^{a_3}) \cup V(C_g^{a_7})) = \emptyset$ (see Fig. 8): Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1 - \{(\bar{C}_g^u, \bar{C}_g^{f(u)})\}$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1 - \{(\bar{C}_g^u, \bar{C}_g^{f(u)})\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - W_1$ and it does not contain e.
Let $S = \{X_f(C_g^y, C_g^{f(y)})|y = g^{2k-1}(a_3);\ 1 \leq k \leq (n - 1)/2\}$. Hence, $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S$ forms $(n - 1)/2$ disjoint 4-cycles. Let $S_1 = \{(f(y), g^{-1}(f(y)))|y = g^{2k-1}(a_3);\ 1 \leq k \leq (n - 1)/2\}$. Consequently, S_1 is a subset of both $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v)$ and $E(C_g^T)$. Therefore,

$$C_e = \bigcup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S \cup E(C_g^T) - S_1$$

forms a cycle of length $n2^n - 3n - 1$ spanning $V(BF_n) - V(\mathcal{B}_1) - \{x\}$ and it does not contain e. Since the length of $\mathcal{B}_1$ is $3n$, there exists $z \in V(\mathcal{B}_1)$ such that $X_f(C_g^z, C_g^{f(z)})$ is fault free and $(z, f(z))$ joins some vertex in both C_e and $\mathcal{B}_1$. Therefore,

$$(E(C_e) \cup E(\mathcal{B}_1) \cup X_f(C_g^z, C_g^{f(z)})) - X_g(C_g^z, C_g^{f(z)})$$

forms a Hamiltonian cycle of $BF_n - F$.

CASE 1.2. $\{u, f(u)\} \cap (V(C_g^{a_3}) \cup V(C_g^{a_7})) \neq \emptyset$ (see Fig. 9): Let $S = \{X_f(C_g^y, C_g^{f(y)})|y = g^{2k-1}(x);\ 1 \leq k \leq (n - 1)/2\}$. Hence, $e \notin S$. Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x\}$ is connected. Let T be any spanning tree of $BF_n^G - \{\bar{C}_g^x\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - V(C_g^x)$. Since S is fault free, $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S$ forms $(n - 1)/2$ disjoint 4-cycles. Let $S_1 = \{(f(y), g^{-1}(f(y)))|y = g^{2k-1}(x);\ 1 \leq k \leq (n - 1)/2\}$. Hence, S_1 is a subset of both $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v)$ and $E(C_g^T)$. Therefore,

$$\bigcup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S \cup E(C_g^T) - S_1$$

forms a Hamiltonian cycle of $BF_n - F$.

CASE 2. e is a g-edge: Let $e = (u, g(u))$ for some $u \in V(BF_n)$.

CASE 2.1. $\bar{C}_g^u = \bar{C}_g^x$: Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1$. Constructing a Hamiltonian cycle for this case is very similar to Case 1.1 except that the chosen vertex z is vertex u when $e \in E(\mathcal{B}_1)$.

CASE 2.2. $\bar{C}_g^u \neq \bar{C}_g^x$ and $\bar{C}_g^u$ is not connected with $\bar{C}_g^x$ in BF_n^G: Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x\}$ is connected. Let T be a spanning tree of $BF_n^G - \{\bar{C}_g^x\}$ such that $(\bar{C}_g^u, \bar{C}_g^{f(u)})$

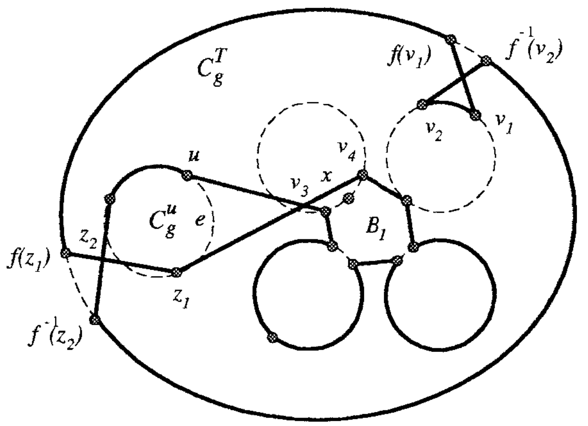

FIG. 10. An illustration for the Case 2.3.1 with $n = 3$, $S = \{(v_1, f(v_1)), (v_2, f^{-1}(v_2))\}$, $X_f(C_g^u, C_g^{f(u)}) = \{(u, v_3), (z_1, v_4)\}$, $X_f(C_g^z, C_g^{f(z)}) = \{(z_1, f(z_1)), (z_2, f^{-1}(z_2))\}$, $X_g(C_g^u, C_g^{f(u)}) = \{(u, z_1), (v_3, v_4)\}$, and $X_g(C_g^z, C_g^{f(z)}) = \{(z_1, z_2), (f(z_1), f^{-1}(z_2))\}$.

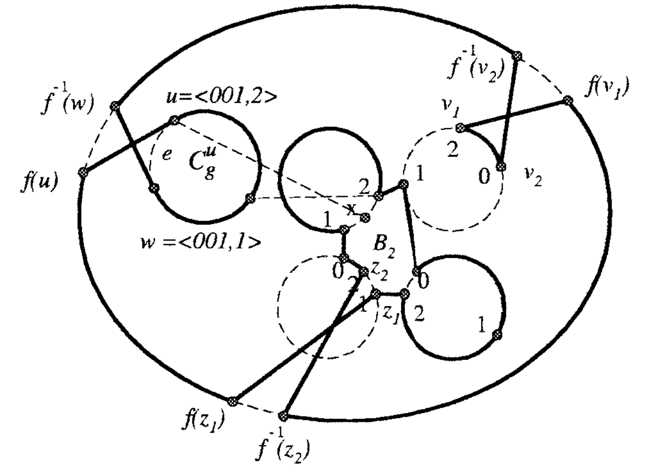

FIG. 11. An illustration for the Case 2.3.2 with $n = 3$, $e = (\langle 001, 2\rangle, \langle 001, 1\rangle)$, and $S_2 = \emptyset$.

$\in E(T)$. Then, the Hamiltonian cycle of $BF_n - F$ can be constructed by the same method used in Case 1.2.

Case 2.3. $(\bar{C}_g^u, \bar{C}_g^x) \in E(BF_n^G)$ (i.e., $\bar{C}_g^u$ and $\bar{C}_g^x$ are connected):

Case 2.3.1. $\bar{C}_g^u \neq \bar{C}_g^{a_3}$ and $\bar{C}_g^u \neq \bar{C}_g^{a_7}$ (see Fig. 10): Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1 - \{\bar{C}_g^u\}$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1 - \{\bar{C}_g^u\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - W_1 - V(C_g^u)$. Let $S = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(a_3);\ 1 \leq k \leq (n-1)/2\}$. Hence, $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S$ forms $(n-1)/2$ disjoint 4-cycles. Let $S_1 = \{(f(y), g^{-1}(f(y))) | y = g^{2k-1}(a_3);\ 1 \leq k \leq (n-1)/2\}$. Accordingly, S_1 is a subset of both $\cup_{(u,v)\in S} X_g(C_g^u, C_g^v)$ and $E(C_g^T)$. Therefore,

$$C_e = \bigcup_{(u,v)\in S} X_g(C_g^u, C_g^v) \cup S \cup E(C_g^T) - S_1$$

forms a cycle spanning $V(BF_n) - V(\mathcal{B}_1) - V(C_g^u) - \{x\}$. Since the length of $\mathcal{B}_1$ is $3n$, we can choose a vertex $z \in V(C_g^u)$ such that $f(z) \in V(C_e)$ if $f(u) \in V(\mathcal{B}_1)$ or $f(z) \in V(\mathcal{B}_1)$ if $f(u) \notin V(\mathcal{B}_1)$. Therefore,

$$(E(C_e) \cup E(\mathcal{B}_1) \cup E(C_g^u) \cup X_f(C_g^u, C_g^{f(u)}) \cup X_f(C_g^z, C_g^{f(z)})) - X_g(C_g^u, C_g^{f(u)}) - X_g(C_g^z, C_g^{f(z)})$$

forms a Hamiltonian cycle of $BF_n - F$.

Case 2.3.2. $\bar{C}_g^u = \bar{C}_g^{a_3}$ (see Fig. 11): Let $S = \{X_g(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(a_3);\ 1 \leq k \leq (n-1)/2\}$.

Assume that $e \notin S$. Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1$. Then, the Hamiltonian cycle of BF_n − F can be constructed by the same method used in Case 1.1.

Assume that $e \in S$. Let $W_2 = V(C_g^{b_1}) \cup V(C_g^{b_3}) \cup V(C_g^{b_5}) \cup V(C_g^{b_7})$ and $\bar{W}_2 = \{\bar{C}_g^{b_1}, \bar{C}_g^{b_3}, \bar{C}_g^{b_5}, \bar{C}_g^{b_7}\}$. Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_2$ is connected. Let T be a spanning tree of $BF_n^G - \bar{W}_2$ such that $(\bar{C}_g^u, \bar{C}_g^{f(u)}) \in E(T)$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - W_2$. Let $S_2 = \{X_f(C_g^y, C_g^{f(y)}) | y = g^{2k-1}(b_8);\ 1 \leq k \leq (n-3)/2\}$. Then,

$$C_e = \left(E(C_g^T) \cup S_2 \cup \bigcup_{(u,v)\in S_2} X_g(C_g^u, C_g^v) \cup X_f(C_g^{g(b_3)}, C_g^{f(g(b_3))}) \cup X_g(C_g^{g(b_3)}, C_g^{f(g(b_3))}) \right)$$
$$-\{(f(y), g^{-1}(f(y))) | y = g^{2k-1}(b_8);\ 1 \leq k \leq (n-3)/2\} - \{(f(g(b_3)), f^{-1}(g^2(b_3)))\}$$

forms a cycle spanning $V(BF_n) - V(\mathcal{B}_2) - \{x\}$ and it does not contain e. Since the length of $\mathcal{B}_2$ is $3n$, there exists $w \in V(\mathcal{B}_2)$ such that $(w, (f(w))$ joins some vertex in both C_e and $\mathcal{B}_2$. Obviously, $X_f(C_g^w, C_g^{f(w)})$ is fault free. Then,

$$(E(C_e) \cup E(\mathcal{B}_2) \cup X_f(C_g^w, C_g^{f(w)})) - X_g(C_g^w, C_g^{f(w)})$$

forms a Hamiltonian cycle of $BF_n - F$.

Case 2.3.3. $\bar{C}_g^u = \bar{C}_g^{a_7}$.

Suppose that $e \in X_g(C_g^x, C_g^{a_7})$. Hence, $e = (\langle 10 \ldots 00, 0\rangle, \langle 100 \ldots 0, 1\rangle)$. We observe that $e \in E(\mathcal{B}_1)$ and $e \notin E(\mathcal{B}_2)$. Using the same method used in the situation $e \in S$ of Case 2.3.2, the Hamiltonian cycle of $BF_n - F$ can be constructed.

Suppose that $e \notin X_g(C_g^x, C_g^{a_7})$. Since $n \geq 3$ and BF_n^G is

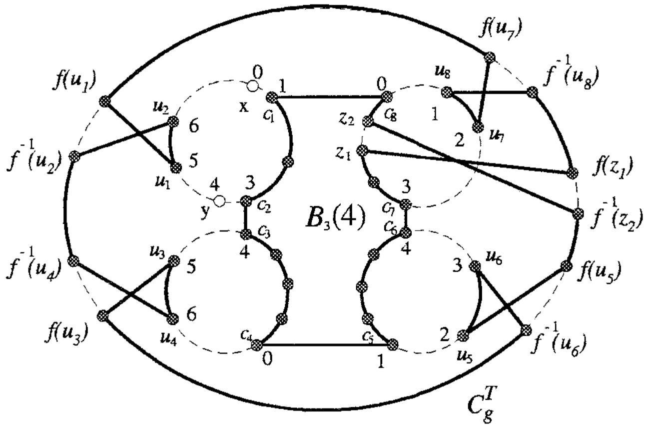

FIG. 12. An illustration for the Case 1 with $j = 4$, $n = 7$, $S_1 = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$, $S_2 = \{(u_3, f(u_3)), (u_4, f^{-1}(u_4))\}$, $S_3 = \{(u_5, f(u_5)), (u_6, f^{-1}(u_6))\}$, and $S_4 = \{(u_7, f(u_7)), (u_8, f^{-1}(u_8))\}$.

isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_1$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_1$ and $e = (u, g(u))$ for some $u \in V(BF_n)$. Constructing a Hamiltonian cycle for this case is very similar to Case 1.1 except that the chosen vertex z is vertex u when $e \in E(\mathcal{B}_1)$. ∎

Lemma 12. *For any odd integer $n \geq 3$, $BF_n - F$ is Hamiltonian if F consists of two vertices.*

Proof. Since BF_n is vertex transitive, we may assume that one faulty vertex is $x = \langle 00\ldots0, 0\rangle$ and the other is y. Let $\bar{W}_3 = \{\bar{C}_g^{c_1}, \bar{C}_g^{c_3}, \bar{C}_g^{c_5}, \bar{C}_g^{c_7}\}$.

CASE 1. $\bar{C}_g^x = \bar{C}_g^y$ (see Fig. 12): Let $y = \langle \underbrace{00\ldots0}_{n}, j\rangle$ for some $1 \leq j \leq n-1$. Since n is an odd integer, without loss of generality, we may assume that j is an even integer. Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \bar{W}_3$ is connected. Let T be any spanning tree of $BF_n^G - \bar{W}_3$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - W_3$. Let

$$S_1 = \left\{ X_f(C_g^y, C_g^{f(y)}) \,\middle|\, y = g^{2k}(c_2);\ 1 \leq k \leq \frac{n-j-1}{2} \right\},$$

$$S_2 = \left\{ X_f(C_g^y, C_g^{f(y)}) \,\middle|\, y = g^{2k-1}(c_3);\ 1 \leq k \leq \frac{n-j-1}{2} \right\},$$

$$S_3 = \left\{ X_f(C_g^y, C_g^{f(y)}) \,\middle|\, y = g^{2k-1}(c_5);\ 1 \leq k \leq \frac{j}{2} - 1 \right\}, \text{ and}$$

$$S_4 = \left\{ X_f(C_g^y, C_g^{f(y)}) \,\middle|\, y = g^{2k-1}(c_8);\ 1 \leq k \leq \frac{j}{2} - 1 \right\}.$$

One can observe that $S_i \cap S_j = \emptyset$ for $1 \leq i \neq j \leq 4$ and $\bigcup_{(u,v)\in S_1\cup S_2\cup S_3\cup S_4} X_g(C_g^u, C_g^v) \cup S_1 \cup S_2 \cup S_3 \cup S_4$ forms $n-3$ disjoint 4-cycles. Let

$$S_5 = \left\{ (f(y), g^{-1}(f(y))) \,\middle|\, y = g^{2k}(c_2);\ 1 \leq k \leq \frac{n-j-1}{2} \right\},$$

$$S_6 = \left\{ (f(y), g^{-1}(f(y))) \,\middle|\, y = g^{2k-1}(c_3);\ 1 \leq k \leq \frac{n-j-1}{2} \right\},$$

$$S_7 = \left\{ (f(y), g^{-1}(f(y))) \,\middle|\, y = g^{2k-1}(c_5);\ 1 \leq k \leq \frac{j}{2} - 1 \right\}, \text{ and}$$

$$S_8 = \left\{ (f(y), g^{-1}(f(y))) \,\middle|\, y = g^{2k-1}(c_8);\ 1 \leq k \leq \frac{j}{2} - 1 \right\}.$$

Consequently, $S_5 \cap S_6 \cap S_7 \cap S_8 = \emptyset$, and $S_5 \cup S_6 \cup S_7 \cup S_8$ is a subset of both $E(C_g^T)$ and $\bigcup_{(u,v)\in S_1\cup S_2\cup S_3\cup S_4} X_g(C_g^u, C_g^v)$. Therefore,

$$C_e = \bigcup_{(u,v)\in S_1\cup S_2\cup S_3\cup S_4} X_g(C_g^u, C_g^v) \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup E(C_g^T) - S_5 - S_6 - S_7 - S_8$$

forms a cycle of length $n2^n - 2n - 6$ spanning $V(BF_n) - V(\mathcal{B}_3(j)) - \{x, y\}$. Since the length of $\mathcal{B}_3(j)$ is $2n + 4$, there exists $z \in V(\mathcal{B}_3(j))$ such that $X_f(C_g^z, C_g^{f(z)})$ is fault free and $(z, f(z))$ joins some vertex in both C_e and $\mathcal{B}_3(j)$. Then,

$$(E(C_e) \cup E(\mathcal{B}_3(j)) \cup X_f(C_g^z, C_g^{f(z)})) - X_g(C_g^z, C_g^{f(z)})$$

forms a Hamiltonian cycle of $BF_n - F$.

CASE 2. $\bar{C}_g^x \neq \bar{C}_g^y$ and $\bar{C}_g^x$ is not connected with $\bar{C}_g^y$ in BF_n^G (See Fig. 13): Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^y\}$ is connected. Let T be any spanning tree of $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^y\}$. By Lemma 4, C_g^T is a cycle spanning $V(BF_n) - V(C_g^x) - V(C_g^y)$. Let $S_1 = \{X_f(C_g^s, C_g^{f(s)}) | s = g^{2k-1}(x);\ 1 \leq k \leq (n-1)/2\}$ and $S_2 = \{X_f(C_g^t, C_g^{f(t)}) | t = g^{2k-1}(y);\ 1 \leq k \leq (n-1)/2\}$. Since $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2$ is fault free, $\bigcup_{(u,v)\in S_1\cup S_2} X_g(C_g^u, C_g^v) \cup S_1 \cup S_2$ forms $n-1$ disjoint 4-cycles. Let $S_3 = \{(f(s), g^{-1}(f(s))) | s = g^{2k-1}(x);\ 1 \leq k \leq (n-1)/2\}$ and $S_4 = \{(f(t), g^{-1}(f(t))) | t = g^{2k-1}(y);\ 1 \leq k \leq (n-1)/2\}$. Hence, $S_3 \cap S_4 = \emptyset$ and $S_3 \cup S_4 \subset E(C_g^T)$. At the same time, $S_3 \cup S_4 \subset \bigcup_{(u,v)\in(S_1\cup S_2)} X_g(C_g^u, C_g^v)$. Therefore,

$$\bigcup_{(u,v)\in(S_1\cup S_2)} X_g(C_g^u, C_g^v) \cup S_1 \cup S_2 \cup E(C_g^T) - S_3 - S_4$$

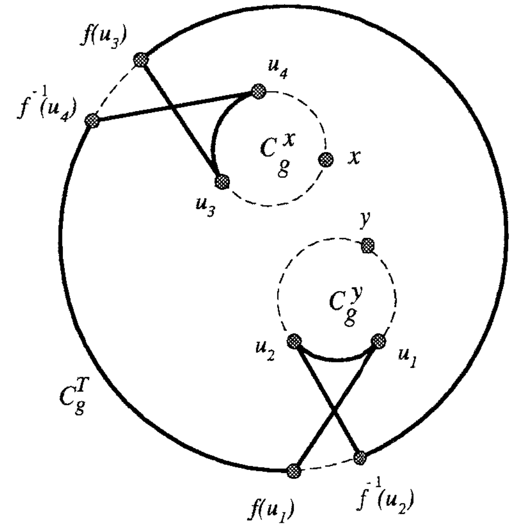

FIG. 13. An illustration for the Case 2 with $n = 3$, $S_1 = \{(u_3, f(u_3)), (u_4, f^{-1}(u_4))\}$, and $S_2 = \{(u_1, f(u_1)), (u_2, f^{-1}(u_2))\}$.

forms a Hamiltonian cycle of $BF_n - F$.

CASE 3. $(\bar{C}_g^x, \bar{C}_g^y) \in E(BF_n^G)$ (i.e., $\bar{C}_g^x$ and $\bar{C}_g^y$ are connected):

CASE 3.1. $\bar{C}_g^y \neq \bar{C}_g^{a_3}$ and $\bar{C}_g^y \neq \bar{C}_g^{a_7}$ (see Fig. 7): Since BF_n^G is an n-dimensional hypercube with $n \geq 3$, $C_g^{a_3}$, $C_g^{a_5}$, and $C_g^{a_7}$ are fault free. As a result, $\bar{C}_g^{a_3}$ is not adjacent to $\bar{C}_g^y$ in BF_n^G. Since $BF_n^G - \bar{W}_1 - \{\bar{C}_g^y\}$ is connected, let T be any spanning tree of $BF_n^G - \bar{W}_1 - \{\bar{C}_g^y\}$. The Hamiltonian cycle of the graph $BF_n - F$ can be constructed by the same method used in Case 2.2.2 of Lemma 10.

CASE 3.2. $\bar{C}_g^y = \bar{C}_g^{a_3}$ or $\bar{C}_g^y = \bar{C}_g^{a_7}$: Let $S_1 = \{\langle \underbrace{00\ldots0}_{n}\ 1, 2k - 1\rangle | 1 \leq k \leq (n-1)/2\}$ *and* $S_2 = \{\langle 1\ \underbrace{00\ldots0}_{n-1}, 2k\rangle | 1 \leq k \leq (n-1)/2\}$. Therefore, there does not exist any edge in $X_f(C_g^u, C_g^{f(u)})$ which joins vertices of C_g^x and C_g^y for all $u \in S_1 \cup S_2$.

CASE 3.2.1. $y \notin S_1 \cup S_2$ (See Fig. 13): Since $n \geq 3$ and BF_n^G is isomorphic to an n-dimensional hypercube, $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^y\}$ is connected. Let T be any spanning tree of $BF_n^G - \{\bar{C}_g^x, \bar{C}_g^y\}$. With the same method used in Case 2, the Hamiltonian cycle of the graph $BF_n - F$ can be constructed.

CASE 3.2.2. $y \in S_1 \cup S_2$:

Given an integer k with $0 \leq k < n$, the mapping σ_k from $V(BF_n)$ into $V(BF_n)$ can be defined by $\sigma_k(\langle a_0 a_1 \ldots a_{n-1}, l\rangle) = \langle a_k a_{k+1} \ldots a_{n-1} a_0 a_1 \ldots a_{k-1}, (l - k) \mathbf{mod}\ n\rangle$. Similarly, we can define the mapping φ_i from $V(BF_n)$ into $V(BF_n)$ as $\varphi_i(\langle a_0 a_1 \ldots, a_i a_{i+1}, \ldots, a_{n-1}, l\rangle) = \langle a_0 a_1 \ldots \bar{a}_i a_{i+1} \ldots a_{n-1}, l\rangle$, where $0 \leq i < n$. Hence, σ_k and φ_i are two automorphisms of BF_n.

Suppose that $y \in S_1$. Then, $y = \langle \underbrace{00\ldots0}_{n-1}\ 1, l\rangle$ for some odd l, $1 \leq l \leq n - 2$. Accordingly, $\varphi_{n-1-l} \circ \sigma_l$ is an automorphism of BF_n such that $\varphi_{n-1-l} \circ \sigma_l(y) = x$ and $\phi_{n-1-l} \circ \sigma_l(x) = \underbrace{00\ldots0}_{n-1-L}\ 1\ \underbrace{00\ldots0}_{l}, n - l\rangle = z$. Consequently, $z \notin S_1 \cup S_2$. Therefore, we can construct a Hamiltonian cycle C in $BF_n - \{x, z\}$ by using the same method as in Case 3.2.1. Finally, $(\varphi_{n-1-l} \circ \sigma_l)^{-1}(C)$ also forms a Hamiltonian cycle of $BF_n - \{x, y\}$.

Suppose that $y \in S_2$. Then, $y = \langle 1\ \underbrace{00\ldots0}_{n-1}, l\rangle$ for some even l, $2 \leq l \leq n - 1$. Hence, $\varphi_{n-1} \circ \sigma_l$ is an automorphism of BF_n such that $\varphi_{n-1} \circ \sigma_l(y) = x$ and $\varphi_{n-l} \circ \sigma_l(x) = \langle \underbrace{00\ldots0}_{n-1}\ 1\rangle\ 1\ \underbrace{00\ldots0}_{l-1}, n - l\rangle = z$. Consequently, $z \notin S_1 \cup S_2$. With the same method used in Case 3.2.1, we can construct a Hamiltonian cycle C in $BF_n - \{x, z\}$. Hence, $(\varphi_{n-l} \circ \sigma_l)^{-1}(C)$ also forms a Hamiltonian cycle of $BF_n - \{x, y\}$. ■

Combining Lemmas 8–12, we have the following theorem:

Theorem 1. *For any integer n with $n \geq 3$, let $F \subset V(BF_n) \cup E(BF_n)$, $f_v = |F \cap V(BF_n)|$, and $|F| \leq 2$. $BF_n - F$ contains a cycle of length $n \times 2^n - 2f_v$. In addition, $BF_n - F$ contains a Hamiltonian cycle if n is an odd integer.*

4. CONCLUSIONS

It is known that wrapped butterfly graphs are very suitable for VLSI implementation because they are regular and have a constant degree. In this paper, the properties of wrapped butterfly graphs are described and investigated. Unlike the previous studies [3, 6] which considered faults on either vertices or edges, this paper showed that there exists a cycle of length $n2^n - 2$ in the faulty wrapped butterfly graph when there is one vertex and one edge fault. When BF_n is not a bipartite graph, we show that it contains a Hamiltonian cycle if it has at most two faults containing at least one vertex fault. These results are optimal because wrapped butterfly graphs are bipartite if and only if n is even. We say two vertices u and v have the same color, black or white, if and only if u and v are in the same partite set. When a wrapped butterfly graph BF_n is a bipartite graph, we have two conjectures:

1. $BF_n - F$ contains a Hamiltonian cycle if F consists of two vertices with different colors.
2. $BF_n - F$ is Hamiltonian if F consists of two black colored vertices and two white colored vertices.

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