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On the log-Sobolev constant for the simple random walk on the n-cycle: the even cases

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Abstract

Consider the simple random walk on the n-cycle $\mathbb{Z}_n$. For this example, Diaconis and Saloff-Coste (Ann. Appl. Probab. 6 (1996) 695) have shown that the log-Sobolev constant α is of the same order as the spectral gap λ. However the exact value of α is not known for $n>4$. (For $n=2$, it is a well known result of Gross (Amer. J. Math. 97 (1975) 1061) that α is $\frac{1}{2}$. For $n=3$, Diaconis and Saloff-Coste (Ann. Appl. Probab. 6 (1996) 695) showed that $\alpha = \frac{1}{2\log 2} < \frac{\lambda}{2} = 0.75$. For $n=4$, the fact that $\alpha = \frac{1}{2}$ follows from $n=2$ by tensorization.) Based on an idea that goes back to Rothaus (J. Funct. Anal. 39 (1980) 42; 42 (1981) 110), we prove that if $n \geqslant 4$ is even, then the log-Sobolev constant and the spectral gap satisfy $\alpha = \frac{\lambda}{2}$. This implies that $\alpha = \frac{1}{2}(1 - \cos\frac{2\pi}{n})$ when n is even and $n \geqslant 4$.

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1. Introduction

Consider a finite state space $\mathscr{S}$ equipped with an irreducible Markov kernel $K(x, y)$, which is reversible with respect to a probability measure π on $\mathscr{S}$ (i.e., $\pi(x)K(x, y) = \pi(y)K(y, x)$ for all $x, y \in \mathscr{S}$). Define an inner product on complex-valued functions on $\mathscr{S}$ by $\langle f, g \rangle = \sum_{s \in \mathscr{S}} f(s)\overline{g(s)}\pi(s)$. The Dirichlet form associated with (K, π) is then given by the formula

$$\mathscr{E}(f, g) = \mathscr{R}\langle (I - K)f, g \rangle,$$

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where I is the identity matrix, f and g are two complex-valued functions, and $\mathcal{R}z$ is the real part of a complex number z. Set

$$E_\pi f = \sum_{s \in \mathcal{S}} f(s)\pi(s)$$

and

$$Var_\pi(f) = ||f - E_\pi f||_2^2.$$

Here $||\cdot||_2$ is the usual l^2-norm with respect to the measure π. The spectral gap λ of (K, π) is defined by

$$\lambda = \min\left\{\frac{\mathcal{E}(f,f)}{Var_\pi(f)}; Var_\pi(f) \neq 0\right\}. \tag{1.1}$$

Since (K, π) is reversible, it is easy to show that the spectral gap λ is the smallest non-zero eigenvalue of $I - K$.

For every function f on $\mathcal{S}$, consider the entropy-like quantity

$$\mathcal{L}(f) = \sum_{s \in \mathcal{S}} |f(s)|^2 \left(\log \frac{|f(s)|^2}{||f||_2^2}\right)\pi(s). \tag{1.2}$$

(Clearly we have $\mathcal{L}(f) \geqslant 0$ and $\mathcal{L}(f) = 0$ only if f is a constant function.) A log-Sobolev inequality is an inequality of the type

$$\mathcal{L}(f) \leqslant C\mathcal{E}(f,f) \tag{1.3}$$

holding for all functions f. We say that α is the log-Sobolev constant of K if $\frac{1}{\alpha}$ is the smallest constant C such that inequality (1.3) holds. In other words,

$$\alpha = \inf\left\{\frac{\mathcal{E}(f,f)}{\mathcal{L}(f)}; \mathcal{L}(f) \neq 0\right\} \tag{1.4}$$

(cf. (1.1)). Notice that $\mathcal{L}(f) = \mathcal{L}(|f|)$ and

$$\begin{aligned}
\mathcal{E}(f,f) &= ||f||_2^2 - \mathcal{R}(\langle Kf, f\rangle) \\
&= \frac{1}{2}\sum_{x,y} (|f(x)|^2 - 2\mathcal{R}(\overline{f(x)}f(y)) + |f(y)|^2)K(x,y)\pi(x) \\
&= \frac{1}{2}\sum_{x,y} |f(x) - f(y)|^2 K(x,y)\pi(x) \\
&\geqslant \frac{1}{2}\sum_{x,y} ||f(x)| - |f(y)||^2 K(x,y)\pi(x) \\
&= \mathcal{E}(|f|,|f|).
\end{aligned}$$

Hence in the definition of the log-Sobolev constant α one can restrict f to be real non-negative function. The following well-known result compares the log-Sobolev constant to the spectral gap. It is a special case of a result proved first by Simon and later independently by Rothaus by a different argument (see a survey paper of Gross [4] or [2]).

Theorem 1. *For any K the log-Sobolev constant α and the spectral gap λ satisfy $2\alpha \leqslant \lambda$.*

The following theorem is a translation of a previous result of Rothaus [6,7]. For a simple proof in our setting, see [8, Theorem 2.2.3].

Theorem 2. *Let K be irreducible and π be its stationary distribution. Then either $2\alpha = \lambda$ or there exists a positive non-constant function f which is a solution of*

$$2f \log f - 2f \log \|f\|_2 - \frac{1}{\alpha}(I - K)f = 0, \tag{1.5}$$

and such that $\alpha = \mathscr{E}(f, f)/\mathscr{L}(f)$.

Inequalities of Poincaré, Cheeger, Sobolev, Nash and log-Sobolev, are advanced techniques for bounded mixing times of finite irreducible reversible Markov chains. However, computing the log-Sobolev constant α exactly is difficult and it has been done only for a handful of examples. Diaconis and Saloff-Coste [2] gave the exact value of the log-Sobolev constant of the chain on a finite space with all rows of K equal to π. (This includes all chains on a two-point space.) We refer to [1,2] for more examples.

In this paper we compute the exact value of the log-Sobolev constant for the simple random walk on the n-cycle. (The exact value of the log-Sobolev constant is well-known for $n \leqslant 4$ (see [2,3])). In Section 3 we prove that if n is even and $n \geqslant 4$, then the log-Sobolev constant α and the spectral gap λ satisfy $2\alpha = \lambda$ (see Theorem 3 below). This implies that $\alpha = \frac{1}{2}(1 - \cos\frac{2\pi}{n})$. Our main result (Theorem 3) follows from Theorems 1 and 2 by showing that if $2\alpha < \lambda$, then there is no positive non-constant function f satisfying (1.5) and such that $\alpha = \frac{\mathscr{E}(f,f)}{\mathscr{L}(f)}$ (this approach was also used earlier in a different context by Mueller and Weissler [5]).

2. The log-Sobolev constant for n-cycle

Consider a simple random walk on the n-cycle $\mathbb{Z}_n$ and write $\mathbb{Z}_n = \{1, 2, \dots, n\}$. Clearly the corresponding Markov kernel K is given by $K(x, x \pm 1) = \frac{1}{2}$ and the uniform distribution on $\mathbb{Z}_n$ is its unique stationary distribution. (For $n = 2$, we have $K(1,2) = K(1,1) = K(2,1) = K(2,2) = \frac{1}{2}$. It is easy to check that the spectral gap of K is 1. Also it follows from a result of Gross [3] that $\alpha = \frac{1}{2}$. Therefore we obtain that $\alpha = \frac{1}{2} = \frac{\lambda}{2}$ in the case $n = 2$.) Throughout this paper we assume that $n \geqslant 3$.

For every $l = 1, 2, \ldots, n-1$, let

$$\theta_l = \frac{2\pi l}{n}$$

and

$$u_l = \begin{pmatrix} \sin\theta_l \\ \sin 2\theta_l \\ \vdots \\ \sin n\theta_l \end{pmatrix}.$$

Then $u_l \neq 0$ and direct computations imply that $Ku_l = (\cos\theta_l)u_l$ for $l = 1, 2, \ldots, n-1$. Therefore the spectrum of $I - K$ is given by the set

$$\sigma(I-K) = \left\{ 1 - \cos\frac{2\pi l}{n} \,\middle|\, l = 1, 2, \ldots, n \right\}.$$

Since K is reversible, we observe that the spectral gap λ of K is $1 - \cos\frac{2\pi}{n}$.

Denote by α the log-Sobolev constant for the simple random walk on the n-cycle. Note that the log-Sobolev constant for the simple random walk on $\mathbb{Z}_3$ is $\frac{1}{2\log 2}$ (see, e.g., [2]). Thus in this case we have $\alpha = \frac{1}{2\log 2} < \frac{\lambda}{2} = \frac{1}{2}(1 - \cos\frac{2\pi}{3}) = 0.75$. For $n = 4$, we obtain $\alpha = \frac{1}{2}$ from $n = 2$ by tensorization. For $n \geqslant 4$, Diaconis and Saloff-Coste [2] showed that α is of the same order as λ. In particular they proved that

$$\frac{8}{25}\frac{\pi^2}{n^2} \leqslant \alpha \leqslant \frac{2\pi^2}{n^2}.$$

By refining their arguments, we obtain

$$\frac{2}{5}\frac{\pi^2}{n^2} \leqslant \alpha \leqslant \frac{\pi^2}{n^2}.$$

The main result of this paper is as follows.

Theorem 3. *Assume that n is even. Then the log-Sobolev constant for the simple random walk on the n-cycle is just one half of its spectral gap: $\alpha = \frac{\lambda}{2}$ (we will prove Theorem 3 in Section 3).*

To compute the exact value of α, we write functions f on $\mathbb{Z}_n$ as vectors $(f(1), f(2), \ldots, f(n))$ in R^n. For every function $f = (x_1, x_2, \ldots, x_n)$, we have

$$\mathscr{L}(f) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \log\frac{x_i^2}{||f||_2^2} \tag{2.1}$$

and

$$\mathscr{E}(f,f) = \frac{1}{2n}(|x_1 - x_2|^2 + |x_2 - x_3|^2 + \cdots + |x_{n-1} - x_n|^2 + |x_n - x_1|^2). \tag{2.2}$$

Clearly function $\mathscr{L}$ is invariant if we permute the components of f, while function $\mathscr{E}$ is not. For a fixed function f, we investigate the extreme value of $\mathscr{E}$ over all permutations on the components of f.

Consider the function

$$F(x) = |x_1 - x_2|^2 + |x_2 - x_3|^2 + \cdots + |x_{n-1} - x_n|^2 + |x_n - x_1|^2, \tag{2.3}$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$. Moreover to every $x = (x_1, x_2, \ldots, x_n)$ with $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, there corresponds an element $\tilde{x} \in R^n$ given by the formula

$$\tilde{x} = \begin{cases} (x_1, x_3, x_5, \ldots, x_{2k+1}, x_{2k}, \ldots, x_4, x_2) & \text{if } n = 2k+1, \\ (x_1, x_3, x_5, \ldots, x_{2k-1}, x_{2k}, \ldots, x_4, x_2) & \text{if } n = 2k. \end{cases} \tag{2.4}$$

Denote by S_n the set of all permutations on $\{1, 2, \ldots, n\}$ and write $\theta x = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$ for $\theta \in S_n$ and $x \in R^n$.

Proposition 1. *For every $x = (x_1, x_2, \ldots, x_n)$ with $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, we have $F(\theta x) \geqslant F(\tilde{x})$ for all $\theta \in S_n$.*

Proof. We prove this by induction on n. Clearly there is nothing to prove in the case $n = 2$. Assume that it is also true for $n = k$. We consider the case $n = k+1$ and fix $x = (x_1, x_2, \ldots, x_{k+1})$ where $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{k+1}$.

Step 1: Set $y = (x_1, x_2, \ldots, x_k)$ and consider the corresponding $\tilde{y}$ given by (2.4). For every $i = 1, 2, \ldots, k-2$, set

$$\tilde{y}_{i,i+2} = \begin{cases} (x_1, x_3, \ldots, x_i, x_{k+1}, x_{i+2}, \ldots, x_4, x_2) & \text{if } i \text{ is odd}, \\ (x_1, x_3, \ldots, x_{i+2}, x_{k+1}, x_i, \ldots, x_4, x_2) & \text{if } i \text{ is even}. \end{cases} \tag{2.5}$$

Thus $\tilde{y}_{i,i+2}$ is obtained by inserting x_{k+1} in $\tilde{y}$ between x_i and x_{i+2}. Also set $\tilde{y}_{1,2} = (x_1, x_3, \ldots, x_4, x_2, x_{k+1})$ and

$$\tilde{y}_{k-1,k} = \begin{cases} (x_1, x_3, \ldots, x_k, x_{k+1}, x_{k-1}, \ldots, x_4, x_2) & \text{if } k \text{ is odd}, \\ (x_1, x_3, \ldots, x_{k-1}, x_{k+1}, x_k, \ldots, x_4, x_2) & \text{if } k \text{ is even}. \end{cases} \tag{2.6}$$

We claim that

$$F(\tilde{y}_{1,2}) \geqslant F(\tilde{y}_{k-1,k}) \tag{2.7}$$

and

$$F(\tilde{y}_{i,i+2}) \geqslant F(\tilde{y}_{k-1,k}) \quad \text{for all } i = 1, 2, \ldots, k-2. \tag{2.8}$$

Note that for every $1 \leqslant i \leqslant k-2$, we have

$$F(\tilde{y}_{i,i+2}) = F(\tilde{y}) + (x_i - x_{k+1})^2 + (x_{k+1} - x_{i+2})^2 - (x_i - x_{i+2})^2. \tag{2.9}$$

Therefore for $1 \leqslant i \leqslant k-4$, we have

$$\begin{aligned} F(\tilde{y}_{i,i+2}) - F(\tilde{y}_{i+2,i+4}) &= [(x_i - x_{k+1})^2 + (x_{k+1} - x_{i+2})^2 - (x_i - x_{i+2})^2] \\ &\quad - [(x_{i+2} - x_{k+1})^2 + (x_{k+1} - x_{i+4})^2 - (x_{i+2} - x_{i+4})^2] \\ &= 2(x_{k+1} - x_{i+2})(x_{i+4} - x_i) \geqslant 0. \end{aligned} \tag{2.10}$$

Also we have

$$\begin{aligned} F(\tilde{y}_{k-2,k}) - F(\tilde{y}_{k-1,k}) &= [(x_{k+1} - x_{k-2})^2 + (x_{k+1} - x_k)^2 - (x_{k-2} - x_k)^2] \\ &\quad - [(x_{k+1} - x_{k-1})^2 + (x_{k+1} - x_k)^2 - (x_k - x_{k-1})^2] \\ &= 2(x_{k+1} - x_k)(x_{k-1} - x_{k-2}) \geqslant 0 \end{aligned} \tag{2.11}$$

and

$$F(\tilde{y}_{k-3,k-1}) - F(\tilde{y}_{k-1,k}) = 2(x_{k+1} - x_{k-1})(x_k - x_{k-3}) \geqslant 0. \tag{2.12}$$

Combining (2.10)–(2.12) gives (2.8). To prove (2.7), it suffices to show that $F(\tilde{y}_{1,2}) \geqslant F(\tilde{y}_{1,3})$. This follows easily from the fact that

$$\begin{aligned} F(\tilde{y}_{1,2}) - F(\tilde{y}_{1,3}) &= [(x_1 - x_{k+1})^2 + (x_{k+1} - x_2)^2 - (x_1 - x_2)^2] \\ &\quad - [(x_1 - x_{k+1})^2 + (x_{k+1} - x_3)^2 - (x_1 - x_3)^2] \\ &= 2(x_{k+1} - x_1)(x_3 - x_2) \geqslant 0. \end{aligned}$$

Step 2: We prove that for every $\theta \in S_{n+1}$, we have

$$F(\theta x) \geqslant F(\tilde{y}_{k-1,k}) = F(\tilde{x}). \tag{2.13}$$

Fix $\theta \in S_{n+1}$ and set $c = \theta x$. Write $c = (\ldots, x_i, x_{k+1}, x_j, \ldots)$ for some $i < j$ and let $z = (\ldots, x_i, x_j, \ldots) \in \mathbf{R}^n$ be obtained by removing the component x_{k+1} from the vector c. If $1 \leqslant j \leqslant k-2$, we have

$$\begin{aligned} F(c) - F(\tilde{y}_{j,j+2}) &= [F(z) + (x_i - x_{k+1})^2 + (x_j - x_{k+1})^2 - (x_i - x_j)^2] \\ &\quad - [F(\tilde{y}) + (x_j - x_{k+1})^2 + (x_{k+1} - x_{j+2})^2 - (x_j - x_{j+2})^2] \\ &= F(z) - F(\tilde{y}) + 2(x_{k+1} - x_j)(x_{j+2} - x_i) \geqslant 0. \end{aligned} \tag{2.14}$$

(In the last inequality, we use the assumption that $F(z) \geqslant F(\tilde{y})$.) If $j = k-1$, we have

$$\begin{aligned} F(c) - F(\tilde{y}_{k-1,k}) &= [F(z) + (x_i - x_{k+1})^2 + (x_{k-1} - x_{k+1})^2 - (x_i - x_{k-1})^2] \\ &\quad - [F(\tilde{y}) + (x_k - x_{k+1})^2 + (x_{k+1} - x_{k-1})^2 - (x_k - x_{k-1})^2] \\ &= F(z) - F(\tilde{y}) + 2(x_k - x_i)(x_{k+1} - x_{k-1}) \geqslant 0. \end{aligned} \tag{2.15}$$

If $j = k$, we have

$$\begin{aligned} F(c) - F(\tilde{y}_{k-1,k}) &= [F(z) + (x_k - x_{k+1})^2 + (x_i - x_{k+1})^2 - (x_i - x_k)^2] \\ &\quad - [F(\tilde{y}) + (x_k - x_{k+1})^2 + (x_{k+1} - x_{k-1})^2 - (x_k - x_{k-1})^2] \\ &= F(z) - F(\tilde{y}) + 2(x_{k-1} - x_i)(x_{k+1} - x_k) \geqslant 0. \end{aligned} \tag{2.16}$$

Therefore (2.13) follows (2.14), (2.15), (2.16) and (2.8). □

Remark 1. Assume that the minimum α in (1.4) is attained at some positive non-constant function f. By the definition of the log-Sobolev constant and Proposition 1, there exists a minimizer of the form $f = (x_1, x_3, \ldots, x_4, x_2)$ while $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Moreover it is not hard to show that any minimizer of $\frac{\mathscr{E}(g,g)}{\mathscr{L}(g)}$ must satisfy the non-linear equation (1.5).

3. Proof of the main result

Throughout this section we assume that n is even and $n \geqslant 4$. We will argue by contradiction to verify that if $\alpha < \frac{\lambda}{2}$, there is no positive non-constant function f satisfying the non-linear equation (1.5) and such that $\alpha = \frac{\mathscr{E}(f,f)}{\mathscr{L}(f)}$. Then our main result (Theorem 3) follows from Theorems 1 and 2. Before proving the main result, we derive a series of lemmas by some combinatorial arguments.

Define the shift operator σ by

$$\sigma(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}),$$

where $x = (x_1, x_2, \ldots, x_n) \in R^n$. Set $\sigma^j(x) = \sigma(\sigma^{j-1}(x))$ for $j \geqslant 2$ and write σ^{-j} for the inverse of σ^j.

Lemma 1. *Consider a vector of the form*

$$u = (x_1, x_3, \ldots, x_{2k-1}, x_{2k}, \ldots, x_4, x_2)$$

where $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{2k}$ *and write* $\sigma^j(u) = ((\sigma^j(u))_1, (\sigma^j(u))_2, \dots, (\sigma^j(u))_{2k})$. *Then for every* $1 \leqslant j \leqslant k-1$, *we have*

$$(\sigma^j(u))_i \leqslant (\sigma^j(u))_{2k-i+1} \quad \textit{for } i = 1, \dots, k \tag{3.1}$$

and

$$(\sigma^{-j}(u))_i \geqslant (\sigma^{-j}(u))_{2k-i+1} \quad \textit{for } i = 1, \dots, k. \tag{3.2}$$

Proof. Assume $1 \leqslant j \leqslant k-1$. Then we have

$$(\sigma^j(u))_i = \begin{cases} x_{2(j-i+1)} & \text{if } 1 \leqslant i \leqslant j, \\ x_{2(i-j)-1} & \text{if } j+1 \leqslant i \leqslant j+k, \\ x_{2k-2[i-(j+k+1)]} & \text{if } j+k+1 \leqslant i \leqslant 2k. \end{cases}$$

(Case $1 \leqslant i \leqslant j \wedge (k-j)$.) Since $i \leqslant (k-j)$ we get $2k-i+1 \geqslant k+j+1$ and $(\sigma^j(u))_{2k-i+1} = x_{2(i+j)}$. Therefore we observe

$$(\sigma^j(u))_i = x_{2(j-i+1)} \leqslant x_{2(i+j)} = (\sigma^j(u))_{2k-i+1}.$$

(Case $j \vee (k-j) < i \leqslant k$.) Note that $(k-j) < i \leqslant k$ implies $k+1 \leqslant (2k-i+1) \leqslant (k+j)$. We have

$$(\sigma^j(u))_i = x_{2(i-j)-1}$$

and

$$(\sigma^j(u))_{2k-i+1} = x_{2(2k-i-j)+1}.$$

Since $2(2k-i-j)+1 \geqslant 2(i-j)-1$, we get $(\sigma^j(u))_i \leqslant (\sigma^j(u))_{2k-i+1}$.

(Case $j \wedge (k-j) < i \leqslant j \vee (k-j)$.) It is obvious that we only need to consider the situation that $j \neq k-j$. We first consider the case that $j < k-j$. Then we have $j < i \leqslant (k-j)$ and $2k-i+1 \geqslant j-k+2k+1 = k+j+1$. Therefore

$$(\sigma^j(u))_i = x_{2(i-j)-1} \leqslant x_{2(i+j)} = (\sigma^j(u))_{2k-i+1}.$$

On the other hand, if $k-j < j$, then we have $k-j < i \leqslant j$. This implies that

$$(\sigma^j(u))_i = x_{2(j-i+1)} \leqslant x_{2(2k-i-j)+1} = (\sigma^j(u))_{2k-i+1}.$$

This completes the proof of (3.1). The proof of (3.2) can be done by similar arguments. Here we omit it. □

Lemma 2. *Let* $u = (u_1, u_2, \dots, u_{2k-1}, u_{2k})$ *be a vector with* $u_i > 0$ *for all* $1 \leqslant i \leqslant 2k$. *Assume further that there exist two positive constants*, c *and* d, *such that*

$$2u_i - (u_{i-1} + u_{i+1}) = cu_i \log du_i^2 \tag{3.3}$$

for all $i = 1, \dots, 2k$ *(here we write* $u_0 = u_{2k}$ *and* $u_{2k+1} = u_1$*).*

(a) *If* $u_i \leqslant u_{2k-i+1}$ *for all* $1 \leqslant i \leqslant k$*, then we have*

$$u_1^2 - u_{2k}^2 + u_k^2 - u_{k+1}^2 \geqslant c[(u_1^2 + \cdots + u_k^2) - (u_{k+1}^2 + \cdots + u_{2k}^2)].$$

(b) *If* $u_i \geqslant u_{2k-i+1}$ *for all* $1 \leqslant i \leqslant k$*, then we have*

$$u_{2k}^2 - u_1^2 + u_{k+1}^2 - u_k^2 \geqslant c[(u_{k+1}^2 + \cdots + u_{2k}^2) - (u_1^2 + \cdots + u_k^2)].$$

Proof. (a) Assume that $u_i \leqslant u_{2k-i+1}$ for all $1 \leqslant i \leqslant k$. For every $1 \leqslant i \leqslant k$, rewrite Eq. (3.3) as

$$2 - \frac{u_{i-1} + u_{i+1}}{u_i} = c \log du_i^2.$$

Then we observe that

$$\begin{aligned}\frac{u_{2k-i} + u_{2k-i+2}}{u_{2k-i+1}} - \frac{u_{i-1} + u_{i+1}}{u_i} &= \frac{u_i(u_{2k-i} + u_{2k-i+2}) - u_{2k-i+1}(u_{i-1} + u_{i+1})}{u_i u_{2k-i+1}} \\ &= c\left(2 \log \frac{u_i}{u_{2k-i+1}}\right) \geqslant c\left(\frac{u_i}{u_{2k-i+1}} - \frac{u_{2k-i+1}}{u_i}\right).\end{aligned} \tag{3.4}$$

(In the last inequality we use the fact that $2 \log t \geqslant t - \frac{1}{t}$ for every $0 < t \leqslant 1$.) Inequality (3.4) implies that

$$(u_i u_{2k-i+2} - u_{i-1} u_{2k-i+1}) + (u_i u_{2k-i} - u_{i+1} u_{2k-i+1}) \geqslant c(u_i^2 - u_{2k-i+1}^2)$$

for all $i = 1, \dots, k$. Our result follows by summing up the above k inequalities.

(b) Assume that $u_i \geqslant u_{2k-i+1}$ for all $1 \leqslant i \leqslant k$. For every i, set $v_i = u_{2k-i+1}$. Then our result follows by applying (a) to the vector $v = (v_1, v_2, \dots, v_{2k})$. $\square$

Lemma 3. *Consider the following* $k \times k$ *matrices*:

$$A = \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & 1 & 0 \\ \vdots & & \ddots & 1 & 2 & 1 \\ 0 & \cdots & \cdots & 0 & 2 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2 & 1 & 0 \\ \vdots & & \ddots & 1 & 2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

(a) *If* $t < 2(1 - \cos\frac{\pi}{2k})$, *then* $P_A(t) = \det(A - tI) > 0$.
(b) *If* $t < 2(1 - \cos\frac{\pi}{2k+1})$, *then* $P_B(t) = \det(B - tI) > 0$.

Proof. (a) For every $1 \leqslant l \leqslant k$, let $\theta_l = \frac{(2l-1)\pi}{2k}$ and

$$v_l = \begin{bmatrix} \sin\theta_l \\ \sin 2\theta_l \\ \vdots \\ \sin k\theta_l \end{bmatrix}.$$

Routine calculation shows that $Av_l = 2(1 + \cos\theta_l)v_l$ for $1 \leqslant l \leqslant k$. Therefore $\{2(1 + \cos\theta_l) | 1 \leqslant l \leqslant k\}$ is the set of all real roots of the characteristic polynomial $P_A(t)$. Note that $(-t)^k$ is the highest order term of $P_A(t)$. This implies that $\lim_{t\to-\infty} P_A(t) = \infty$. Since $2(1 - \cos\frac{\pi}{2k})$ is the smallest real root of $P_A(t)$, we observe that $P_A(t) > 0$ for all $t < 2(1 - \cos\frac{\pi}{2k})$.

(b) The proof of (b) is the same as that of (a) where value of θ_l is replaced by $\frac{2l\pi}{2k+1}$. □

Lemma 4. (a) *Consider the following system of inequalities*:

$$\begin{cases} A_j - A_{j+1} \geqslant 4t(A_1 + \cdots + A_j), \quad j = 1, \ldots, k-1, \\ A_k \geqslant 2t(A_1 + \cdots + A_k). \end{cases} \tag{3.5}$$

If $t < \frac{1}{2}(1 - \cos\frac{\pi}{2k})$, *then system* (3.5) *has no solution* $(A_1, A_2, \ldots, A_k)$ *with* $A_1 < 0$.

(b) *Consider the following system of inequalities*:

$$\begin{cases} A_j - A_{j+1} \geqslant 4t(A_1 + \cdots + A_j), \quad j = 1, \ldots, k-1, \\ A_k \geqslant 4t(A_1 + \cdots + A_k). \end{cases} \tag{3.6}$$

If $t < \frac{1}{2}(1 - \cos\frac{\pi}{2k+1})$, *then the system* (3.6) *has no solution* $(A_1, A_2, \ldots, A_k)$ *with* $A_1 < 0$.

Proof. (a) Let $f_1(t) = 2 - 4t$ and $g_1(t) = 4t$. For every $1 \leqslant l \leqslant k-1$, put

$$f_{l+1}(t) = (1-4t)f_l(t) - g_l(t) \tag{3.7}$$

and

$$g_{l+1}(t) = 4tf_l(t) + g_l(t). \tag{3.8}$$

Clearly (3.7)–(3.8) imply

$$\begin{aligned} g_{l+1}(t) - g_l(t) &= 4tf_l(t) \\ &= f_l(t) - g_l(t) - f_{l+1}(t). \end{aligned}$$

Hence we have $f_l(t) = g_{l+1}(t) + f_{l+1}(t)$ for $1 \leqslant l \leqslant k-1$. Moreover for $2 \leqslant l \leqslant k-1$, we obtain

$$\begin{aligned} f_{l+1}(t) &= (2-4t) f_l(t) - (f_l(t) + g_l(t)) \\ &= (2-4t) f_l(t) - f_{l-1}(t). \end{aligned}$$

Note that $f_1(t) = 2-4t,\ f_2(t) = (1-4t)f_1(t) - g_1(t) = (2-4t)^2 - 2$. Therefore we observe

$$f_l(t) = \det(M_l - 4tI_l), \quad 1 \leqslant l \leqslant k, \tag{3.9}$$

where I_l is the $l \times l$ identity matrix and M_l is the $l \times l$ matrix of the same form as that in Lemma 3(a).

Assume that $t < \frac{1}{2}(1 - \cos\frac{\pi}{2k})$ and $(A_1, A_2, \ldots, A_k)$ satisfies the system of inequalities (3.5). Since $t < \frac{1}{2}(1 - \cos\frac{\pi}{2l})$ for $1 \leqslant l \leqslant k$, Lemma 3(a) and (3.9) imply that $f_l(t) > 0$ for all $l = 1, 2, \ldots, k$.

For every $1 \leqslant i \leqslant k-1$, we have, by (3.5),

$$A_{k-i} - A_{k-i+1} \geqslant 4t(A_1 + \cdots + A_{k-i}).$$

For $1 \leqslant j \leqslant k$, we claim that

$$f_j(t)A_{k-j+1} \geqslant g_j(t)(A_1 + \cdots + A_{k-j}). \tag{3.10}$$

Clearly (3.10) holds for $j = 1$. Assume it also holds for some i with $1 \leqslant i \leqslant k-1$. Since $f_i(t) > 0$, we get

$$\begin{aligned} f_i(t)A_{k-i} &= f_i(t)(A_{k-i} - A_{k-i+1}) + f_i(t)A_{k-i+1} \geqslant (4tf_i(t) + g_i(t))(A_1 + \cdots + A_{k-i}) \\ &= g_{i+1}(t)(A_1 + \cdots + A_{k-i-1}) + (4tf_i(t) + g_i(t))A_{k-i}. \end{aligned}$$

The above inequality implies that (3.10) also holds for $j = i + 1$. Hence (3.10) is true for $1 \leqslant j \leqslant k$. Plugging $j = k$ into (3.10) gives $f_k(t)A_1 \geqslant 0$. Since $f_k(t) > 0$, we observe that $A_1 \geqslant 0$. This completes the proof of (a).

(b) The proof of (b) follows word by word that of (a) while replacing $f_1(t)$ by $1 - 4t$. $\square$

Proof of Theorem 3. By Theorems 1 and 2, it suffices to show that if $\alpha < \frac{\lambda}{2}$, then there is no positive non-constant function f satisfying the non-linear equation (1.5) and such that $\alpha = \frac{\mathscr{E}(f,f)}{\mathscr{L}(f)}$. We argue by contradiction. Suppose that $\alpha < \frac{\lambda}{2} = \frac{1}{2}(1 - \cos \frac{2\pi}{n})$ and there exists a positive non-constant unit function f satisfying the non-linear equation (1.5) and such that $\alpha = \frac{\mathscr{E}(f,f)}{\mathscr{L}(f)}$. By Remark 1, we can assume further that $f = (x_1, x_3, \ldots, x_{n-1}, x_n, \ldots, x_4, x_2)$, where $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ and $x_1 < x_n$. Moreover the function f satisfies the equations:

$$2x_i - (x_i^{(1)} + x_i^{(2)}) = 2\alpha x_i \log n x_i^2, \quad 1 \leqslant i \leqslant n,$$

where $x_i^{(1)}$ and $x_i^{(2)}$ are the two nearest neighbors of x_i.

Recall that σ is the shift operator and $\sigma^j = \sigma(\sigma^{j-1})$ for $j \geqslant 2$. Write $n = 4k$ or $n = 4k + 2$. For $j = 1, \ldots, k$, we have

$$\sigma^j(f) = (x_{2j}, \ldots, x_2, x_1, \ldots, x_{n-2j-1}, x_{n-2j+1}, \ldots, x_{n-1}, x_n, \ldots, x_{2j+2})$$

and

$$\sigma^{-j}(f) = (x_{2j+1}, \ldots, x_{n-1}, x_n, \ldots, x_{n-2j+2}, x_{n-2j}, \ldots, x_2, x_1, \ldots, x_{2j-1}).$$

By Lemmas 1 and 2(a), we get

$$\begin{aligned}
&(x_{2j}^2 - x_{2j+2}^2 + x_{n-2j-1}^2 - x_{n-2j+1}^2) \\
&\quad \geqslant 2\alpha[(x_2^2 + x_4^2 + \cdots + x_{2j}^2 + x_1^2 + x_3^2 + \cdots + x_{n-2j-1}^2) \\
&\qquad - (x_{n-2j+1}^2 + x_{n-2j+3}^2 + \cdots + x_{n-1}^2 + x_{2j+2}^2 + x_{2j+4}^2 + \cdots + x_n^2)].
\end{aligned}$$

Similarly Lemmas 1 and 2(b) imply that

$$\begin{aligned}
&(x_{2j-1}^2 - x_{2j+1}^2 + x_{n-2j}^2 - x_{n-2j+2}^2) \\
&\quad \geqslant 2\alpha[(x_1^2 + x_3^2 + \cdots + x_{2j-1}^2 + x_2^2 + x_4^2 + \cdots + x_{n-2j}^2) \\
&\qquad - (x_{2j+1}^2 + x_{2j+3}^2 + \cdots + x_{n-1}^2 + x_{n-2j+2}^2 + x_{n-2j+4}^2 + \cdots + x_n^2)].
\end{aligned}$$

Note that $n-2j-1 \geqslant 2j+1$ and $n-2j \geqslant 2j+2$ for $1 \leqslant j \leqslant k$. Summing up the above two inequalities gives

$$(x_{2j-1}^2 + x_{2j}^2 - x_{2j+1}^2 - x_{2j+2}^2) + (x_{n-2j-1}^2 + x_{n-2j}^2 - x_{n-2j+1}^2 - x_{n-2j+2}^2)$$
$$\geqslant 4\alpha[(x_1^2 + x_2^2 + \cdots + x_{2j}^2) - (x_{n-2j+1}^2 + x_{n-2j+2}^2 + \cdots + x_n^2)].$$

Let $A_i = x_{2i-1}^2 + x_{2i}^2 - x_{n-2i+1}^2 - x_{n-2i+2}^2$ for $1 \leqslant i \leqslant k$. If $n = 4k$, then we have

$$\begin{cases} A_j - A_{j+1} \geqslant 4\alpha(A_1 + A_2 + \cdots + A_j), & j = 1, \ldots, k-1, \\ A_k \geqslant 2\alpha(A_1 + A_2 + \cdots + A_k). \end{cases}$$

If $n = 4k+2$, then we observe that

$$\begin{cases} A_j - A_{j+1} \geqslant 4\alpha(A_1 + A_2 + \cdots + A_j), & j = 1, \ldots, k-1, \\ A_k \geqslant 4\alpha(A_1 + A_2 + \cdots + A_k). \end{cases}$$

Note that $\alpha < \frac{1}{2}(1 - \cos\frac{2\pi}{n})$ and $A_1 = x_1^2 + x_2^2 - x_{n-1}^2 - x_n^2 \leqslant x_1^2 - x_n^2 < 0$. By Lemma 4, we get a contradiction. This completes the proof. □

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