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European Journal of Operational Research 149 (2003) 229–244

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

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Production, Manufacturing and Logistics

Optimal replenishment for a periodic review inventory system with two supply modes

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Received 1 October 1999; accepted 3 May 2002

Abstract

In this paper, we study periodic inventory systems with long review periods. We develop dynamic programming models for these systems in which regular orders as well as emergency orders can be placed periodically. We identify two cases depending on whether or not a fixed cost for placing an emergency order is present. We show that if the emergency supply mode can be used, there exists a critical inventory level such that if the inventory position at a review epoch falls below this critical level, an emergency order is placed. We also develop simple procedures for computing the optimal policy parameters. In all cases, the optimal order-up-to level is obtained by solving a myopic cost function. Thus, the proposed ordering policies are easy to implement.

Keywords: Inventory; Periodic review; Dynamic programming; Ordering policy

1. Introduction

In this paper, we study a periodic review inventory system in which there are two resupply modes: namely a regular mode and an emergency mode. Orders placed via the emergency mode, compared to orders placed via the regular mode, have a shorter lead time but are subject to larger variable costs and/or fixed order costs.

Many studies in the literature address this problem. See, e.g., Veinott (1966) and Whittmore and Saunders (1977) for periodic review models, and Moinzadeh and Nahmias (1988) and Moinzadeh and Schmidt (1991) for continuous review models. Earlier periodic review models, however, have focused on the situation in which supply lead times are a multiple of a review period. Such models could be regarded as an approximation of continuous review models, as the review periods can be modeled as small as, say, one working day. In contrast with earlier studies, we consider periodic inventory systems in which the review periods are long so that they are *possibly larger* than the supply lead times. For example, a retailer may

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0377-2217/03/$ - see front matter
doi:10.1016/S0377-2217(02)00446-0

place regular replenishment orders biweekly or monthly while the supply lead time is of the order of one week. This class of problems is important since periodic systems with long review periods are commonly used to achieve economics in the inventory review as well as in the coordination and consolidation of orders for different items. A common way to implement the coordination mechanism is through a periodic system in which on a given day the inventory status of all items from the same supplier are reviewed and the appropriate orders are placed.

We will develop dynamic programming models for the periodic system described above, and devise optimal ordering policies that minimize the expected discounted cost over an infinite horizon, which includes item cost, inventory holding cost, and shortage cost. Note that Chiang and Gutierrez (1996) study the exact same periodic system as in this paper. They find that either only the regular supply mode is used, or there exists an indifference inventory level such that if the inventory position at a review epoch falls below this level, an emergency order is placed. This result is similar to the results obtained in this paper. However, Chiang and Gutierrez investigate only the case in which emergency orders have same variable costs (as the regular orders) but larger fixed order costs. Moreover, they assume the use of an order-up-to-R policy and allow the placement of a regular order *or* an emergency order (not *both*) at a periodic review epoch while minimizing the *expected total cost per period*. The ordering policy developed in their model thus is *not* an optimal policy. In contrast, this paper considers more general cases in which placing an emergency order incurs larger variable costs and/or fixed order costs. Furthermore, we develop *optimal* policies that allow for possible regular *and* emergency orders at a periodic review epoch while minimizing the *expected discounted cost over an infinite horizon*. See Table 1 for the differences between these two papers.

In addition, Chiang and Gutierrez (1998) and Chiang (2001) study a different inventory system with long review periods. They assume that while regular orders are placed periodically (as in this paper), emergency orders can be placed *continuously* since the inventory status of items is known at any time. However, the inventory status of items may not be continuously updated in practice. For example, a retailer may review

Table 1
Differences among the models in Chiang and Gutierrez (1996), Chiang and Gutierrez (1998), and this paper

	Chiang and Gutierrez (1996)	This paper	Chiang and Gutierrez (1998)
Periodic systems considered	Long review periods	Long review periods	Long review periods (also called order cycles) within which there are a number of short periods (of length of one working day)
How the inventory status of items is updated?	On a periodic basis	On a periodic basis	On a continuous basis
When emergency orders are placed?	At periodic epochs	At periodic epochs	At the beginning of each short period
Cost of placing an emergency order	A positive fixed cost only	A positive fixed cost and/or a larger variable cost	A larger variable cost only
How inventory carrying cost is charged?	Based on average inventory	Based on average inventory	Based on inventory at the end of each short period
How shortage cost is charged?	Charged at the time shortage is satisfied	Charged at the time shortage is satisfied	Charged at the end of each short period as long as shortage exists
Criterion used	Minimize expected total cost per period	Minimize expected discounted cost over an infinite horizon	Minimize expected discounted cost over an infinite horizon
Is the ordering policy developed optimal?	Not optimal	Optimal	Optimal

inventory only before coordinating the orders for a group of items to a distribution center biweekly or monthly. Also, reviewing inventory and placing orders on a periodic basis is desirable in situations such as when vendors make routine visits to customers to take new orders (see, e.g., Chase and Aquilano, 1995). In addition, if the supplier of a line of items accepts orders only at particular times (see, e.g., Starr, 1996), there is no reason why we should review inventory frequently. Moreover, if items are time-consuming, difficult, or expensive to count, they should be reviewed only periodically to save costs. Consequently, for all such situations, if inventory on hand for an item is dangerously low at any time between two periodic reviews, an emergency order could be placed only later when a periodic review takes place. Finally, it is also possible that placing emergency orders at other than review epochs is significantly more expensive for the same reason that a periodic policy is usually followed in the first place: consolidation of orders to a single vendor. To summarize, while Chiang and Gutierrez's (1998) model allows the placement of emergency orders virtually at any time between two periodic reviews, our model places emergency orders only at periodic epochs. See Table 1 for other differences between these two papers.

We will consider two cases depending on whether or not a fixed cost for placing an emergency order is present. We show that under certain conditions, the emergency supply channel is never used, and that under other conditions, the emergency channel can be used. For the latter case, we develop optimal policies and identify the critical inventory level at a review epoch such that if the inventory position at a review epoch falls below this critical level, an emergency order is placed. We also will develop simple procedures for computing optimal policy parameters. In all cases, the optimal order-up-to level is obtained by solving a myopic cost function. Thus, the proposed ordering policies are easy to implement.

2. Assumptions and notation

Suppose that we now review inventory at a periodic epoch t with n periods remaining (until the end of the planning horizon) and want to decide whether to place a regular order and/or an emergency order. Denote the length of a period by T. In this paper, as in Chiang and Gutierrez (1996), we assume that T is not a decision variable and is exogenously determined. Denote the unit item cost of the regular and emergency supply modes by c_1 and c_2 respectively, where $c_1 \leqslant c_2$ and the lead time of the two modes by τ_1 and τ_2 respectively, where $\tau_2 < \tau_1$. Assume, as in Chiang and Gutierrez (1996), that $\tau_2 < T$ and $\tau_3 = \tau_1 - \tau_2 < T$ (i.e., there is no order cross-over). Assume also that there is a fixed cost K for placing an emergency order, which represents the extra expense of making a special arrangement with the supplier. The fixed cost of placing a regular order is assumed to be negligible or zero. One possible reason for this is that a regular order for an item is part of a joint order (due to the consolidation of orders for different items, as mentioned in Section 1) which consists of a mix of products shipped in the same truck, and the ordering cost (including the transportation cost) for a joint order is incurred every time a joint order is placed and thus is irrelevant to individual items. Other reasons include the routine administrative work (for placing a regular order) and the use of information technology such as EDI. As a result, we will focus on the following two cases in our analysis: (i) $K = 0$ and (ii) $K > 0$. It is assumed in the first case that $c_1 < c_2$. Finally, there is an inventory holding cost at a rate of h per unit held per unit time and a shortage cost at π per unit, charged at the time shortage is satisfied.

In addition to the cost and operation parameters defined above, we will refer to the following notation.

$(x)^+$	$\max\{x, 0\}$.
$\mathrm{D}f$	the first derivative of the function f.
$\delta(x)$	1 if x is positive and 0 if x is 0.
α	the discount factor.
X_0	demand during the upcoming time interval $[t, t+T)$ (i.e., X_0 is the demand during a review period).

X_1 demand during the upcoming time interval $[t, t+\tau_1)$ (i.e., X_1 is the demand during the regular supply lead time).

X_2 demand during the upcoming time interval $[t, t+\tau_2)$ (i.e., X_2 is the demand during the emergency supply lead time).

X_3 demand during the time interval $[t+\tau_2, t+\tau_1)$ (i.e., X_3 is the demand during the inter-arrival time between the two supply modes). Since $X_1 = X_2 + X_3$, X_1 is stochastically larger than X_2.

X_4 demand during the time interval $[t+\tau_1, t+T+\tau_2)$.

$f_i(X_i)$ the probability density function (p.d.f.) of X_i, $i = 0, 1, 2$.

$G(H,r,R)$ the expected total cost per period (excluding a possible fixed ordering cost of K) given that the inventory position (i.e., inventory on hand plus inventory on order minus backorder) at the review epoch is H, the inventory position after a possible emergency order is r, and the inventory position after a possible emergency order and a possible regular order is R. See Fig. 1 for a realization of the inventory process that depicts the model considered in this paper.

$C_n(H)$ the expected discounted cost with n periods remaining given that the inventory position is H and an optimal ordering policy is used at every review epoch. $C_n(H)$ satisfies the functional equation

$$C_n(H) = \min_{H \leqslant r \leqslant R} \{K\delta(r-H) + G(H,r,R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} \tag{1}$$

where $C_0(H) \equiv 0$.

Other assumptions made in our analysis include the following:

(1) Shortage (if any) should be satisfied as early as possible. This means that if the inventory position is negative at a review epoch, an emergency order should be placed immediately and its size should be sufficient to raise the inventory position to a non-negative level, i.e., r should be greater than or equal to zero. In the next section, however, r is allowed to be negative to get a complete picture of the function $G(H,r,R)$.

(2) Demand is non-negative and independently distributed in disjoint time intervals. Demand X in a time interval of length τ has mean $\lambda\tau$, where λ is the mean demand rate per unit time (e.g., $E(X_3) = \lambda\tau_3$). Also, demand is assumed to follow the normal, Poisson, gamma, or geometric distribution.

(3) R is assumed to be greater than zero. In addition, let $\underline{R}$ be the minimum order-up-to level satisfying $Pr(X_1 > R) \leqslant \varphi_R$, where φ_R is a maximum allowable probability that a regular order placed fails to clear any backorders ($\varphi_R = 0.01$, for example). We assume throughout the analysis that $R \geqslant \underline{R}$,

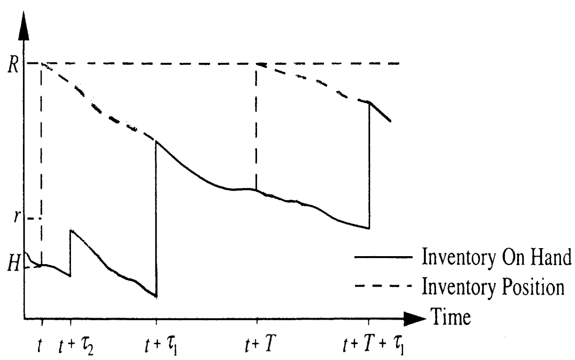

Fig. 1. A realization of the inventory process for the model considered in this paper.

i.e., a regular order when arriving will clear any backorders with probability almost 1. Note that the ordinary (R, T) model, by assuming first that backorders are incurred only in very small quantities, also assumes that a regular order when arriving is almost always sufficient to meet any outstanding backorders (see, e.g., Hadley and Whitin, 1963). The assumption $R \geqslant \underline{R}$ is reasonable if the ratio h/π (which determines the optimal probability of a stockout in the Newsboy problem) is not too large.

(4) In computing the inventory holding cost within a time interval, it is assumed that the inventory on hand decreases with a constant rate. In addition, the exact holding cost expressions which involve a variable in the denominator will be approximated by a lower bound (as in Chiang and Gutierrez, 1996). This assumption is similar to the one in the ordinary (R, T) model in which the expected on-hand inventory is approximately equal to the expected net inventory. As these approximations underestimate the true holding cost, the optimal order-up-to levels derived may be somewhat a little large.

3. An expression for $G(H, r, R)$

In this section, we develop an explicit expression for $G(H, r, R)$. Notice that at the current review epoch t, the expected cost incurred before time $T + \tau_2$ has been determined by the decision made in the previous review. Thus, $G(H, r, R)$, which consists of item cost, inventory holding cost, and shortage cost, is derived for the upcoming time interval $[t + \tau_2, t + T + \tau_2)$. The item cost is simply $c_2(r - H) + c_1(R - r)$. If a regular order is placed at the current epoch, inventory on hand will increase at time $t + \tau_1$ Hence, in developing the inventory holding or shortage cost expression, we need to consider two disjoint time intervals $[t + \tau_2, t + \tau_1)$ and $[t + \tau_1, t + T + \tau_2)$ separately.

We first evaluate the inventory holding cost for the time interval $[t + \tau_2, t + \tau_1)$. If $r < X_2$ the on-hand inventory is zero for any time during the interval $[t + \tau_2, t + \tau_1)$. However, if $r \geqslant X_2$ and the demand X_3 during the time interval $[t + \tau_2, t + \tau_1)$ is less than or equal to $r - X_2$, then the average inventory holding cost over the interval $[t + \tau_2, t + \tau_1)$ is $h\tau_3(r - X_2 - 0.5X_3)$. On the other hand, if $r \geqslant X_2$ and $X_3 > r - X_2$ the average inventory holding cost is $0.5h\tau_3(r - X_2)^2/X_3$. As the variable X_3 appears in the denominator, this exact expression will substantially complicate the subsequent analysis. To overcome this problem, as mentioned in Section 2, we use a lower bound $h\tau_3(r - X_2 - 0.5X_3)$ for this expression, i.e., the same one as when $X_3 \leqslant r - X_2$. To summarize, if $r \geqslant X_2$, the average inventory holding cost over the time interval $[t + \tau_2, t + \tau_1)$ is $h\tau_3(r - X_2 - 0.5X_3)$ and when taking the expected value of X_3, becomes $h\tau_3(r - X_2 - 0.5\lambda\tau_3)$.

Similarly, if $R \geqslant X_1$, the average inventory holding cost over the time interval $[t + \tau_1, t + T + \tau_2]$ is $h(T - \tau_3)[R - X_1 - 0.5\lambda(T - \tau_3)]$. However, as $R \geqslant \underline{R}$ is assumed, $Pr(X_1 > R)$ is approximately zero, i.e., $Pr(R \geqslant X_1)$ approximately equals 1. Thus, the average holding cost over the interval $[t + \tau_1, t + T + \tau_2)$ is approximated by $h(T - \tau_3)[R - \lambda\tau_1 - 0.5\lambda(T - \tau_3)]$.

Next, we develop the shortage cost expression of $G(H, r, R)$. Since shortage cost is assumed to be charged at the time shortage is satisfied, we consider two upcoming epochs $t + \tau_2$ and $t + \tau_1$ when an emergency order (if placed) and a regular order (if placed) arrive respectively. We first evaluate the shortage cost incurred at the epoch $t + \tau_2$. There are two possible cases if shortage occurs: first, the emergency order will clear the backorders (i.e., if $X_2 \leqslant r$), and second, the emergency order fails to clear the backorders (i.e., if $X_2 > r$). In the former case, the amount of shortage is $(X_2 - H)^+$, while in the latter case, the amount of shortage is the quantity ordered, i.e., $r - H$. Similarly, if the regular order placed clears the backorders (i.e., if $X_1 \leqslant R$), the amount of shortage is $(X_1 - r)^+$; otherwise (i.e., if $X_1 > R$), it is the quantity ordered $(R - r)$. Again, as $R \geqslant \underline{R}$ is assumed, $Pr(X_1 > R)$ is approximately zero. Thus, the latter case is not considered.

Combining the above analysis, we obtain

$$\begin{aligned} G(H,r,R) &= c_2(r-H)+c_1(R-r)+\int_0^r h\tau_3(r-X_2-0.5\lambda\tau_3)f_2(X_2)\,\mathrm{d}X_2 \\ &\quad + h(T-\tau_3)[R-\lambda\tau_1-0.5\lambda(T-\tau_3)] \\ &\quad + \pi\left\{\int_H^r (X_2-H)f_2(X_2)\,\mathrm{d}X_2 + (r-H)\int_r^\infty f_2(X_2)\,\mathrm{d}X_2 + \int_r^\infty (X_1-r)f_1(X_1)\,\mathrm{d}X_1\right\} \\ &= c_2(r-H)+c_1(R-r)+\int_0^r h\tau_3(r-X_2-0.5\lambda\tau_3)f_2(X_2)\,\mathrm{d}X_2 \\ &\quad + h(T-\tau_3)[R-\lambda\tau_1-0.5\lambda(T-\tau_3)] \\ &\quad + \pi\left\{\int_H^\infty (X_2-H)f_2(X_2)\,\mathrm{d}X_2 - \int_r^\infty (X_2-r)f_2(X_2)\,\mathrm{d}X_2 + \int_r^\infty (X_1-r)f_1(X_1)\,\mathrm{d}X_1\right\}. \end{aligned} \tag{2}$$

Let

$$G_1(H) = -c_2H + \pi\int_H^\infty (X_2-H)f_2(X_2)\,\mathrm{d}X_2, \tag{3}$$

$$\begin{aligned} G_2(r) &= (c_2-c_1)r + \int_0^r h\tau_3(r-X_2-0.5\lambda\tau_3)f_2(X_2)\,\mathrm{d}X_2 \\ &\quad + \pi\left\{\int_r^\infty (X_1-r)f_1(X_1)\,\mathrm{d}X_1 - \int_r^\infty (X_2-r)f_2(X_2)\,\mathrm{d}X_2\right\}, \end{aligned} \tag{4}$$

$$G_3(R) = c_1R + h(T-\tau_3)[R-\lambda\tau_1-0.5\lambda(T-\tau_3)]. \tag{5}$$

Then $G(H,r,R)$ can be expressed as

$$G(H,r,R) = G_1(H)+G_2(r)+G_3(R). \tag{6}$$

Notice that $G_1(H)$ is easily seen to be convex. Also, $G_3(R)$ in (5) is a linear (thus convex) function, since it is derived based on the assumption $R \geqslant \underline{R}$. We next analyze the functional form of $G_2(r)$. It follows from (4) that for $r \leqslant 0$, $G_2(r)$ is equal to $(c_2-c_1)r + \pi(\lambda\tau_1-\lambda\tau_2)$, i.e., it is increasing on r (it is constant if $c_2=c_1$). For $r>0$, the first and second derivatives of $G_2(r)$ are given by

$$\mathrm{D}G_2(r) = (c_2-c_1) + h\tau_3\left\{\int_0^r f_2(X_2)\,\mathrm{d}X_2 - 0.5\lambda\tau_3 f_2(r)\right\} + \pi\left\{\int_r^\infty f_2(X_2)\,\mathrm{d}X_2 - \int_r^\infty f_1(X_1)\,\mathrm{d}X_1\right\}, \tag{7}$$

$$\mathrm{DD}G_2(r) = h\tau_3\{f_2(r) - 0.5\lambda\tau_3\,\mathrm{D}f_2(r)\} + \pi\{f_1(r)-f_2(r)\}. \tag{8}$$

It can be seen from (7) that $\mathrm{D}G_2(r) = (c_2-c_1)+h\tau_3 > 0$ as r becomes very large, and $\mathrm{D}G_2(r) = (c_2-c_1) - 0.5h\tau_3\lambda\tau_3 f_2(r)$ as r approaches zero from the right. For the normal or Poisson demand, there is at most one positive r, denoted by r_{I} equating (8) to zero (see Appendix). It can be shown that this also holds for the gamma or geometric demand. If r_{I} does not exist, $G_2(r)$ is convex on $r>0$; otherwise if r_{I} exists, $G_2(r)$ is concave for $r \in (0, r_{\mathrm{I}})$ and convex on $r > r_{\mathrm{I}}$ (see Appendix for details). Consequently, it follows that if $c_2 > c_1$, then either $G_2(r)$ is increasing on all r, or there exist $\hat{r}$ and r^*, which are respectively the (unique) local maximum and minimum of $G_2(r)$, such that $G_2(r)$ is increasing on $r<\hat{r}$ decreasing on $r \in (\hat{r}, r^*)$, and increasing on $r > r^*$ (see Fig. 2). Note that $\hat{r} \geqslant 0$ and $r^* > 0$. On the other hand if $c_2 = c_1$, then either $G_2(r)$ is non-decreasing on all r, or it is constant on $r \leqslant 0$, decreasing on $r \in (0, r^*)$, and increasing on $r > r^*$ (see Fig. 3). Notice that $\mathrm{D}G_2(r)$ approximately equals $(c_2-c_1)+h\tau_3$ for $r \geqslant \underline{R}$, since by assumption, $Pr(X_1 > r)$ is approximately zero for $r \geqslant \underline{R}$ (and so is $Pr(X_2 > r)$). This implies that if $G_2(r)$ has the shape as depicted in Fig. 2 or Fig. 3, then $r^* < \underline{R}$.

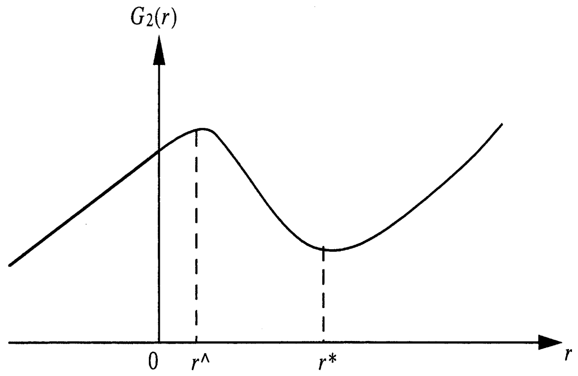

Fig. 2. $G_2(r)$ with $c_2 > c_1$.

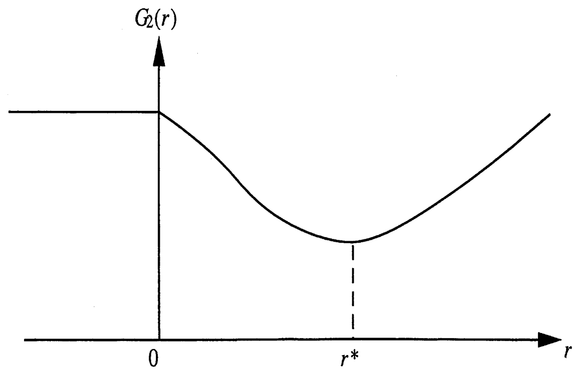

Fig. 3. $G_2(r)$ with $c_2 = c_1$.

4. The case where fixed cost of ordering is zero

We first analyze the case in which there is no fixed cost for placing an emergency order, i.e., $K = 0$ (and thus $c_2 > c_1$). We show that under a certain condition, the emergency supply channel is never used, and that under other conditions, the emergency channel can be used. For the latter situation, we show that there exists a critical level such that if the inventory position at a review epoch is below this critical level, an emergency order is placed. We also suggest a simple myopic cost function to solve for the optimal policy parameters. Hence, the proposed inventory control policy is easy to implement.

We first state in Theorem 1 the conditions under which the emergency supply channel is never used and can be used respectively.

Theorem 1. *Suppose that $K = 0$. If $G_2(r)$ is increasing on all r, the emergency supply channel will never be used; otherwise if $G_2(r)$ is increasing on $r < \hat{r}$, decreasing on $r \in (\hat{r}, r^*)$, and increasing on $r > r^*$, the emergency channel can be used, i.e., if $H < r^*$, an emergency order is placed.*

To verify Theorem 1, we see from (1) and (6) that $C_n(H)$ is expressed as

$$C_n(H) = \min_{H \leqslant r \leqslant R} \{G_2(r) + G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} + G_1(H). \tag{9}$$

If $G_2(r)$ is increasing on all r, then (9) reduces to

$$C_n(H) = \min_{H \leqslant R} \{G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} + G_1(H) + G_2(H), \tag{10}$$

i.e., $r = H$. This implies that the emergency supply channel will never be used, and thus the ordinary order-up-to-R policy is optimal. It can be seen from (7) that if π is very small (in comparison to $c_2 - c_1$), $DG_2(r)$ may be positive for all $r > 0$.

Thus, we assume for the remaining of this section that $G_2(r)$ has the shape as depicted in Fig. 2. We show in the following that if the inventory position at a review epoch falls below r^*, an emergency order is placed. Notice that (9) can be written as

$$C_n(H) = \min_{H \leqslant R} \{ \min_{H \leqslant r \leqslant R} \{G_2(r)\} + G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} + G_1(H). \tag{11}$$

Since we assume that $R \geqslant \underline{R}$, $\min_{H \leqslant r \leqslant R} G_2(r)$ reduces to $\min_{H \leqslant r} G_2(r)$ due to $r^* < \underline{R}$. Hence, (11) simplifies to

$$C_n(H) = \min_{H \leqslant R} \{G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} + G_1(H) + \min_{H \leqslant r} G_2(r). \tag{12}$$

As we can see from Fig. 2, there exists r_0 such that $G_2(r_0) = G_2(r^*)$. It follows that $\min_{H \leqslant r} G_2(r) = G_2(H)$ for $H \leqslant r_0$, $\min_{H \leqslant r} G_2(r) = G_2(r^*)$ for $H \in (r_0, r^*)$, and $\min_{H \leqslant r} G_2(r) = G_2(H)$ for $H \geqslant r^*$. In summary, $\min_{H \leqslant r} G_2(r)$ is a non-decreasing function of H. Define for each review epoch $J_n(R)$ as

$$J_n(R) = G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0). \tag{13}$$

Denote by $R_n(H)$ the value of R minimizing $J_n(R)$ for $R \geqslant H$ (note that $R_n(H) \geqslant \underline{R}$ as $R \geqslant \underline{R}$ is required) and denote by R_n^* the value of R minimizing $J_n(R)$ for all $R \geqslant \underline{R}$ (i.e., R_n^* is the global minimum). If $R_n(H)$ or R_n^* is not an unique minimum, the smallest such value is chosen. In general, (12) is expressed as

$$C_n(H) = J_n(R_n(H)) + G_1(H) + \min_{H \leqslant r} G_2(r) \tag{14}$$

(note that $J_n(R_n(H))$ is also a non-decreasing function of H). In particular, for $H \leqslant R_n^*$,

$$C_n(H) = J_n(R_n^*) + G_1(H) + \min_{H \leqslant r} G_2(r). \tag{15}$$

The optimal ordering policy at a review epoch with n periods remaining can be described in three different cases: (i) if $H \in (r_0, r^*)$, order amounts $r^* - H$ and $R_n^* - r^*$ at unit cost c_2 and c_1 respectively, (ii) if $H \leqslant R_n^*$ but $H \notin (r_0, r^*)$, order an amount $R_n^* - H$ at unit cost c_1, and (iii) if $H > R_n^*$, order an amount $R_n(H) - H$ at unit cost c_1. Notice that the reason of not placing an emergency order for $H \leqslant r_0$ is because the incremental item cost $(c_2 - c_1)(r^* - H)$ is too large and shortages which have occurred can still be satisfied by a regular order. However, we assume that if shortages occur, they should be satisfied as early as possible. Hence, as long as $H < r^*$ an emergency order is always placed to bring the inventory position to r^*. As a note, the probability of $H \leqslant r_0$ is virtually zero as r_0 is usually a large negative number (this is shown in the computation).

We observe in practice that in most cases there exists a minimum divisible quantity and demand occurs in a multiple of this quantity. Since demand in a review period is non-negative and bounded, if follows that the state space for H is finite. Note that even if there does not exist a minimum divisible quantity and demand can occur in any finite non-negative real amount, the state space must be discretized when implemented on a digital computer. Moreover, the action space is also finite since in practice the order quantity is also bounded and orders will be placed in a multiple of a divisible quantity. As the dynamic program defined by (9) is a Markov decision process, it follows (Blackwell, 1962) that there exists an optimal policy that is stationary if the planning horizon is infinite. Hence, the sequence $\{R_n^*\}$ can be replaced by a single number R^* for the infinite horizon problem. Consequently, r^* and R^* constitute the optimal policy parameters for the infinite horizon model. Thus, if $H < r^*$, order amounts $r^* - H$ and $R^* - r^*$ at unit cost c_2 and c_1 respectively; otherwise if $H \in [r^*, R^*]$, order an amount $R^* - H$ at unit cost c_1. Notice that H will never go above R^* as demand is non-negative.

Next, we propose a simple model to compute the optimal order-up-to level R^*. (r^* can be obtained by decreasing r from $\underline{R}$ until $G_2(r)$ starts increasing.) Denote by $C(H,R)$ the expected cost per period for an inventory position of H if r^* and an order-up-to level R are used for an infinite horizon. Then, it follows that

$$C(H,R) = G_1(H) + G_2(r^*) + G_3(R) \quad \text{for } H < r^*,$$
$$C(H,R) = G_1(H) + G_2(H) + G_3(R) \quad \text{for } H \geqslant r^*.$$

Define $F(H)$ as

$$F(H) \equiv G_1(H) + G_2(r^*) \quad \text{for } H < r^*,$$
$$F(H) \equiv G_1(H) + G_2(H) \quad \text{for } H \geqslant r^*.$$

Then $C(H,R) = F(H) + G_3(R)$, i.e., the expected cost per period decomposes into a function of H (the state) and a function of R (the action). Also, since the inventory position at a review epoch will never go above R, the order-up-to level R is always feasible at a review epoch. Moreover, the state H at a review epoch depends only on the action R at the previous review epoch and does not depend on the state at the previous epoch. As a result, all the conditions of Theorem 1 in Sobel (1981) are satisfied. It follows that minimizing a myopic cost function given by

$$J(R) = G_3(R) + \alpha E_{X_0} F(R - X_0) \tag{16}$$

will minimize the expected discounted cost over an infinite horizon. To minimize $J(R)$, we note that $J(R)$ is a convex function as $F(H)$ and $G_3(R)$ are convex. Thus, the optimum R^* can be obtained by, for example, increasing R from $\underline{R}$ until $J(R)$ starts increasing. In summary, we use the following method to obtain the optimal policy parameter r^* and R^*.

Step 1. Decrease r from $\underline{R}$ (or a large value of r) until $r = r^*$ where $\mathrm{D}G_2(r^*) = 0$. As a note, if $\mathrm{D}G_2(r)$ never becomes negative as r decreases, the emergency channel is never used.

Step 2. Increase R from $\underline{R}$ (or decrease R from a large value of R) until $R = R^*$ (and $J(R)$ starts increasing) where $J(R^*)$ is the minimum of $J(R)$.

Note that if we let $C_0(H) \equiv F(H)$, the expected discounted cost over an infinite horizon is equal to $F(H) + J(R^*)/(1-\alpha)$ for any $H \leqslant R^*$ which depends on the initial state H. Hence, in Section 6, we will use $J(R)$ only for computing the savings obtained from employing two supply modes. Incidentally, if only the regular supply mode is used, the optimal order-up-to level is obtained by solving also (16) except that $F(H) \equiv G_1(H) + G_2(H)$ for all $H \leqslant R$.

5. The case where fixed cost of ordering is positive

In this section, we analyze the case in which there is a positive fixed cost for placing an emergency order, i.e., $K > 0$. We show that a similar optimal policy exists, i.e., if the emergency supply channel can be used, there exists a critical level such that if the inventory position at a review epoch falls below this critical level, an emergency order is placed. We also propose a myopic cost function to solve for the optimal policy parameters.

We first state the condition under which the emergency supply channel is never used, no matter whether $c_2 = c_1$ or not.

Lemma 1. *Suppose that $K > 0$. If $G_2(r)$ is non-decreasing on r, the emergency supply channel will never be used.*

To prove Lemma 1, we see from (1) and (6) that $C_n(H)$ is expressed by

$$\begin{aligned} C_n(H) &= \min_{H \leqslant r \leqslant R} \{K\delta(r-H) + G_2(r) + G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\} + G_1(H) \\ &= \min_{H \leqslant r} \{K\delta(r-H) + G_2(r) + \min_{r \leqslant R} \{G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0)\}\} + G_1(H). \end{aligned} \tag{17}$$

Define $J_n(R)$ as

$$J_n(R) = G_3(R) + \alpha E_{X_0} C_{n-1}(R - X_0) \tag{18}$$

and let $F_n(r) = \min_{r \leqslant R} \{J_n(R)\}$. Then $C_n(H)$ is expressed by

$$C_n(H) = \min_{H \leqslant r} \{K\delta(r-H) + G_2(r) + F_n(r)\} + G_1(H). \tag{19}$$

Denote by $R_n(r)$ the (smallest) value of R minimizing $J_n(R)$ for $R \geqslant r$ (as before, $R_n(r) \geqslant \underline{R}$ since $R \geqslant \underline{R}$ is required) and R_n^* the (smallest) value of R minimizing $J_n(R)$ for all $R \geqslant \underline{R}$. In general, $F_n(r) = J_n(R_n(r))$. In particular, for $r \leqslant R_n^*$, $F_n(r) = J_n(R_n^*)$. $F_n(r)$ is a non-decreasing function of r. Thus, if $G_2(r)$ is also non-decreasing on r, it follows from (19) that the emergency supply channel will never be used. Notice that this condition is similar to the condition for employing only the regular channel in Theorem 1.

We assume in the subsequent analysis that $G_2(r)$ has the shape as depicted in Fig. 2 or Fig. 3. We first assume that $c_2 > c_1$. We state in Theorem 2 the conditions under which the emergency supply channel is never used and can be used respectively.

Theorem 2. *Suppose that $K > 0$ and $c_2 > c_1$. Suppose also that $G_2(r)$ is increasing on $r < \hat{r}$, decreasing on $r \in (\hat{r}, r^*)$, and increasing on $r > r^*$. If $K \geqslant G_2(\hat{r}) - G_2(r^*)$, the emergency channel will never be used. Otherwise if $K < G_2(\hat{r}) - G_2(r^*)$, the emergency channel can be used, i.e., there exists a critical level which is less than r^* (i.e., the critical level when $K = 0$, other things being equal), such that if H falls below this level, an emergency order is placed.*

To show Theorem 2, we observe that as $F_n(r)$ is non-decreasing on r, if $K \geqslant G_2(\hat{r}) - G_2(r^*)$ in Fig. 2, then $r = H$, i.e., the emergency mode will never be used at any review epoch. We assume that $K < G_2(\hat{r}) - G_2(r^*)$. Define $r_{\mathrm{U}} = \inf\{r > \hat{r} | G_2(r) \leqslant K + G_2(r^*)\}$ and $r_{\mathrm{L}} = \sup\{r < \hat{r} | G_2(r) \leqslant K + G_2(r^*)\}$. As r_{U} and r_{L} are independent of n, they are the same for all review epochs. Since $r^* < \underline{R}$ and $F_n(r)$ is constant for $r < \underline{R}$, it follows that the optimal policy at a review epoch with n periods remaining is (i) if $H \in (r_{\mathrm{L}}, r_{\mathrm{U}})$, order amounts $r^* - H$ and $R_n^* - r^*$ at unit cost c_2 (plus a fixed cost of K) and c_1 respectively, i.e.,

$$C_n(H) = K + G_2(r^*) + F_n(r^*) + G_1(H) = K + G_2(r^*) + J_n(R_n^*) + G_1(H); \tag{20}$$

(ii) if $H \leqslant R_n^*$ but $H \notin (r_{\mathrm{L}}, r_{\mathrm{U}})$, order an amount $R_n^* - H$ at unit cost c_1, i.e.,

$$C_n(H) = G_2(H) + F_n(H) + G_1(H) = G_2(H) + J_n(R_n^*) + G_1(H); \tag{21}$$

and finally (iii) if $H > R_n^*$, order an amount $R_n(H) - H$ at unit cost c_1, i.e.,

$$C_n(H) = G_2(H) + F_n(H) + G_1(H) = G_2(H) + J_n(R_n(H)) + G_1(H). \tag{22}$$

The reason of not placing an emergency order in case of $H \leqslant r_L$ is the large incremental item cost and/or fixed ordering cost. Since we assume that shortages (if any) should be satisfied as early as possible, we require that as long as $H < r_U$, an emergency order is always placed to bring the inventory position to r^* (note again that the probability of $H \leqslant r_L$ is approximately zero as r_L is usually negative). Notice (by the definition of r_U) that the critical level r_U is a decreasing function of K as long as $K < G_2(\hat{r}) - G_2(r^*)$. If we use the same reasoning as for the case of $K = 0$, we see that there also exists an optimal policy that is stationary if the planning horizon is infinite. Hence, the sequence $\{R_n^*\}$ can be replaced by a single number R^* for the infinite horizon problem. Consequently, r_U, r^*, and R^* constitute the optimal policy parameters for the infinite horizon model. Thus, if $H < r_U$, the optimal policy is to order amounts $r^* - H$ and $R^* - r^*$ at unit cost c_2 (plus a fixed cost of K) and c_1 respectively; otherwise if $H \in [r_U, R^*]$, the optimal policy is to order an amount $R^* - H$ at unit cost c_1.

As in the case of $K = 0$, we suggest a simple model to compute the optimal order-up-to level R^*. Denote by $\hat{C}(H,R)$ the expected cost per period for an inventory position of H if r_U, r^*, and an order-up-to level R are used for an infinite horizon. It follows that

$$\hat{C}(H,R) = K + G_1(H) + G_2(r^*) + G_3(R) \quad \text{for } H < r_U,$$
$$\hat{C}(H,R) = G_1(H) + G_2(H) + G_3(R) \quad \text{for } H \geqslant r_U.$$

Define $\hat{F}(H)$ as

$$\hat{F}(H) \equiv K + G_1(H) + G_2(r^*) \quad \text{for } H < r_U,$$
$$\hat{F}(H) \equiv G_1(H) + G_2(H) \quad \text{for } H \geqslant r_U.$$

Then $\hat{C}(H,R) = \hat{F}(H) + G_3(R)$. By verifying the conditions of Theorem 1 in Sobel (1981) as we did for the case of $K = 0$, it follows that minimizing a myopic cost function given by

$$\hat{J}(R) = G_3(R) + \alpha E_{X_0} \hat{F}(R - X_0) \tag{23}$$

attains optimality for the infinite horizon model.

To minimize $\hat{J}(R)$, we observe that $\hat{J}(R)$ is not a convex function as $\hat{F}(H)$ is not. However, we see that $\mathrm{D}\hat{F}(H) < \mathrm{D}F(H)$ for $H \in (r_U, r^*)$ and $\mathrm{D}\hat{F}(H) = \mathrm{D}F(H)$ for $H < r_U$ or $H \geqslant r^*$. This implies that $\mathrm{D}\hat{J}(R) \leqslant \mathrm{D}J(R)$. If R^* is the optimal order-up-to level that minimizes $J(R)$, then $\mathrm{D}J(R^*) = 0$ if $R^* \neq \underline{R}$ (the following theorem is also true if $R^* = \underline{R}$), which indicates $\mathrm{D}\hat{J}(R) \leqslant \mathrm{D}J(R) < 0$ for $R < R^*$ (due to the convexity of $J(R)$). This completes the proof the following theorem.

Theorem 3. *The optimal order-up-to level minimizing $\hat{J}(R)$ is greater than or equal to the optimal order-up-to level minimizing $J(R)$.*

To obtain the optimum R^* that minimizes $\hat{J}(R)$, Theorem 3 gives a lower bound on R^* while a large value of R serves as an upper bound. Thus, we can perform a simple search on the values of R between these two bounds and compare $\hat{J}(R)$. In summary, we use the following method to obtain the optimal policy parameters r_U, r^* and R^*.

Step 1. Obtain r^* by using the same procedure as in the cost of $K = 0$.

Step 2. Decrease r from r^* until $r = r_U$ where $G_2(r_U) \leqslant K + G_2(r^*)$ and $G_2(r_U - 1) > K + G_2(r^*)$. As a note, if there does not exist r_U, indicating $K \geqslant G_2(\hat{r}) - G_2(r^*)$, the emergency channel is never used.

Step 3. Set on R^* a lower bound (e.g., $\underline{R}$ or the value of R minimizing $J(R)$) as well as an upper bound. Search exhaustively on the values of R between these two bounds and select the R minimizing $\hat{J}(R)$.

Likewise, we can derive the optimal ordering policy for the case of $c_2 = c_1$. As we see from Fig. 3, if $K \geqslant G_2(0) - G_2(r^*)$, the emergency supply channel will never be used. We assume that $K < G_2(0) - G_2(r^*)$. Define $r_B = \inf\{r > 0 | G_2(r) \leqslant K + G_2(r^*)\}$. Then the optimal policy for the infinite horizon model is (i) if $H < r_B$, order amounts $r^* - H$ and $R^* - r^*$ at unit cost c_2 (plus a fixed cost of K) and c_1 respectively, and (ii) otherwise if $H \in [r_B, R^*]$, order an amount $R^* - H$ at unit cost c_1. To obtain the optimal order-up-to level R^* in this case, we also minimize (23) except that $\hat{F}(H) \equiv K + G_1(H) + G_2(r^*)$ for $H < r_B$ and $\hat{F}(H) \equiv G_1(H) + G_2(H)$ for $H \geqslant r_B$. A lower bound on R^* is obtained by pretending $K = 0$ and solving the resulting convex function of (23) for the optimal R. A similar solution procedure (to the one for the case of $c_2 > c_1$) also can be employed.

6. Computational results

In this section, we present some computational results for the optimal ordering policies we develop in Sections 4 and 5.

We first assume that the fixed ordering cost K is zero. We investigate the effect of demand variance, unit shortage cost, emergency unit cost, and carrying cost/unit/unit time on the performance of the proposed model, as compared to that of the regular-mode-only model. Table 2 gives the sensitivity results for the first

Table 2
Computational results: sensitivity to demand variance and unit shortage cost

Parameters		Regular-mode-only model		Proposed model				Savings (%)
σ^2	π	R^*	$J(R^*)$	r_0	r^*	R^*	$J(R^*)$	
10,000	$1.25	310	2701.7	[a]				
	2.5	416	2781.0	−25	136	415	2775.5	0.20
	5	489	2844.4	−200	195	458	2821.5	0.81
	10	545	2897.0	−658	234	491	2855.3	1.44
	40	629	2980.6	−3591	290	542	2909.6	2.38
	70	657	3008.7	−6568	309	559	2928.3	2.67
	100	673	3025.4	−9554	320	568	2939.4	2.84
2500	1.25	343	2671.1	[a]				
	2.5	404	2717.2	−42	144	387	2709.8	0.27
	5	444	2751.8	−265	173	406	2729.5	0.81
	10	474	2779.8	−744	192	422	2746.9	1.18
	40	519	2825.2	−3711	220	447	2775.2	1.77
	70	533	2841.0	−6699	230	457	2785.1	1.97
	100	542	2850.5	−9692	235	462	2791.1	2.08
1250	1.25	350	2662.5	[a]				
	2.5	402	2698.4	−53	146	381	2689.8	0.31
	5	431	2723.2	−284	166	395	2705.3	0.66
	10	452	2743.1	−770	180	405	2717.7	0.93
	40	484	2775.2	−3746	200	423	2737.7	1.35
	70	494	2786.4	−6738	206	429	2744.6	1.50
	100	501	2793.1	−9733	210	433	2748.8	1.59

Data: $\lambda = 250.0$, $T = 1.0$, $\tau_1 = 0.6$, $\tau_2 = 0.2$, $K = \$0$, $h = \$1.0$/unit/unit time, $c_1 = \$10$, $c_2 = \$11$, $\alpha = 0.98$.
[a] Indicates that only the regular supply mode is used.

Table 3
Computational results: sensitivity to emergency unit cost and carrying cost/unit/unit time

Parameters		Regular-mode-only model		Proposed model				Savings (%)
c_2	h	R^*	$J(R^*)$	r_0	r^*	R^*	$J(R^*)$	
\$10.5	\$0.5	533	2700.8	−7697	232	448	2664.0	1.36
	1.0	519	2825.2	−7645	228	427	2753.8	2.53
	2.0	501	3057.8	−7545	221	407	2918.3	4.56
11.0	0.5	533	2700.8	−3735	223	478	2677.4	0.87
	1.0	519	2825.2	−3711	220	447	2775.2	1.77
	2.0	501	3057.8	−3664	216	418	2947.4	3.61
12.0	0.5	533	2700.8	−1759	212	502	2687.3	0.50
	1.0	519	2825.2	−1748	210	477	2795.5	1.05
	2.0	501	3057.8	−1726	207	442	2986.0	2.35

Data: $\lambda = 250.0$, $\sigma^2 = 2500$, $T = 1.0$, $\tau_1 = 0.6$, $\tau_2 = 0.2$, $K =\$0$, $c_1 =\$10$, $\pi =\$40$/unit, $\alpha = 0.98$.

two parameters and Table 3 gives the results for the last two parameters. The average CPU time of obtaining the optimal order-up-to level for the proposed model is 0.33 seconds on an IBM 3081. Demand is assumed to be normal with mean $\lambda\tau$ and variance $\sigma^2\tau$ for a time interval of length τ.

It is clear from Table 2 that as the unit shortage cost π increases (other things being equal), the percentage savings of the proposed model in comparison to the regular-mode-only model increases, i.e., the proposed model becomes more attractive. The same phenomenon is also observed by the changes in the demand variance σ^2. Other things being equal, the proposed model performs better as σ^2 increases. This is because a small amount of safety stock may be enough to avoid stockouts if demand variability is low. On the other hand, the use of a faster supply mode may also be needed in addition to carrying a relatively large amount of safety stock if demand variability is high. These findings agree with the ones in Chiang and Gutierrez (1996). Notice that r_0 is negative in our computation and $Pr(H \leqslant r_0)$ is virtually zero.

Next, we see from Table 3 that other things being equal, the percentage savings of the proposed model increases as the carrying cost h per unit and unit time increases. Intuitively, if h is high, it is not advantageous to carry a large amount of inventory for a long period of time and thus the use of the emergency mode may be needed. In addition, we observe from Table 3 that as the emergency unit cost c_2 increases (other things being equal), the proposed model becomes less attractive. Accompanying this is the result that as c_2 increases, the optimum R^* of the proposed model increases while the optimum r^* decreases, reflecting the fact that the probability of using the emergency mode, i.e., $Pr(X_0 > R^* - r^*)$, becomes smaller. It can be seen from (7) that as c_2 increases beyond a certain value (other things being equal), $\mathrm{D}G_2(r) \geqslant 0$, i.e., only the regular mode is used.

We next assume that there is a positive fixed ordering cost K. Intuitively, if $K > 0$ (other things being equal), the expected cost over an infinite horizon will be larger than when $K = 0$. For example, if we compare the savings of the case of $\tau_1 = 0.6$ and $c_2 =\$11$ in Table 4 to the savings of the same case in Table 3, the savings is down from 1.77% to below 1.57%; and the larger the K (other things being equal), the smaller the savings of the proposed model. It is of no doubt that if the incremental ordering cost and/or unit cost is larger, the emergency mode is of less value to employ. Note that r_{L} is negative in the computation and $Pr(H \leqslant r_{\mathrm{L}})$ is virtually zero.

Finally, for the case of $c_1 = c_2$, we compare the performance of the proposed model and Chiang and Gutierrez's model. To facilitate the comparison, we compute the expected cost per period for the proposed model (using the optimal policy parameters r_{B}, r^* and R^* obtained for minimizing the expected discounted cost over an infinite horizon) as follows:

$$E_{X_0}\hat{C}(H,R) = E_{X_0}\hat{F}(R - X_0) + G_3(R). \tag{24}$$

Table 4
Computational results: optimal ordering policy when fixed ordering cost is positive

Parameters			Regular-mode-only model		Proposed model					Savings (%)
τ_1	c_2	K	R^*	$J(R^*)$	r_L	r_U/r_B	r^*	R^*	$\hat{J}(R^*)$	
0.4	\$10	\$10	461	2804.9	–	158	181	408	2747.4	2.05
		40	461	2804.9	–	144	181	413	2759.6	1.62
		160	461	2804.9	–	123	181	434	2783.2	0.77
		640	461	2804.9	–	93	181	455	2800.9	0.14
	11	10	461	2804.9	−1799	147	159	421	2767.8	1.32
		40	461	2804.9	−1769	136	159	427	2775.2	1.06
		160	461	2804.9	−1649	118	159	441	2789.4	0.55
		640	461	2804.9	−1169	90	159	457	2801.9	0.11
0.6	\$10	\$10	519	2825.2	–	219	240	413	2729.9	3.37
		40	519	2825.2	–	204	240	418	2753.8	2.53
		160	519	2825.2	–	181	240	484	2803.3	0.78
		640	519	2825.2	–	147	240	513	2821.5	0.13
	11	10	519	2825.2	−3701	206	220	453	2780.9	1.57
		40	519	2825.2	−3671	195	220	469	2793.0	1.14
		160	519	2825.2	−3551	175	220	497	2810.8	0.51
		640	519	2825.2	−3071	143	220	514	2822.3	0.10
0.8	\$10	\$10	576	2844.5	–	277	297	419	2732.9	3.92
		40	576	2844.5	–	261	297	420	2761.6	2.91
		160	576	2844.5	–	237	297	548	2827.7	0.59
		640	576	2844.5	–	201	297	571	2841.9	0.09
	11	10	576	2844.5	−5591	264	278	517	2809.5	1.23
		40	576	2844.5	−5561	252	278	535	2819.5	0.88
		160	576	2844.5	−5441	231	278	559	2833.6	0.38
		640	576	2844.5	−4961	196	278	573	2842.6	0.07

Data: $\lambda = 250.0$, $\sigma^2 = 2500$, $T = 1.0$, $\tau_2 = 0.2$, $h =$ \$1.0/unit/unit time, $c_1 =$ \$10, $\pi =$ \$40/unit, $\alpha = 0.98$.

Table 5
Comparison of computational results obtained in Chiang and Gutierrez (1996) and this paper

Parameters		Chiang and Gutierrez's model			Proposed model				Savings (%)
τ_1	π	I[a]	R^*	Cost per period	r_B	r^*	R^*	Cost per period	
0.4	\$10	99	385	189.3	110	146	360	178.5	5.71
	40	120	403	206.0	128	158	384	194.6	5.53
	70	126	410	211.7	134	162	392	200.5	5.29
0.6	10	141	448	199.7	163	198	356	173.7	13.02
	40	169	471	220.7	185	214	381	191.3	13.32
	70	178	478	227.7	192	218	390	197.5	13.26
0.8	10	183	507	206.9	214	249	360	176.8	14.55
	40	218	534	231.3	240	269	384	194.7	15.82
	70	228	542	239.4	248	275	392	201.3	15.91

Data: $\lambda = 250.0$, $\sigma^2 = 1250$, $T = 1.0$, $\tau_2 = 0.2$, $h =$ \$1.0/unit/unit time, $K =$ \$40.
[a] *Note*: I is the critical inventory level below which an emergency order is placed.

Expression (24) excludes the item cost (by subtracting $c_1\lambda$), since Chiang and Gutierrez's model does not include it (as it is constant). As we see from Table 5, the proposed model yields smaller costs than Chiang and Gutierrez's model, given that all parameters are the same.

7. Concluding remarks

In this paper, we develop dynamic programming models for an inventory system where regular orders as well as emergency orders can be placed periodically. We identify two important cases depending on whether or not a fixed cost for placing an emergency order is present. We show that if the emergency supply channel can be used, there exists a critical inventory level such that if the inventory position at a review epoch falls below this level, an emergency order is placed. We also develop simple procedures for computing the optimal policy parameters. In all cases, the optimal order-up-to level is obtained by solving a myopic cost function. Thus, the proposed ordering policies are easy to implement.

Notice that we assume throughout the entire analysis that the cost of placing a regular order is zero. If there is also a positive fixed cost for placing a regular order, the ordering policies developed in this paper are no longer optimal. The reason is that the policies developed will place at least a regular order as long as the demand during the preceding period is not zero. However, it is obvious that we do not order at all if the cost of placing a regular order is positive and the demand during the preceding period is very small (which is theoretically possible as demand is stochastic). For example, if the demand during the preceding period is only a few units compared to a mean demand of several hundred, then ordering these few units at a review epoch is apparently not economical. This is just like the ordinary case of using only the regular supply mode. The ordering-up-to-R policy is optimal with zero ordering costs. On the other hand, the ordering-up-to-R policy is no longer optimal with positive ordering costs. There exists an additional parameter that triggers the placement of an order (it is well known that the (s, S) type policy is then optimal).

How to employ the two supply modes when a fixed cost of placing a regular order is also present provides a future research direction. A preliminary investigation shows that the form of the optimal ordering policy is rather complex. Nevertheless, since the review periods studied in this paper are long, ordering always up to R^* (after a possible emergency order is placed) as described in Section 5 should be a nearly optimal policy if the fixed cost of placing a regular order is not large (see, e.g., Hax and Candea, 1984), which is especially true nowadays when information technology such as EDI has increasingly gained use in industry.

Appendix

We show that for normal or Poisson demand, there is at most one positive r, denoted by r_1, equating (8) to zero.

For normal demand (with mean $\lambda\tau$ and variance $\sigma^2\tau$ for a time interval of length τ), (8) reduces to

$$\begin{aligned}\mathrm{DD}G_2(r) &= h\tau_3 f_2(r)\{1.0 - (\lambda\tau_3(\lambda\tau_2 - r)/2\sigma^2\tau_2)\} + \pi\{f_1(r) - f_2(r)\} \\ &= \pi f_2(r)\big\{(h\tau_3/\pi) - (h\tau_3\lambda\tau_3(\lambda\tau_2 - r)/2\sigma^2\tau_2\pi) + (f_1(r)/f_2(r)) - 1.0\big\} \\ &= \pi f_2(r)\Big\{(h\tau_3/\pi) - (h\tau_3\lambda\tau_3(\lambda\tau_2 - r)/2\sigma^2\tau_2\pi) \\ &\qquad + (t_1/t_2)(\tau_2/\tau_1)^{0.5}\exp\{0.5\tau_3(r^2 - \lambda^2\tau_1\tau_2)/\sigma^2\tau_1\tau_2\} - 1.0\Big\}\end{aligned} \tag{A.1}$$

where t_1 and t_2 are normalizing constants (greater than 1) as demand is non-negative. As the expression within the big parentheses increases on r, there exists at most one positive r that can equate (A.1) to zero.

For Poisson demand, (8) (interpreted as the second difference) is expressed by

$$\begin{aligned} & h\tau_3 P_2(r)\{1.0 - (\tau_3(\lambda\tau_2 - r)/2\tau_2)\} + \pi\{P_1(r) - P_2(r)\} \\ & \quad = \pi P_2(r)\{(h\tau_3/\pi) - (h\tau_3\tau_3(\lambda\tau_2 - r)/2\tau_2\pi) + (P_1(r)/P_2(r)) - 1.0\} \\ & \quad = \pi P_2(r)\{(h\tau_3/\pi) - (h\tau_3\tau_3(\lambda\tau_2 - r)/2\tau_2\pi) + (\tau_1/\tau_2)^r \exp\{-\lambda\tau_3\} - 1.0\} \end{aligned} \tag{A.2}$$

where $P_1(\cdot)$ (resp. $P_2(\cdot)$) is the Poisson density function for demand during the time interval $[kT, kT + \tau_1)$ (resp. $[kT, kT + \tau_2)$). As before, the expression within the big parentheses increases on r. Thus, there exists at most one positive r that can equate (A.2) to zero.

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