A General Construction for Nonblocking Crosstalk-free Photonic Switching Networks

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The graph representation $G(M)$ of a multistage switching network M is well known. Lea [IEEE Trans Commun 38 (1990), 529–538] observed that link-disjoint paths in M correspond to node-disjoint paths in $G(M)$. He proposed $G(M)$ as a network by treating nodes as crossbars to transfer the node-disjoint property to the crosstalk-free property essential for photonic networks using directional couplers as components. However, such a network has its peculiarities and is not commonly used. In this paper, we will show how to take advantage of this correspondence to construct nonblocking crosstalk-free networks using the vertical stacking method. Our construction simplifies the proofs of many existing results, as well as establishing some new results.

Keywords: Log(N, k, p) network; banyan network; Benes network; photonic switching network; strictly nonblocking network; crosstalk-free

1. INTRODUCTION

As an electronic switching network has electronic crossbars for its switching components, a photonic switching network has directional couplers as counterparts. Functionally, a directional coupler is similar to a 2×2 crossbar in the sense that it has two states: The "straight" state has input 1 connected to output 1 and input 2 to output 2, while the "cross" state has input 1 connected to output 2 and input 2 to output 1 (see Fig. 1).

Due to the popularity of directional couplers as components, photonic switching networks often use 2×2 switching elements. In particular, the class of *$Log_2(N, k, p)$ networks* covers the whole spectrum of extra-stage banyan networks, including the banyan network at one extreme with no extra stage and the Benes network at the other extreme with the maximum number of extra stages (these networks are to be defined later). $\text{Log}_2(N, k, p)$ networks are often used as the underlying networks for photonic switching.

A major disadvantage of directional couplers is the *crosstalk* problem, namely, when a directional coupler carries two signals, even by different channels, the signals may still spill over between the channels and produce unwanted noise. Many studies have been done to design photonic switching networks with directional couplers but no crosstalk, that is, only one signal is allowed to go through a coupler.

A network is *strictly nonblocking* if, regardless of how existing connections are routed, a new connection can always be routed by a path link-disjoint with all existing paths. It is *wide-sense nonblocking* if the above can be achieved under a routing algorithm. It is *rearrangeable* if link-disjoint paths exist for any set of connections (routed simultaneously). The network is called *crosstalk-free* strictly nonblocking (wide-sense nonblocking, rearrangeable) if node-disjoint replaces link-disjoint.

A popular method of constructing nonblocking networks is to vertically stack up enough copies of a blocking network and, essentially, to identify their inputs and outputs [2, 3, 5, 6, 8, 9]. On the other hand, Lea [5] observed that link-disjoint paths in M correspond to node-disjoint paths in its graph version. In this paper, we will show how to take advantage of this correspondence to construct nonblocking crosstalk-free networks using the vertical stacking method. Our construction simplifies the proofs of many existing results, as well as establishing some new results.

2. THE EXTRA-STAGE INVERSE BANYAN NETWORK AND ITS GRAPH VERSION

A banyan network of order n has $N = 2^n$ inputs, 2^n outputs, and n stages, while each stage has 2^{n-1} 2×2 crossbars. To describe the linking pattern between two adjacent stages, it is convenient to label the $N/2$ crossbars in each stage by the binary numbers $0, 1, \ldots, N/2 - 1$. Then, the bipartite graph between stages i and $i + 1$

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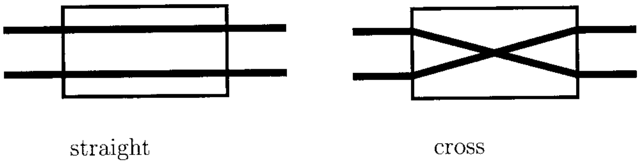


FIG. 1. Two states of a directional coupler.

connects crossbars which agree in all bits except perhaps bit i (see Fig. 2).

Let $BY^{-1}(n, k)$ denote the multistage network obtained by adding k extra stages to the n-stage inverse banyan network where the linking pattern of the k extra stages is a mirror image of the first k stages. The networks $BY^{-1}(3, 1)$ and $BY^{-1}(3, 2)$ are shown in Figure 3. $BY^{-1}(n, n - 1)$ is also known as the Benes network of order 2^n and denoted by $B(n)$.

Shyy and Lea [8] introduced the $\text{Log}_2(N, k, p)$ network constructed by vertically stacking p copies of $BY^{-1}(n, k)$, $N = 2^n$, and adding an extra stage of N inputs and an extra stage of N outputs, where input (output) i is connected to input (output) i of each $BY^{-1}(n, k)$. Figure 4 illustrates the $\text{Log}_2(4, 1, 3)$ network which is also known as a Cantor network [2]. Hwang [3] extended $\text{Log}_2(N, k, p)$ networks to $\text{Log}_d(N, k, p)$ networks by replacing the 2×2 crossbars with $d \times d$ crossbars.

A multistage network, though looking very much like a graph by treating its crossbars as nodes, is not a graph since the inputs and outputs of the network are dangling links from nodes not connecting to other nodes. But there is a well-known graph representation $G(M)$ of a network M obtained by treating the network links as nodes of $G(M)$; an arc between two nodes in $G(M)$ represents that the two corresponding links can be connected through a switch in M. Figure 5 illustrates a network and its graph representation.

A network is called *unitary* if each input crossbar has one input and each output crossbar one output. In fact, we can even remove the unique input (output) and treat the input (output) switch itself as the input (output). Note that by treating each node in $G(M)$ as a switch then $G(M)$ can also be viewed as an unitary network, which Lea [5] called a *bipartite graph network.* It is easily seen that if M has s stages then $G(M)$ has $s + 1$ stages, and if M is d-nary (every crossbar is $d \times d$), then $G(M)$ is also d-nary, except that every input (output) switch has exactly one input (output). Lea [5] observed the following:

Lemma 1 (the basic lemma). *If M is a strictly nonblocking (wide-sense nonblocking, rearrangeable) network, then $G(M)$ is a crosstalk-free strictly nonblocking (wide-sense nonblocking, rearrangeable) network.*

Proof. Since a node in $G(M)$ corresponds to a link in M, two link-disjoint paths in M become two node-disjoint paths in $G(M)$. ∎

The problem with using $G(M)$ as a photonic network is that it is not a network commonly studied in the literature so we do not know its other properties besides being crosstalk-free. Furthermore, if M is a d-nary network using $d \times d$ crossbars as components, then $G(M)$ is also d-nary except that input switches and output switches are unitary. Therefore, we propose a different approach by studying the well-known $\text{Log}_d(N, k, p)$ network for its crosstalk-free properties. While $\text{Log}_d(N, k, p)$ is not the graph representation of some network, we show that it is closely related to a $G(M)$ and its crosstalk-free property can be derived from the latter. This work is made much easier by the recent work of Hwang and Yen [4] who characterized $G(M)$ for any bit permutation network M (including all extra-stage banyan-type networks). Let M^* denote M with inputs and outputs removed from input switches and output switches. In particular, Hwang and Yen [4] proved

Lemma 2. $BY^{-1}(n + 1, k)^* = G(BY^{-1}(n, k))$ *for* $0 \le k \le n - 1$.

Traditionally, the Benes network is defined only for an odd number of stages. Padmanabhan and Netravali [7] showed that the same expansion scheme also works for an even numbers of stages, which they called *dilated Benes networks,* denoted by $DB(n)$ (see Fig. 6).

It is easily seen that $DB(n) = BY^{-1}(n, n - 2)$. Hence,

Corollary 3. $DB(n)^* = G(B(n - 1))$.

Note that Lemma 2 does not include representing $B(n)^* = BY^{-1}(n, n - 1)^*$ as $G(M)$ for some M. To do this, we

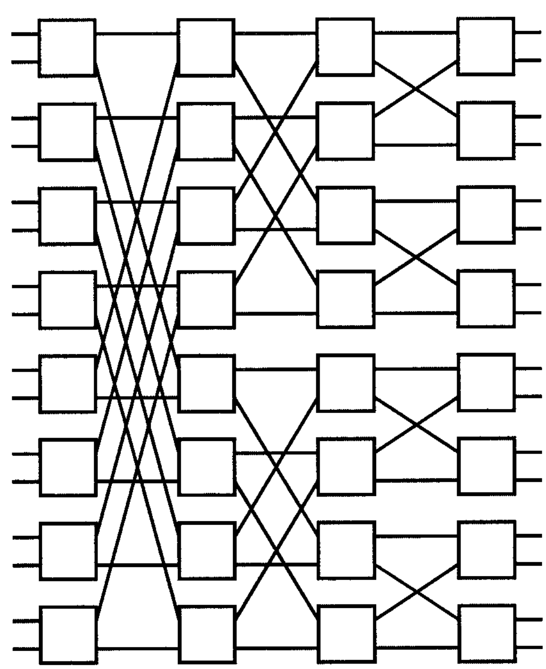
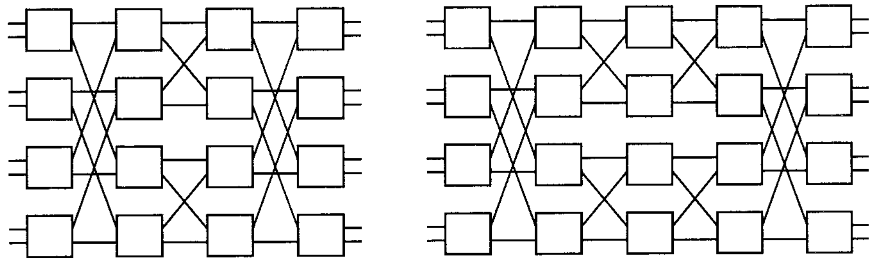


FIG. 2. An inverse banyan network of order 4.



$BY^{-1}(3,1)$ $BY^{-1}(3,2)$

FIG. 3. Two inverse banyan networks.

need to add a dummy stage in the middle of $B(n-1)$ which connects the two outputs of a crossbar to the two inputs of a crossbar in the next stage. $B(n)$, after this addition will be denoted by $AB(n)$. It is easily verified that

Lemma 4. $B(n+1)^* = G(AB(n))$.

Figure 7 gives an illustration of Lemma 4.

3. THE MAIN RESULTS

Let p-M denote the network similar to $\mathrm{Log}_2(N, k, p)$ except replacing $BY^{-1}(n, k)$ with M. Suppose that M is d-nary for some d and $G(p\text{-}M)$ has stages 1 to s. Since p-M is unitary, each stage-2 (stage-$(s-1)$) crossbar has one input (output) in $G(p\text{-}M)$. Hence, $G(p\text{-}M)$ cannot be represented as p-M' for some d-nary M' and we cannot apply Lemma 1 directly to p-M type networks. We will remedy this situation:

Consider a request from input i to output j. The *channel graph* of (i, j) is the union of all paths from i to j. We say the (i, j) connection is link(node)-blocked if every path from i to j contains a busy link (node).

For convenience, we define the following quantities to indicate the degree of "blockingness" of a given network M: (1) Suppose that t is the number of paths of the (i, j) channel graph and s is the maximum number of (i, j) paths being link(node)-blocked by paths of other (disjoint) input–output pairs. Then, the *link(node)-blockingness of the* (i, j) *connection in* M is defined as s/t. (2) The *link(node)-blockingness of M* is the maximum link(node)-blockingness over all (i, j) pairs. Note that an (i, j) path may be blocked more than once when we compute s. For example, Figure 8 shows the channel graph of (1, 1) in $BY_3^{-1}(2, 1)$. There are four paths blocking the (1, 1) paths, with two of them blocking the top path.

It is clear that $(\lfloor s/t \rfloor + 1)$ - M is (crosstalk-free) strictly nonblocking if the link (node)-blockingness of M is s/t, since at most $\lfloor s/t \rfloor$ copies are blocked for any request. Note that if only a fraction of a copy is blocked then the (i, j) connection can still be made through that copy. The reason that we keep the fraction instead of rounding it down is to maintain accuracy in multiplication. The following theorem derives strictly nonblocking crosstalk-free networks p-M' from strictly nonblocking networks p-M where M' and M are d-nary:

Theorem 5. *For two d-nary networks M_1 and M_2 with $M_2^* = G(M_1)$, if the link-blockingness of M_1 is b_1, then p_2-M_2 is a strictly nonblocking crosstalk-free network for $p_2 \geq \lfloor db_1 \rfloor + 2(d-1) + 1$.*

Proof. By Lemma 1, the link-blockingness of M_1 being b_1 means that the node-blockingness of M_2^* is b_1 too.

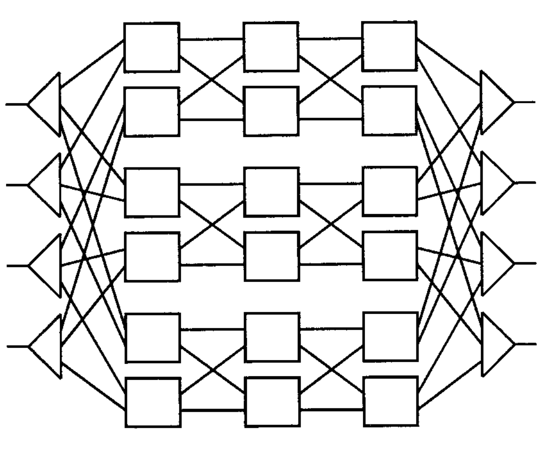
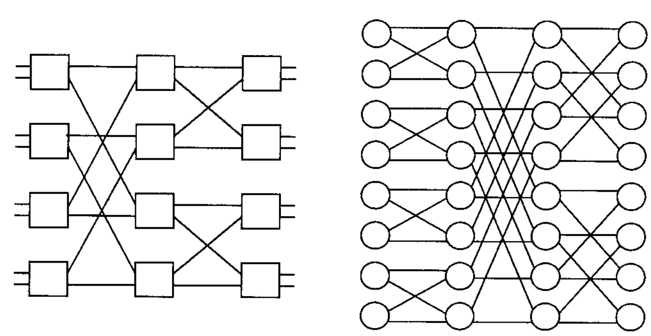


FIG. 4. The $\mathrm{Log}_2(4, 1, 3)$ network.



M $G(M)$

FIG. 5. A network M and its graph $G(M)$.

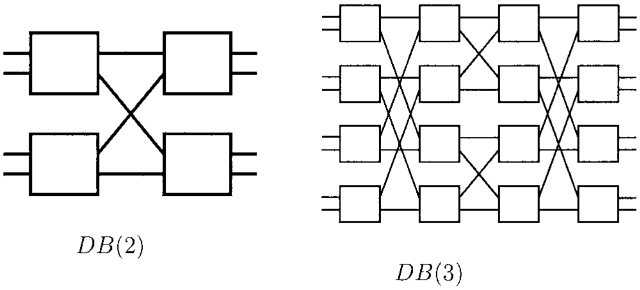


FIG. 6. Dilated Benes networks, $DB(2)$ and $DB(3)$.

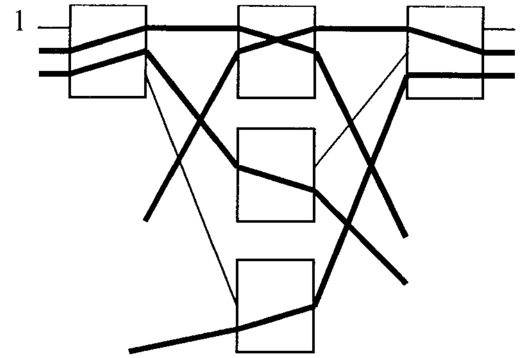


FIG. 8. The link-blockingness of $BY_3^{-1}(2, 1)$ is 4/3.

Consider a connection (x, y) in p_2-M_2 going through the input switch i and output switch j of M_2. Then, in M_2, paths not from the input switch i nor to the output switch j can contribute at most db_1 to the node-blockingness of the (x, y) connection in M_2. Suppose this is not true. Since these paths can be d-colored [1] such that all paths from a given input switch or output switch are colored differently, there exists one color whose paths contribute more than b_1 to the node-blockingness of the (x, y) connection in M_2, namely, the link-blockingness of the (i, j) connection in M_1 is more than b_1, a contradiction. Finally, at most $d - 1$ connections going through input switch i (output switch j) can contribute $d - 1$ to the node-blockingness of the (x, y) connection in M_2. Therefore, the node-blockingness of M_2 is not greater than $db_1 + 2(d - 1)$, which means there are at most $\lfloor db_1 \rfloor + 2(d - 1)$ copies such that the (x, y) connection cannot be made, that is, p_2-M_2 is crosstalk-free strictly nonblocking if $p_2 \geq \lfloor db_1 \rfloor + 2(d - 1) + 1$. ∎

Corollary 6. *For two binary networks M_1 and M_2 with $M_2^* = G(M_1)$, if the link-blockingness of M_1 is b_1, then p_2–M_2 is a strictly nonblocking crosstalk-free network for $p_2 \geq \lfloor 2b_1 \rfloor + 3$.*

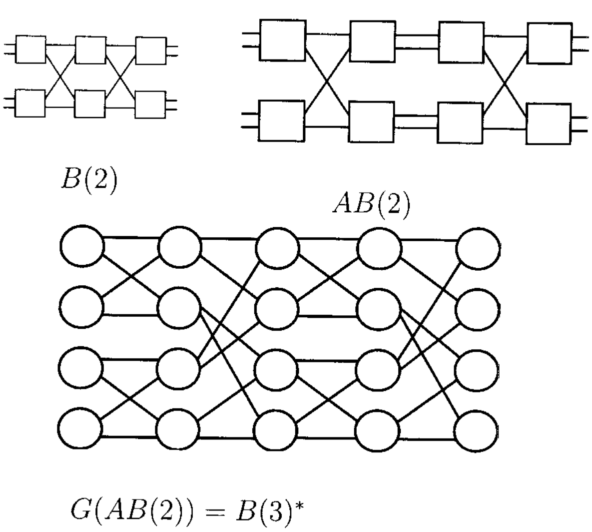


FIG. 7. $B(3)^*$ as a graph.

Figure 9 illustrates the concept of Corollary 6.

Theorem 7. *For two d-nary networks M_1 and M_2 with $M_2^* = G(M_1)$, if p_1-M_1 is a rearrangeable multistage network, then p_2-M_2 is a rearrangeable crosstalk-free network for $p_2 \geq dp_1$.*

Proof. Let R denote the bipartite graph where nodes are input and output crossbars in M_2 and a link between nodes i and j implies a request from an input of switch i to an output of switch j. Then, R has maximum degree d and it is well known [1] that R can be d-colored. The subgraph of each color can be routed by p_1-M_2. Hence, dp_1-M_2 suffices for routing R. ∎

We first show that many existing results about nonblocking crosstalk-free networks can be obtained through the results of this section.

Theorem 8 (Padmanabhan and Netravali [7]). *The unitarized dilated Benes network is crosstalk-free rearrangeable.*

Proof. This follows from Corollary 3 and the well-known fact [1] that the Benes network is rearrangeable. ∎

Theorem 9 (Vaez and Lea [9]). *$Log_2(2^n, 0, p)$ is crosstalk-free strictly nonblocking if*

$$p \geq \begin{cases} 2^{(n+2)/2} - 1 \text{ for } n \text{ even}, \\ 3 \cdot 2^{(n-1)/2} - 1 \text{ for } n \text{ odd}. \end{cases}$$

Proof. Theorem 9 follows from Lemma 2, Corollary 6, and the fact [8] that the link-blockingness of $BY^{-1}(n-1, 0)$ is no more than

$$\begin{cases} 3 \cdot 2^{(n-3)/2} - 2 \text{ for } n - 1 \text{ even}, \\ 2^{n/2} - 2 \text{ for } n - 1 \text{ odd}. \end{cases}$$

∎

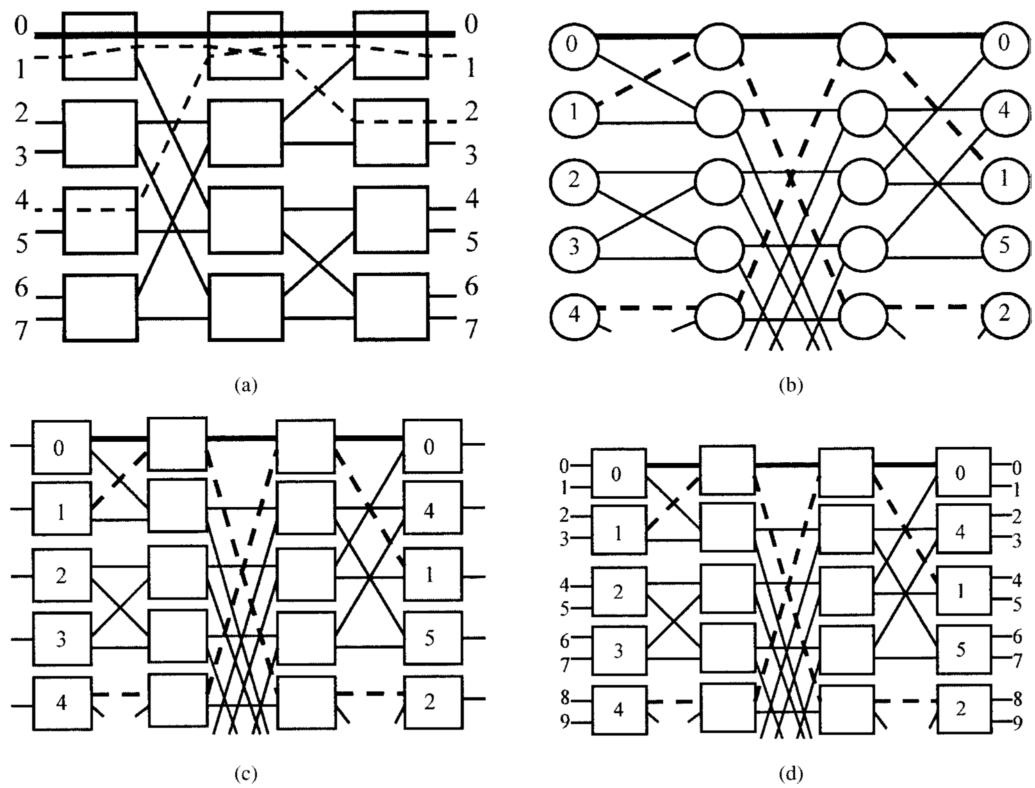


FIG. 9. The concept of Corollary 6. (a) In p_1-BY^{-1} (3), request (0, 0) and existing connections (1, 2) and (4, 1) blocking (0, 0) in their planes. (b) In $G(BY^{-1}(3))$, paths (1, 2) and (4, 1) intersect nodes lying on path (0, 0). (c) In $BY^{-1}(4)^*$, two routes intersect switches lying on a connection from input switch 0 to output switch 0. (d) Connections (2, 8) (3, 9) (8, 4) (9, 5) (1, 2) and (4, 1) can node-block the (0, 0) connection.

Theorem 10 (Vaez and Lea [9]). *$Log_2(2^n, k, p)$ is crosstalk-free strictly nonblocking for* $0 \le k \le n-2$ *if*

$$p \ge \begin{cases} 2k + 2^{(n-k+2)/2} - 1 \text{ for } n + k \text{ even}, \\ 2k + 3 \cdot 2^{(n-k-1)/2} - 1 \text{ for } n + k \text{ odd}. \end{cases}$$

For $k = n - 1$, *the condition becomes* $p \ge 2n - 1$.

Proof. For $0 \le k \le n - 2$, Theorem 10 follows from Lemma 2, Corollary 6, and the fact [8] that the link-blockingness of $BY^{-1}(n-1, k)$ is no more than

$$\begin{cases} k + 3 \cdot 2^{(n-k-3)/2} - 2 \text{ for } n + k - 1 \text{ even}, \\ k + 2^{(n-k)/2} - 2 \text{ for } n + k - 1 \text{ odd}. \end{cases}$$

For $k = n - 1$, that is, the Cantor network constructed with $B(n)$, Theorem 10 follows from Lemma 4, Corollary 6, and the fact that the link-blockingness of $AB(n - 1)$ is no more than $n - 2$ [since $AB(n - 1)$ is equivalent to $B(n - 1)$]. ■

Next, we give some new results:

Theorem 11. *$Log_2(2^n, k, p)$ is crosstalk-free rearrangeable if* $p \ge 2^{\lfloor (n-k-1)/2 \rfloor + 1}$ *for* $0 \le k \le n - 1$.

Proof. For $0 \le k \le n - 2$, Theorem 11 follows from Lemma 2, Theorem 7, and the fact that $\mathrm{Log}_2(2^{n-1}, k, p)$ is a rearrangeable crossbar network [6] if $p \ge 2^{\lfloor (n-k-1)/2 \rfloor}$.

For $k = n - 1$, that is, the Cantor network constructed with $B(n)$, Theorem 11 follows from Lemma 4, Theorem 7, and the fact that p-$AB(n - 1)$ is a rearrangeable crossbar network if $p \ge 1$ [since $AB(n - 1)$ is equivalent to $B(n - 1)$], where $2^{\lfloor (n-(n-1)-1)/2 \rfloor + 1} = 2$. ■

Theorem 12. *$Log_d(d^n, k, p)$ is crosstalk-free strictly nonblocking for* $0 \le k \le n - 2$ *if*

$$p \ge \begin{cases} 2k(d-1) + 2d^{(n-k)/2} - 1 \text{ for } n + k \text{ even}, \\ 2k(d-1) + (d+1)d^{(n-k-1)/2} - 1 \text{ for } n + k \text{ odd}. \end{cases}$$

Proof. This follows from Theorem 5 and the fact derived by Hwang [3] that the link-blockingness of $BY_d^{-1}(n-1, k)$ is no more than

$$\begin{cases} \dfrac{2k(d-1)}{d} + (d+1)d^{(n-k-3)/2} - 2 \text{ for } n + k - 1 \text{ even}, \\ \dfrac{2k(d-1)}{d} + 2d^{(n-k-2)/2} - 2 \text{ for } n + k - 1 \text{ odd}. \end{cases}$$

■

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