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Super-connectivity and super-edge-connectivity for some interconnection networks ☆

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Abstract

Let $G = (V, E)$ be a k-regular graph with connectivity κ and edge connectivity λ. G is *maximum connected* if $\kappa = k$, and G is *maximum edge connected* if $\lambda = k$. Moreover, G is *super-connected* if it is a complete graph, or it is maximum connected and every minimum vertex cut is $\{x|(v,x) \in E\}$ for some vertex $v \in V$; and G is *super-edge-connected* if it is maximum edge connected and every minimum edge disconnecting set is $\{(v,x)|(v,x) \in E\}$ for some vertex $v \in V$. In this paper, we present three schemes for constructing graphs that are super-connected and super-edge-connected. Applying these construction schemes, we can easily discuss the super-connected property and the super-edge-connected property of hypercubes, twisted cubes, crossed cubes, möbius cubes, split-stars, and recursive circulant graphs.

Keywords: Connectivity; Edge connectivity; Super-connectivity; Super-edge-connectivity

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1. Introduction

For the graph definitions and notations we follow [2]. $G = (V, E)$ is a simple graph if V is a finite set and E is a subset of $\{(a,b)|(a,b)$ is an *unordered pair of* $V\}$. We say that V is the *vertex set* and E is the *edge set*. The *neighborhood* of v, $N_G(v)$, is $\{x|(v,x) \in E\}$. The *neighbor-edge* of v, $NE_G(v)$, is $\{(v,x)|(v,x) \in E\}$. The *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of vertices in $N_G(v)$. A graph G is k-regular if $\deg_G(v) = k$, for every vertex $v \in V$.

A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one connected component. It is known that only complete graphs do not have vertex cuts. The *connectivity* of G, written $\kappa(G)$, is defined as the minimum size of a vertex cut if G is not a complete graph, and $\kappa(G) = |V(G)| - 1$ if otherwise. A graph G is k-connected if $k \leqslant \kappa(G)$. Assume that G is a k-regular graph with connectivity κ. We say that G is *maximum connected* if $\kappa = k$; and G is *super-connected* if it is a complete graph, or it is maximum connected and every minimum vertex cut is $N_G(v)$ for some vertex v.

An *edge disconnecting set* is a set $F \subseteq E(G)$ such that $G - F$ has more than one connected component. A graph is k-edge-connected if every disconnecting set has at least k edges. The *edge connectivity* of G, written $\lambda(G)$, is the minimum size of an edge disconnecting set. Assume that G is a k-regular graph with edge connectivity λ. A graph G is k-edge-connected if $k \leqslant \lambda(G)$. We say that G is *maximum edge connected* if $\lambda = k$; and G is *super-edge-connected* if it is maximum edge connected and every minimum edge disconnecting cut is $NE_G(v)$ for some vertex v.

The architecture of an interconnection network is usually represented by a graph. There are numerous mutually conflicting requirements in designing the topology of interconnection networks. Network reliability is one of the major factors in designing the topology of an interconnection network. It has been shown that a network is more reliable if it is super-connected [3,4,9]. Some important families of interconnection networks have been proven to be super-connected [3,4,9]. In this paper, we present three schemes to construct super-connected and super-edge-connected graphs. With these construction schemes, we can easily discuss the super-connected property and the super-edge-connected property of hypercubes, twisted cubes, crossed cubes, möbius cubes, split-stars, and recursive circulant graphs.

2. The first constructing scheme

Assume that t is a positive integer. Let G_1 and G_2 be two graphs with t vertices, and M be any arbitrary *perfect matching* between the vertices of G_1 and G_2; i.e., a set of t edges with one endpoint in G_1, and the other endpoint in G_2. The graph $G(G_1, G_2; M)$ is defined as a graph with the vertex set

$V(G(G_1, G_2; M)) = V(G_1) \cup V(G_2)$, and edge set $E(G(G_1, G_2; M)) = E(G_1) \cup E(G_2) \cup M$. We note that the cartesian product of a graph H and a complete graph K_2 can be viewed as a $G(H, H; M)$ for some M.

Theorem 1. *Assume that t is a positive integer. Let G_1 and G_2 be two k-regular maximum connected graphs with t vertices, and M be any perfect matching between $V(G_1)$ and $V(G_2)$. Then, $G(G_1, G_2; M)$ is $(k+1)$-regular super-connected if and only if* (1) $t > k+1$ *or* (2) $t = k+1$ *with* $k = 0, 1, 2$.

Proof. Since G_1 and G_2 are k-regular connected graphs, $t \geqslant k+1$. By definition, $G(G_1, G_2; M)$ is a $(k+1)$-regular graph. To prove $G(G_1, G_2; M)$ is super-connected, we need to check if $G(G_1, G_2; M) - F$ is connected for any $F \subset V(G(G_1, G_2; M))$ such that $|F| = k+1$ and $F \neq N_{G(G_1,G_2;M)}(v)$ for any vertex $v \in V(G(G_1, G_2; M))$.

Suppose that $t = k+1$. Obviously, G_1 and G_2 are isomorphic to the complete graph K_{k+1}. Moreover, $G(G_1, G_2; M)$ is isomorphic to the cartesian product of K_{k+1} and K_2. Without loss of generality, we assume that $V(G_1) = \{a_0, a_1, \ldots, a_k\}$ and $V(G_2) = \{b_0, b_1, \ldots, b_k\}$, where b_i is the vertex matched with a_i under M for every i. By brute force, we can check that $G(G_1, G_2; M)$ is super-connected for $k = 0, 1, 2$. When $k \geqslant 3$, we set $F = \{a_0, a_1\} \cup \{b_i | 2 \leqslant i \leqslant k\}$. It is easy to see that $|F| = k+1$, $F \neq N_{G(G_1,G_2;M)}(v)$, and F is a vertex cut of $G(G_1, G_2; M)$. So $G(G_1, G_2; M)$ is not super-connected.

Now, assume that $t > k+1$. Since K_1 is the only connected 0-regular graph and K_2 is the only connected 1-regular graph, let $k \geqslant 2$. We set $X_1 = F \cap V(G_1)$ and $X_2 = F \cap V(G_2)$.

Case 1. $|X_1| < k$ and $|X_2| < k$. Thus, both $G_1 - X_1$ and $G_2 - X_2$ are connected. Since $t = |M| > k+1$ and $|F| = k+1$, there exists $a \in V(G_1) - F$ and $b \in V(G_2) - F$ such that $(a, b) \in M$. Thus, $G(G_1, G_2; M) - F$ is connected.

Case 2. Either $k \leqslant |X_1| \leqslant k+1$ or $k \leqslant |X_2| \leqslant k+1$. We assume without loss of generality that $k \leqslant |X_1| \leqslant k+1$. Hence, $|X_2| \leqslant 1$. Since $k \geqslant 2$, $G_2 - X_2$ is connected. Let C be any connected component of $G_1 - X_1$. We will claim that there exists $a \in C$ and $b \in V(G_2) - F$ such that $(a, b) \in M$. With this claim, $G(G_1, G_2; M) - F$ is connected.

First, if C consists of only one vertex a, then $N_{G_1}(a) \subset F$. Let b be the vertex in G_2 with $(a, b) \in M$. Since $F \neq N_{G(G_1,G_2;M)}(a)$, $b \in V(G_2) - F$. Thus, the claim holds. Now, if C contains at least two vertices a and a'. Let b, b' be the matched vertices of a, a' in G_2, respectively. Since at most one vertex of G_2 is in F, we may assume $b \notin F$. Thus, our claim holds.

Therefore, the theorem is proved. □

A similar argument leads to the following theorem for super-edge-connected, and the corollary.

Theorem 2. *Assume that t is a positive integer. Let* G_1 *and* G_2 *be two k-regular maximum-edge-connected graphs with t vertices, and M is any perfect matching between* $V(G_1)$ *and* $V(G_2)$. *Then,* $G(G_1, G_2; M)$ *is* $(k+1)$*-regular super-edge-connected if and only if* (1) $t > k+1$ *or* (2) $t = k+1$ *with* $k = 0$.

Corollary 1. *Assume that t is a positive integer. Let* G_1 *and* G_2 *be two k-connected and k′-edge connected graphs with t vertices, and M is any perfect matching between* $V(G_1)$ *and* $V(G_2)$. *Then,* $G(G_1, G_2; M)$ *is* $(k+1)$*-connected and* $(k'+1)$*-edge connected.*

Network topology is always represented by a graph where vertices represent processors and edges represent links between processors. Among these topologies, the binary hypercube [7], Q_n, is one of the most popular topology. The hypercube Q_n can be recursively defined as $Q_1 = K_2$ and Q_n is the cartesian product of Q_{n-1} and K_2. The super-connected property and the super-edge-connected property of hypercubes are discussed in [4,9]. Here, we reprove this result. Recursively applying Theorems 1 and 2, we can easily prove that Q_n is super-connected for every n and super-edge-connected if $n \neq 2$.

Twisted cubes [1], crossed cubes [6], and möbius cubes [5] are derived by changing the connection of some hypercube links according to some specified rules.

In [1], the twisted n-cube TQ_n is defined for odd values of n. The vertex set of the twisted n-cube TQ_n is the set of all binary strings of length n. Let $u = u_{n-1}u_{n-2}\dots u_1u_0$ be any vertex in TQ_n. For $0 \leqslant i \leqslant n-1$, let the ith parity function be $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive-or operation. We can recursively define TQ_n as follows: A twisted 1-cube, TQ_1, is a complete graph with two vertices 0 and 1. Suppose that $n \geqslant 3$. We can decompose the vertices of TQ_n into four sets, $TQ^{0,0}_{n-2}$, $TQ^{0,1}_{n-2}$, $TQ^{1,0}_{n-2}$ and $TQ^{1,1}_{n-2}$ where $TQ^{i,j}_{n-2}$ consists of those vertices u with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i,j) \in \{(0,0),(0,1),(1,0),(1,1)\}$, the induced subgraph of $TQ^{i,j}_{n-2}$ in TQ_n is isomorphic to TQ_{n-2}. The edges that connect these four subtwisted cubes can be described as follows: Any vertex $u_{n-1}u_{n-2}\cdots u_1u_0$ with $P_{n-3}(u) = 0$ is connected to $\bar{u}_{n-1}\bar{u}_{n-2}\cdots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\dots u_1u_0$; and to $u_{n-1}\bar{u}_{n-2}\dots u_1u_0$ and $\bar{u}_{n-1}u_{n-2}\dots u_1u_0$ if $P_{n-3}(u) = 1$.

From the definition, both the subgraph induced by $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and the subgraph induced by $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ are isomorphic to $TQ_{n-2} \times K_2$, where K_2 is the complete graph with two vertices. Moreover, the edges joining $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ form a perfect matching of TQ_n. Recursively applying Theorems 1 and 2, we can easily prove that TQ_n is super-connected and super-edge-connected for every odd n.

Two two-digit binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair related, denoted by $x \sim y$, if and only if $(x,y) \in \{(00,00),(10,10),(01,11),(11,01)\}$. An n-dimension crossed cube CQ_n [6] is a graph $CQ_n = (V,E)$ that is recursively constructed as follows: CQ_1 is a complete graph with two vertices labeled by 0 and 1. CQ_n consists of two identical $(n-1)$-dimension crossed cubes, CQ^0_{n-1} and CQ^1_{n-1}. The vertex $u = 0u_{n-2}\cdots u_0 \in V(CQ^0_{n-1})$ and vertex $v = 1v_{n-2}\cdots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if (1) $u_{n-2} = v_{n-2}$ if n is even; and (2) for $0 \leqslant i < \lfloor (n-1)/2 \rfloor$, $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$.

From the definition, CQ_n can be viewed as $G(CQ_{n-1}, CQ_{n-1}; M)$ for some perfect matching M. Recursively applying Theorems 1 and 2, we can easily prove that CQ_n is super-connected for every n and super-edge-connected if $n \neq 2$.

The möbius cube [5], $MQ_n = (V,E)$, of dimension n has 2^n vertices. Each vertex is labeled by a unique n-bit binary string as its address and has connections to n other distinct vertices. The vertex with address $X = x_{n-1}x_{n-2}\cdots x_0$ connects to n other vertices Y_i, $0 \leqslant i \leqslant n-1$, where the address of Y_i satisfies (1) $Y_i = (x_{n-1}\cdots x_{i+1}\bar{x}_i\cdots x_0)$ if $x_{i+1} = 0$; or (2) $Y_i = (x_{n-1}\cdots x_{i+1}\overline{x_i\cdots x_0})$ if $x_{i+1} = 1$.

From the above definition, X connects to Y_i by complementing the bit x_i if $x_{i+1} = 0$, or by complementing all bits of $x_i\cdots x_0$ if $x_{i+1} = 1$. For the connection between X and Y_{n-1}, we can assume that the unspecified x_n is either 0 or 1, which gives slightly different topologies. If x_n is 0, we call the network generated the "0-möbius cube", denoted by 0-MQ_n; and if x_n is 1, we call the network generated the "1-möbius cube", denoted by 1-MQ_n.

According to the above definition, 0-MQ_{n+1} and 1-MQ_{n+1} can be recursively constructed from a 0-MQ_n and a 1-MQ_n by adding a perfect matching. Recursively applying Theorems 1 and 2, we can easily prove that every 0-MQ_n or 1-MQ_n is super-connected for every n and super-edge-connected if $n \neq 2$.

Assume n is a positive integer. The *alternating graph* A_n [3] is an attractive interconnection graph topology. $V(A_n) = \{p \mid p = p_1p_2\cdots p_{n-2}$ with $p_i \in \{1, 2, \ldots, n\}$ for $1 \leqslant i \leqslant n-2$ and $p_i \neq p_j$ if $i \neq j\}$ and $E(A_n) = \{(p,q) \mid$ there exists a unique $i \in \langle n-2 \rangle$ such that $p_i \neq q_i\}$. In [3], *split-star* S^2_n is proposed as an attractive interconnection network. $V(S^2_n) = \{p \mid p = p_0p_1p_2\cdots p_{n-2}$ with $p_0 \in \{0,1\}$, $p_i \in \{1,2,\ldots,n\}$ for $1 \leqslant i \leqslant n-2$, and $p_i \neq p_j$ if $i \neq j \neq 0\}$ and $E(S^2_n) = \{(p,q) \mid$ there exists a unique i with $0 \leqslant i \leqslant n-2$ such that $p_i \neq q_i\}$.

It is pointed out in [3] that S^2_n can be viewed as $G(A_n, A_n; M)$. Moreover, it is proved that A_n is super-connected unless $n = 4$ and super-edge-connected for any n. Applying Theorems 1 and 2, we can easily reprove the result in [3] that S^2_n is super-connected for any n and super-edge-connected unless $n = 3$.

3. The second constructing scheme

Let r and t be positive integers with $r \geqslant 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are graphs with $|V(G_i)| = t$ for $0 \leqslant i \leqslant r-1$. We define $H = G(G_0, G_1, \cdots, G_{r-1}; \mathscr{M})$

with $V(H) = V(G_0) \cup V(G_1) \cup \ldots \cup V(G_{r-1})$ and $E(H) = \mathscr{M} \cup \bigcup_{i=0}^{r-1} E(G_i)$, where $\mathscr{M} = \bigcup_{i=0}^{r-1} M_{i,i+1\,(\mathrm{mod}\,r)}$ with $M_{i,i+1\,(\mathrm{mod}\,r)}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1\,(\mathrm{mod}\,r)})$.

Theorem 3. *Let r and t be positive integers with $r \geqslant 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are k-regular maximum connected graphs with $|V(G_i)| = t$ for $0 \leqslant i \leqslant r-1$ and $\mathscr{M} = \bigcup_{i=0}^{r-1} M_{i,i+1\,(\mathrm{mod}\,r)}$, where $M_{i,i+1\,(\mathrm{mod}\,r)}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1\,(\mathrm{mod}\,r)})$. Then, $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ is $(k+2)$-regular super-connected if and only if* (1) $k \geqslant 1$ *or* (2) $k = 0$ *and $r = 3, 4, 5$.*

Proof. Because $G_0, G_1, \ldots, G_{r-1}$ are k-regular connected graphs, $t \geqslant k+1$, by definition, H is a $(k+2)$-regular graph. To prove H is super-connected, we need to check if $H - F$ is connected for any vertex subset F of H such that $|F| = k+2$ and $F \neq N_H(v)$ for any vertex $v \in V(H)$. We set $X_i = F \cap V(G_i)$ for $i = 0, 1, \ldots, r-1$.

Case 1. $|X_i| < k$ for $i = 0, 1, \cdots, r-1$. Then, $G_i - X_i$ is connected for every i.

Suppose that $|X_i \cup X_{i+1\,(\mathrm{mod}\,r)}| \leqslant k$ for $0 \leqslant i \leqslant r-1$. By Theorem 1, there exist $x_i \in V(G_i) - F$ and $x_{i+1\,(\mathrm{mod}\,r)} \in V(G_{i+1\,(\mathrm{mod}\,r)}) - F$ such that $(x_i, x_{i+1\,(\mathrm{mod}\,r)}) \in M_{i,i+1\,(\mathrm{mod}\,r)}$. Thus, $H - F$ is connected.

Suppose that $k+1 \leqslant |X_i \cup X_{i+1\,(\mathrm{mod}\,r)}| \leqslant k+2$ for some i. Without loss of generality, we may assume that $k+1 \leqslant |X_0 \cup X_1| \leqslant k+2$ and $|X_0| \geqslant |X_1|$. Since $|X_1| < k$, $|X_0| \geqslant 2$. Thus, $|X_i \cup X_{i+1\,(\mathrm{mod}\,r)}| \leqslant k$ for $1 \leqslant i \leqslant r-1$. Suppose that $k \leqslant 1$. Then, $|X_i| = 0$ for every i. Since $F = \bigcup_{i=1}^{r-1} X_i$, $|F| = 0$. This is impossible. Thus, $k \geqslant 2$. By Theorem 1, $G(G_i, G_{i+1\,(\mathrm{mod}\,r)}; M_{i,i+1\,(\mathrm{mod}\ r)}) - (X_i \cup X_{i+1\,(\mathrm{mod}\,r)})$ is connected for $1 \leqslant i \leqslant r-1$. Hence, $H - F$ is connected.

Case 2. $|X_i| \geqslant k$ for some i. Without loss of generality, we assume that $k \leqslant |X_0| \leqslant k+2$, $|X_0| \geqslant |X_i|$ for $1 \leqslant i \leqslant r-1$, and $|X_1| \geqslant |X_{r-1}|$.

Subcase 2.1. $t > k+1$. Thus, G_i is not a complete graph. Hence, $k > 1$. By Theorem 1, $G(G_i, G_{i+1\,(\mathrm{mod}\,r)}; M_{i,i+1\,(\mathrm{mod}\,r)})$ is $(k+1)$-regular super-connected for $0 \leqslant i \leqslant r-1$.

Suppose that $|X_0 \cup X_1| = k$. Then, $|X_0| = k$, $|X_1| = |X_{r-1}| = 0$, and $|X_i \cup X_{i+1}| \leqslant k$ for $0 \leqslant i \leqslant r-2$. Then, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $0 \leqslant i \leqslant r-2$. Hence, $H - F$ is connected.

Suppose that $|X_0 \cup X_1| = k+1$. Then, $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is connected unless $X_0 \cup X_1 = N_{G(G_0,G_1;M_{0,1})}(x)$ for some vertex $x \in V(G_0) \cup V(G_1)$.

Suppose that $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is connected. $H - F$ is connected because $G(G_i, G_{i+1\,(\mathrm{mod}\,r)}; M_{i,i+1\,(\mathrm{mod}\,r)}) - (X_i \cup X_{i+1\,(\mathrm{mod}\,r)})$ is also connected for $1 \leqslant i \leqslant r-1$.

Suppose that $X_0 \cup X_1 = N_{G,(G_0,G_1;M_{0,1})}(x)$ for some vertex $x \in V(G_0) \cup V(G_1)$. Since $|X_0| \geqslant k, x \in V(G_0)$. Let x_1 be the vertex in $V(G_1)$ such that $(x, x_1) \in M_{0,1}$ and x_{r-1} be the vertex in $V(G_{r-1})$ such that $(x_{r-1}, x) \in M_{r-1,0}$. Obviously, x_1 is the only vertex in X_1. Since G_1 is k-connected with $k > 1$, $G_1 - \{x_1\}$ is connected. Let y be any vertex of $V(G_0)$ with $y \neq x$, and y_1 be the vertex in $V(G_1)$ such that $(y, y_1) \in M_{0,1}$. Hence, $y_1 \notin F$ since $y_1 \neq x_1$. Therefore, $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1 \cup \{x\})$ is connected. Since $F \neq N_H(x), x_{r-1} \notin F$. Since $|X_i| \leqslant 1$ for $1 \leqslant i \leqslant r-1$, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $1 \leqslant i \leqslant r-2$ follows from Theorem 1. Thus, $H - F$ is connected.

Suppose that $|X_0 \cup X_1| = k+2$. Then, $|X_i| = 0$ for $2 \leqslant i \leqslant r-1$. Let x_0 be any vertex of $V(G_0) - F$ and x_{r-1} be the vertex such that $(x_{r-1}, x_0) \in M_{r-1,0}$. Obviously, $x_{r-1} \notin F$. Similarly, let x_1 be any vertex of $V(G_1) - F$ and x_2 be the vertex such that $(x_1, x_2) \in M_{1,2}$. Obviously, $x_2 \notin F$. Since $|X_i| = 0$ for $2 \leqslant i \leqslant r-1$, either $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $2 \leqslant i \leqslant r-2$ with $r > 3$ or $G_2 - X_2$ is connected with $r = 3$. Hence, $H - F$ is connected.

Subcase 2.2. $t = k+1$. Thus, every G_i is isomorphic to complete graph K_{k+1}. Since K_{k+1} contains $k+1$ vertices, $k \leqslant |X_0| \leqslant k+1$.

Suppose that $k = 0$. Then, H is isomorphic to the cycle C_r. It is easy to check H is super-connected if and only if $r = 3, 4, 5$. Thus, we consider $k \geqslant 1$.

Suppose that $|X_0| = k+1$. Thus, $|\bigcup_{i=1}^{r-1} X_i| = 1$. By Corollary 1, $G(G_i, G_{i+1}; M_{i,i+1})$ is $(k+1)$-connected for $1 \leqslant i \leqslant r-2$. Thus, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $1 \leqslant i \leqslant r-2$. Hence, $H - F$ is connected.

Suppose that $|X_0| = k$. Thus, there is only one vertex a_0 in $V(G_0) - F$.

Assume that $k = 1$. Then, $|X_i| \leqslant 1$ for $0 \leqslant i \leqslant r-1$. Suppose that $|X_i \cup X_{i+1}| \leqslant 1$ for $0 \leqslant i \leqslant r-2$. Thus, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $0 \leqslant i \leqslant r-2$. So $H - F$ is connected. Suppose $|X_i \cup X_{i+1}| = 2$ for some i. We may without loss of generality assume that $|X_0 \cup X_1| = 2$. In this case, we may label the vertices of $V(G_i)$ as a_i, a'_i for $0 \leqslant i \leqslant r-2$ with $(a_i, a_{i+1}) \in M_{i,i+1}$ and $(a'_i, a'_{i+1}) \in M_{i,i+1}$.

Suppose that $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is disconnected. Without loss of generality, we assume that $a'_0 \in F$ and $a_1 \in F$.

Suppose that $r = 3$. Since $F \neq N_H(a'_1)$, $a'_2 \notin F$. Thus, $G(G_1, G_2; M_{1,2}) - (X_1 \cup X_2)$ is connected. $a_2 \in F$ since $|X_2| = 1$. So $a'_2 \in N_H(a_0)$ since $F \neq N_H(a_0)$. Thus, $G(G_0, G_2; M_{0,2}) - (X_0 \cup X_2)$ is connected. Hence, $H - F$ is connected.

Suppose that $r > 3$. Since $F \neq N_H(a_0)$ and $F \neq N_H(a'_1)$, $a'_2 \notin F$ and $\hat{a}_{r-1} \notin F$ where $\hat{a}_{r-1}$ is the vertex such that $(a_0, \hat{a}_{r-1}) \in M_{0,r-1}$. Thus, $G(G_1, G_2; M_{1,2}) - (X_1 \cup X_2)$ and $G(G_0, G_{r-1}; M_{0,r-1}) - (X_0 \cup X_{r-1})$ are connected. Obviously, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $2 \leqslant i \leqslant r-2$. Therefore, $H - F$ is connected.

Suppose that $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is connected. We may without loss of generality assume that $a'_0 \in F$ and $a'_1 \in F$.

Suppose $r = 3$. Since $F \neq N_H(a_2')$, either (1) $a_2' \in F$ and $a_2 \notin F$; or (2) $a_2' \notin F$, $a_2 \in F$, and $(a_0, a_2') \in M_{0,2}$. In the first case, $G(G_1, G_2; M_{1,2}) - (X_1 \cup X_2)$ is connected; and in the second case, $G(G_0, G_2; M_{0,2}) - (X_0 \cup X_2)$ is connected. Hence, $H - F$ is connected.

Suppose that $r > 3$. Since $|F - (X_0 \cup X_1)| = 1$, we may without loss of generality assume that $|X_2| = 0$. So $G(G_1, G_2; M_{1,2}) - X_1$ is connected. $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected for $2 \leqslant i \leqslant r-2$ because $|X_i \cup X_{i+1}| \leqslant k$. So $H - F$ is connected.

Now, we consider $k > 1$. Let $a_1 \in V(G_1)$ and $(a_0, a_1) \in M_{0,1}$.

Suppose that $a_1 \notin F$. Thus, $G(G_0, G_1; M_{0,1}) - (X_0 \cup X_1)$ is connected. For $i = 1, 2, \ldots, r-2$, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected because $|X_i \cup X_{i+1}| \leqslant k$. Therefore, $H - F$ is connected.

Suppose that $a_1 \in F$. Let a_{r-1} be a vertex in $V(G_{r-1})$ such that $(a_0, a_{r-1}) \in M_{0,r-1}$. Since $F \neq N_H(a_0), a_{r-1} \notin F$. Thus, $G(G_0, G_{r-1}; M_{0,r-1}) - (X_0 \cup X_{r-1})$ is connected. For $i = 1, 2, \ldots, r-2$, $G(G_i, G_{i+1}; M_{i,i+1}) - (X_i \cup X_{i+1})$ is connected because $|X_i \cup X_{i+1}| \leqslant k$. Therefore, $H - F$ is connected.

The theorem is proved. $\square$

With a similar argument as above, we have the following results.

Theorem 4. *Let r and t be positive integers with $r \geqslant 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are k-regular maximum edge connected graphs with $|V(G_i)| = t$ for $0 \leqslant i \leqslant r-1$ and $\mathscr{M} = \bigcup_{i=0}^{r-1} M_{i,i+1\,(\mathrm{mod}\,r)}$ with $M_{i,i+1\,(\mathrm{mod}\,r)}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1\,(\mathrm{mod}\,r)})$. Then, $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ is $(k+2)$-regular super-edge-connected if and only if* (1) $k \geqslant 2$, (2) $k = 1$ *and* $r \neq 3$, *or* (3) $k = 0$ *and* $r = 3$.

Corollary 2. *Let r and t be positive integers with $r \geqslant 3$. Assume that $G_0, G_1, \ldots, G_{r-1}$ are k-connected and k′-edge connected graphs with $|V(G_i)| = t$ for $0 \leqslant i \leqslant r-1$ and $\mathscr{M} = \cup_{i=0}^{r-1} M_{i,i+1\,(\mathrm{mod}\,r)}$, where $M_{i,i+1\,(\mathrm{mod}\,r)}$ is any arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1\,(\mathrm{mod}\,r)})$. Then, $H = G(G_0, G_1, \ldots, G_{r-1}; \mathscr{M})$ is* $(k+2)$*-connected and* $(k'+2)$*-edge-connected.*

4. The third constructing scheme

Assume that t is an integer with $t \geqslant 2$. Let G_1 and G_2 be two graphs with t vertices such that $V(G_1) = \{a_i | 0 \leqslant i < t\}$ and $V(G_2) = \{b_i | 0 \leqslant i < t\}$. Let $\mathscr{C}$ be a set of edges given by $\mathscr{C} = \{(a_i, b_i) | 0 \leqslant i < t\} \cup \{(b_i, a_{i+1\,(\mathrm{mod}\ t)}) | 0 \leqslant i < t\}$. The graph $G(G_1, G_2; \mathscr{C})$ is defined to be the graph with the vertex set $V(G(G_1, G_2; \mathscr{C})) = V(G_1) \cup V(G_2)$, and edge set $E(G(G_1, G_2; \mathscr{C})) = E(G_1) \cup E(G_2) \cup \mathscr{C}$.

Theorem 5. *Assume that t is an integer with $t \geqslant 2$. Let G_1 and G_2 be two k-regular maximum connected graphs with t vertices such that $V(G_1) = \{a_i | 0 \leqslant i < t\}$ and $V(G_2) = \{b_i | 0 \leqslant i < t\}$. Let $\mathscr{C}$ be a set of edges given by $\mathscr{C} = \{(a_i, b_i) | 0 \leqslant i < t\} \cup \{(b_i, a_{i+1\,(\mathrm{mod}\, t)}) | 0 \leqslant i < t\}$. Then, $G(G_1, G_2; \mathscr{C})$ is $(k+2)$-regular super-connected if* (1) $t > k+1$ *with* $k \geqslant 3$ *or* (2) $t = k+1$ *with* $k = 1, 2, 3$.

Proof. Since G_1 and G_2 are k-regular connected graph, $t \geqslant k+1$. By definition, $G(G_1, G_2; \mathscr{C})$ is $(k+2)$-regular. To prove $G(G_1, G_2; \mathscr{C})$ is super-connected, we need to check if $G(G_1, G_2; \mathscr{C}) - F$ is connected for any $F \subset V(G(G_1, G_2; \mathscr{C}))$ such that $|F| = k+2$ and $F \neq N_{G(G_1,G_2;\mathscr{C})}(V)$ for any vertex $v \in V(G(G_1, G_2; \mathscr{C}))$.

Suppose that $t = k+1$. Obviously, G_1 and G_2 are isomorphic to the complete graph K_{k+1}. By brute force, we can check that $G(G_1, G_2; \mathscr{C})$ is super-connected for $k = 1, 2, 3$. When $k \geqslant 4$, set $F = \{a_0, a_{t-2}, a_{t-1}\} \cup \{b_i | 0 \leqslant i \leqslant t-3\}$. It is easy to see that $|F| = k+2$, $F \neq N_{G(G_1,G_2;\mathscr{C})}(v)$, and F is a vertex cut of $G(G_1, G_2; \mathscr{C})$. Hence, $G(G_1, G_2; \mathscr{C})$ is not super-connected.

Now, assume that $t > k+1$ with $k \geqslant 3$. We set $X_1 = F \cap V(G_1)$ and $X_2 = F \cap V(G_2)$.

Case 1. $|X_1| < k$ and $|X_2| < k$. $G(G_1, G_2; \mathscr{C}) - F$ is connected with the same argument in Theorem 1.

Case 2. Either $k \leqslant |X_1| \leqslant k+2$ or $k \leqslant |X_2| \leqslant k+2$. We assume without loss of generality that $k \leqslant |X_1| \leqslant k+2$, then $|X_2| \leqslant 2$ and $G_2 - X_2$ is connected. Let C be any connected component of $G_1 - X_1$. We will claim that there exists $a_i \in C$ such that at least one of $(a_i, b_{i-1\,(\mathrm{mod}\, t)}), (a_i, b_i)$ is in $\mathscr{C}$. With this claim, $G(G_1, G_2; \mathscr{C}) - F$ is connected.

First, if C consists of only one vertex a_i, then $N_{G_1}(a_i) \subset F$. Since $F \neq N_{G(G_1,G_2;\mathscr{C})}(a_i)$, at most one of $b_i, b_{i-1\,(\mathrm{mod}\, t)}$ lies in F. Thus, the claim holds. Now, if C contains at least two vertices a_i and a_j. Since $|X_2| \leqslant 2$, we may assume $b_{i-1\,(\mathrm{mod}\, t)} \notin F$. Thus, our claim holds.

The theorem is proved. □

A similar argument leads to the following results.

Theorem 6. *Assume that t is an integer with $t \geqslant 2$. Let G_1 and G_2 be two k-regular maximum edge connected graphs with t vertices such that $V(G_1) = \{a_i | 0 \leqslant i < t\}$ and $V(G_2) = \{b_i | 0 \leqslant i < t\}$. Let $\mathscr{C}$ be a set of edges given by $\mathscr{C} = \{(a_i, b_i) | 0 \leqslant i < t\} \cup \{(b_i, a_{i+1\,(\mathrm{mod}\, t)}) | 0 \leqslant i < t\}$. Then, $G(G_1, G_2; \mathscr{C})$ is $(k+2)$-regular super-edge-connected if* (1) $t > k+1$ *with* $k \geqslant 2$ *or* (2) $t = k+1$ *with* $k \geqslant 1$.

Assume that c, d, r are integers with $r \geqslant 0$, $d > 1$, and $1 \leqslant c < d$. It is proposed in [8] that the recursive circulant graph $RC(c, d, r)$ as the circulant graph $G(cd^r; \{1, d, \ldots, d^{\lceil \log_d cd^r \rceil - 1}\})$. For $0 \leqslant i < d$, let V_i^r denote the set $\{j | 0 \leqslant j < cd^r, j = i (\bmod d)\}$. We use $RC_i(c, d, r)$ to denote the subgraph of $RC(c, d, r)$ induced by V_i^r. For a positive integer N, let Z_N be the additive group of residue classes modulo N. We can recursively describe $RC(c, d, r)$ as follows: Assume that $r = 0$. Then $RC(c, d, 0)$ is the graph with $V(RC(c, d, 0)) = \{0\}$ and $E(RC(c, d, 0)) = \emptyset$ if $c = 1$, $V(RC(c, d, 0)) = \{0, 1\}$ and $E(RC(c, d, 0)) = \{(0, 1)\}$ if $c = 2$, and $V(RC(c, d, 0)) = Z_c$ and $E(RC(c, d, 0)) = \{(i, i + 1) | i \in Z_c\}$ if $c \geqslant 3$. Assume that $r \geqslant 1$. The induced subgraph $RC_i(c, d, r)$ is isomorphic to $RC(c, d, r - 1)$ for $0 \leqslant i < d$. More precisely, let f_i^r be the function from $Z_{cd^{r-1}}$ into V_i^r defined by $f_i^r(x) = dx + i$. Then, f_i^r induces an isomorphism from $RC(c, d, r - 1)$ into $RC_i(c, d, r)$. Let $H(c, d, r)$ denote the set of edges of $RC(c, d, r)$ not in $\cup_{i=0}^{d-1}(E(RC_i(c, d, r)))$. Then, $H(c, d, r) = \{(i, i + 1) | i \in Z_{cd^r}\}$.

Suppose $d \geqslant 3$. Then, $RC(c, d, r)$ can be expressed as $G(H, H, \ldots, H; \mathscr{M})$, with d H's, where H is $RC(c, d, r - 1)$ for some $\mathscr{M} = \cup_{i=0}^{d-1} M_{i,i+1\,(\bmod d)}$. Suppose $d = 2$. Then, $RC(c, d, r)$ can be expressed as $G(H, H; \mathscr{C})$, where H is $RC(c, d, r - 1)$. Recursively applying Theorems 3–6, we can easily prove that $RC(c, d, r)$ is super-connected unless (1) $r = 0$ and $c \geqslant 5$ or (2) $r = 1$, $c = 1$, and $d \geqslant 5$; and super-edge-connected unless (1) $r = 0$ and $c \geqslant 4$, (2) $r = 1$, $c = 1$, and $d \geqslant 4$, or (3) $r = 1$, $c = 2$, and $d = 3$.

Corollary 3. *Let t be an integer with $t \geqslant 2$. Assume that G_1 and G_2 are two k-connected and k'-edge connected graphs with t vertices. Then, $G(G_1, G_2; \mathscr{C})$ is $(k + 2)$-connected and $(k' + 2)$-edge connected.*

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