Commun. Math. Phys. 239, 183–240 (2003)
Digital Object Identifier (DOI) 10.1007/s00220-003-0878-5

Communications in
**Mathematical
Physics**

Liouville Action and Weil-Petersson Metric on Deformation Spaces, Global Kleinian Reciprocity and Holography

Leon A. Takhtajan[1], **Lee-Peng Teo**[2]

[1] Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651, USA.
E-mail: leontak@math.sunysb.edu
[2] Department of Applied Mathematics, National Chiao Tung University, 1001, Ta-Hsueh Road, Hsinchu City, 30050, Taiwan, R.O.C. E-mail: lpteo@math.nctu.edu.tw

Received: 5 September 2002 / Accepted: 21 February 2003
Published online: 23 June 2003 –

Abstract: We rigorously define the Liouville action functional for the finitely generated, purely loxodromic quasi-Fuchsian group using homology and cohomology double complexes naturally associated with the group action. We prove that classical action – the critical value of the Liouville action functional, considered as a function on the quasi-Fuchsian deformation space, is an antiderivative of a 1-form given by the difference of Fuchsian and quasi-Fuchsian projective connections. This result can be considered as global quasi-Fuchsian reciprocity which implies McMullen's quasi-Fuchsian reciprocity. We prove that the classical action is a Kähler potential of the Weil-Petersson metric. We also prove that the Liouville action functional satisfies holography principle, i.e., it is a regularized limit of the hyperbolic volume of a 3-manifold associated with a quasi-Fuchsian group. We generalize these results to a large class of Kleinian groups including finitely generated, purely loxodromic Schottky and quasi-Fuchsian groups, and their free combinations.

Contents

1. Introduction

Fuchsian uniformization of Riemann surfaces plays an important role in the Teichmüller theory. In particular, it is built into the definition of the Weil-Petersson metric on Teichmüller spaces. This role became even more prominent with the advent of string theory, started with Polyakov's approach to non-critical bosonic strings [Pol81]. It is natural to consider the hyperbolic metric on a given Riemann surface as a critical point of a certain functional defined on the space of all smooth conformal metrics on it. In string theory this functional is called *the Liouville action functional* and its critical value – *the classical action*. This functional defines *the two-dimensional theory of gravity with cosmological term* on a Riemann surface, also known as *Liouville theory*.

From a mathematical point of view, the relation between Liouville theory and complex geometry of moduli spaces of Riemann surfaces was established by P. Zograf and the first author in [ZT85, ZT87a, ZT87b]. It was proved that the classical action is a Kähler potential of the Weil-Petersson metric on moduli spaces of pointed rational curves [ZT87a], and on Schottky spaces [ZT87b]. In the rational case the classical action is a generating function of accessory parameters of Klein and Poincaré. In the case of compact Riemann surfaces, the classical action is an antiderivative of a 1-form on the Schottky space given by the difference of Fuchsian and Schottky projective connections. In turn, this 1-form is an antiderivative of the Weil-Petersson symplectic form on the Schottky space.

C. McMullen [McM00] has considered another 1-form on Teichmüller space given by the difference of Fuchsian and quasi-Fuchsian projective connections, the latter corresponds to Bers' simultaneous uniformization of a pair of Riemann surfaces. By establishing quasi-Fuchsian reciprocity, McMullen proved that this 1-form is also an antiderivative of the Weil-Petersson symplectic form, which is bounded on the Teichmüller space due to the Kraus-Nehari inequality. The latter result is important in proving that the moduli space of Riemann surfaces is Kähler hyperbolic [McM00].

In this paper we extend McMullen's results along the lines of [ZT87a, ZT87b] by using the homological algebra machinery developed by E. Aldrovandi and the first author in [AT97]. We explicitly construct a smooth function on the quasi-Fuchsian deformation space and prove that it is an antiderivative of the 1-form given by the difference of Fuchsian and quasi-Fuchsian projective connections. This function is defined as a classical action for Liouville theory for a quasi-Fuchsian group. The symmetry property of this function is the global quasi-Fuchsian reciprocity, and McMullen's quasi-Fuchsian reciprocity [McM00] is its immediate corollary. We also prove that this function is a Kähler potential of the Weil-Petersson metric on the quasi-Fuchsian deformation space. As it will be explained below, construction of the Liouville action functional is not a trivial issue and it requires homological algebra methods developed in [AT97]. Furthermore, we show that the Liouville action functional satisfies the *holography principle* in string theory (also called *AdS/CFT correspondence*). Specifically, we prove that the Liouville action functional is a regularized limit of the hyperbolic volume of a 3-manifold associated with a quasi-Fuchsian group. Finally, we generalize these results to a large class of Kleinian groups including finitely generated, purely loxodromic Schottky and quasi-Fuchsian groups, and their free combinations. Namely, we define the Liouville action functional, establish the holography principle, and prove that the classical action is an antiderivative of a 1-form on the deformation space given by the difference of Fuchsian and Kleinian projective connections, thus establishing global Kleinian reciprocity. We also prove that the classical action is a Kähler potential of the Weil-Petersson metric.

Here is a more detailed description of the paper. Let X be a Riemann surface of genus $g > 1$, and let $\{U_\alpha\}_{\alpha\in A}$ be its open cover with charts U_α, local coordinates $z_\alpha : U_\alpha \to \mathbb{C}$, and transition functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$. A (holomorphic) projective connection on X is a collection $P = \{p_\alpha\}_{\alpha\in A}$, where p_α are holomorphic functions on U_α which on every $U_\alpha \cap U_\beta$ satisfy

$$p_\beta = p_\alpha \circ f_{\alpha\beta} \left(f'_{\alpha\beta}\right)^2 + \mathcal{S}(f_{\alpha\beta}),$$

where prime indicates derivative. Here $\mathcal{S}(f)$ is the Schwarzian derivative,

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The space $\mathcal{P}(X)$ of projective connections on X is an affine space modeled on the vector space of holomorphic quadratic differentials on X.

The Schwarzian derivative satisfies the following properties:

SD1 $\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g\, (g')^2 + \mathcal{S}(g)$.
SD2 $\mathcal{S}(\gamma) = 0$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{C})$.

Let $\pi : \Omega \to X$ be a holomorphic covering of a compact Riemann surface X by a domain $\Omega \subset \hat{\mathbb{C}}$ with a group of deck transformations being a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. It follows from **SD1-SD2** that every such covering defines a projective connection on X by $P_\pi = \{\mathcal{S}_{z_\alpha}(\pi^{-1})\}_{\alpha\in A}$. The Fuchsian uniformization $X \simeq \Gamma\backslash\mathbb{U}$ is the covering $\pi_F : \mathbb{U} \to X$ by the upper half-plane $\mathbb{U}$ where the group of deck transformations is a Fuchsian group Γ, and it defines the Fuchsian projective connection P_F. The Schottky uniformization $X \simeq \Gamma\backslash\Omega$ is the covering $\pi_S : \Omega \to X$ by a connected domain $\Omega \subset \hat{\mathbb{C}}$, where the group of deck transformations Γ is a Schottky group – finitely-generated, strictly loxodromic, free Kleinian group. It defines the Schottky projective connection P_S.

Let $\mathfrak{T}_g$ be the Teichmüller space of marked Riemann surfaces of genus $g > 1$ (with a given marked Riemann surface as the origin), defined as the space of marked normalized Fuchsian groups, and let $\mathfrak{S}_g$ be the Schottky space, defined as the space of marked normalized Schottky groups with g free generators. These spaces are complex manifolds of dimension $3g - 3$ carrying Weil-Petersson Kähler metrics, and the natural projection map $\mathfrak{T}_g \to \mathfrak{S}_g$ is a complex-analytic covering. Denote by ω_{WP} the symplectic form of the Weil-Petersson metric on spaces $\mathfrak{T}_g$ and $\mathfrak{S}_g$, and by $d = \partial + \bar{\partial}$ – the de Rham differential and its decomposition. The affine spaces $\mathcal{P}(X)$ for varying Riemann surfaces X glue together to an affine bundle $\mathfrak{P}_g \to \mathfrak{T}_g$, modeled over the holomorphic cotangent bundle of $\mathfrak{T}_g$. The Fuchsian projective connection P_F is a canonical section of the affine bundle $\mathfrak{P}_g \to \mathfrak{T}_g$, invariant under the action of the Teichmüller modular group Mod_g. The Schottky projective connection is a canonical section of the affine bundle $\mathfrak{P}_g \to \mathfrak{S}_g$, and the difference $P_F - P_S$, where P_F is considered as a section of $\mathfrak{P}_g \to \mathfrak{S}_g$, is a $(1,0)$-form on $\mathfrak{S}_g$. This 1-form has the following properties [ZT87b]. First, it is ∂-exact – there exists a smooth function $S : \mathfrak{S}_g \to \mathbb{R}$ such that

$$P_F - P_S = \frac{1}{2}\,\partial S. \tag{1.1}$$

Second, it is a $\bar\partial$-antiderivative, and hence a d-antiderivative by (1.1), of the Weil-Petersson symplectic form on $\mathfrak{S}_g$,

$$\bar\partial(P_F - P_S) = -i\,\omega_{WP}. \tag{1.2}$$

It immediately follows from (1.1) and (1.2) that the function $-S$ is a Kähler potential for the Weil-Petersson metric on $\mathfrak{S}_g$, and hence on $\mathfrak{T}_g$,

$$\partial\bar\partial S = 2i\,\omega_{WP}. \tag{1.3}$$

Arguments using the quantum Liouville theory (see, e.g., [Tak92] and references therein) confirm formula (1.1) with function S given by the classical Liouville action, as was already proved in [ZT87b]. However, the general mathematical definition of the Liouville action functional on a Riemann surface X is a non-trivial problem interesting in its own right (and for rigorous applications to the quantum Liouville theory). Let $\mathcal{CM}(X)$ be a space (actually a cone) of smooth conformal metrics on a Riemann surface X. Every $ds^2 \in \mathcal{CM}(X)$ is a collection $\left\{e^{\phi_\alpha}|dz_\alpha|^2\right\}_{\alpha\in A}$, where functions $\phi_\alpha \in C^\infty(U_\alpha, \mathbb{R})$ satisfy

$$\phi_\alpha \circ f_{\alpha\beta} + \log|f'_{\alpha\beta}|^2 = \phi_\beta \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{1.4}$$

According to the uniformization theorem, X has a unique conformal metric of constant negative curvature -1, called hyperbolic, or Poincaré metric. Gaussian curvature -1 condition is equivalent to the following nonlinear PDE for functions ϕ_α on U_α,

$$\frac{\partial^2\phi_\alpha}{\partial z_\alpha \partial\bar z_\alpha} = \frac{1}{2}\,e^{\phi_\alpha}. \tag{1.5}$$

In string theory this PDE is called the Liouville equation. The problem is to define the Liouville action functional on Riemann surface X – a smooth functional $S : \mathcal{CM}(X) \to \mathbb{R}$ such that its Euler-Lagrange equation is the Liouville equation. At first glance it looks like an easy task. Set $U = U_\alpha$, $z = z_\alpha$ and $\phi = \phi_\alpha$, so that $ds^2 = e^{\phi}|dz|^2$ in U. Elementary calculus of variations shows that the Euler-Lagrange equation for the functional

$$\frac{i}{2}\iint\limits_U \left(|\phi_z|^2 + e^{\phi}\right) dz \wedge d\bar z,$$

where $\phi_z = \partial\phi/\partial z$, is indeed the Liouville equation on U. Therefore, it seems that the functional $\frac{i}{2}\iint_X \omega$, where ω is a 2-form on X such that

$$\omega|_{U_\alpha} = \omega_\alpha = \left(\left|\frac{\partial\phi_\alpha}{\partial z_\alpha}\right|^2 + e^{\phi_\alpha}\right) dz_\alpha \wedge d\bar z_\alpha, \tag{1.6}$$

does the job. However, due to the transformation law (1.4) the first terms in local 2-forms ω_α do not glue properly on $U_\alpha \cap U_\beta$ and a 2-form ω on X satisfying (1.6) does not exist!

Though the Liouville action functional can not be defined in terms of a Riemann surface X only, it can be defined in terms of planar coverings of X. Namely, let Γ be a Kleinian group with the region of discontinuity Ω such that $\Gamma\backslash\Omega \simeq X_1 \sqcup \cdots \sqcup X_n$ – a disjoint union of compact Riemann surfaces of genera > 1 including Riemann surface X. The covering $\Omega \to X_1 \sqcup \cdots \sqcup X_n$ introduces a global "étale" coordinate, and for large variety of Kleinian groups (Class A defined below) it is possible, using methods [AT97], to define the Liouville action functional $S : \mathcal{CM}(X_1 \sqcup \cdots \sqcup X_n) \to \mathbb{R}$ such that its critical value is a well-defined function on the deformation space $\mathfrak{D}(\Gamma)$. In the simplest case when X is a punctured Riemann sphere such a global coordinate exists already on X, and the Liouville action functional is just $\frac{i}{2}\iint_X \omega$, appropriately regularized at the punctures [ZT87a]. When X is compact, one possibility is to use the "minimal" planar cover of X given by the Schottky uniformization $X \simeq \Gamma\backslash\Omega$, as in [ZT87b]. Namely, identify $\mathcal{CM}(X)$ with the affine space of smooth real-valued functions ϕ on Ω satisfying

$$\phi \circ \gamma + \log|\gamma'|^2 = \phi \quad \text{for all} \quad \gamma \in \Gamma, \tag{1.7}$$

and consider the 2-form $\omega[\phi] = (|\phi_z|^2 + e^\phi) dz \wedge d\bar{z}$ on Ω. The 2-form $\omega[\phi]$ can not be pushed forward on X, so that the integral $\frac{i}{2}\iint_F \omega$ depends on the choice of fundamental domain F for the marked Schottky group Γ. However, one can add boundary terms to this integral to ensure the independence of the choice of fundamental domain for the marked Schottky group Γ, and to guarantee that its Euler-Lagrange equation is the Liouville equation on $\Gamma\backslash\Omega$. The result is the following functional introduced in [ZT87b]:

$$\begin{aligned} S[\phi] &= \frac{i}{2}\iint\limits_F \left(|\phi_z|^2 + e^\phi\right) dz \wedge d\bar{z} \\ &+ \frac{i}{2}\sum_{k=1}^g \int_{C_k} \left(\phi - \frac{1}{2}\log|\gamma_k'|^2\right)\left(\frac{\gamma_k''}{\gamma_k'}dz - \frac{\overline{\gamma_k''}}{\overline{\gamma_k'}}d\bar{z}\right) \\ &+ 4\pi\sum_{k=1}^g \log|c(\gamma_k)|^2. \end{aligned} \tag{1.8}$$

Here F is the fundamental domain of the marked Schottky group Γ with free generators $\gamma_1, \ldots, \gamma_g$, bounded by $2g$ non-intersecting closed Jordan curves $C_1, \ldots, C_g, C_1', \ldots, C_g'$ such that $C_k' = -\gamma_k(C_k)$, $k = 1, \ldots, g$, and $c(\gamma) = c$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Classical action $S : \mathfrak{S}_g \to \mathbb{R}$ that enters (1.1) is the critical value of this functional.

In [McM00] McMullen considered the quasi-Fuchsian projective connection P_{QF} on a Riemann surface X which is given by Bers' simultaneous uniformization of X and fixed Riemann surface Y of the same genus and opposite orientation. Similar to formula (1.2), he proved

$$d(P_F - P_{QF}) = -i\,\omega_{WP}, \tag{1.9}$$

so that the 1-form $P_F - P_{QF}$ on $\mathfrak{T}_g$ is a d-antiderivative of the Weil-Petersson symplectic form, bounded in Teichmüller and Weil-Petersson metrics due to the Kraus-Nehari inequality. Part $\bar{\partial}(P_F - P_{QF}) = -i\,\omega_{WP}$ of (1.9) actually follows from (1.1) since

$P_S - P_{QF}$ is a holomorphic $(1,0)$-form on $\mathfrak{S}_g$. Part $\partial(P_F - P_{QF}) = 0$ follows from McMullen's quasi-Fuchsian reciprocity.

Our first result is an analog of the formula (1.1) for the quasi-Fuchsian case. Namely, let Γ be a finitely generated, purely loxodromic quasi-Fuchsian group with region of discontinuity Ω, so that $\Gamma\backslash\Omega$ is the disjoint union of two compact Riemann surfaces with the same genus $g > 1$ and opposite orientations. Denote by $\mathfrak{D}(\Gamma)$ the deformation space of Γ – a complex manifold of complex dimension $6g-6$, and by ω_{WP} – the symplectic form of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$. To every point $\Gamma' \in \mathfrak{D}(\Gamma)$ with the region of discontinuity Ω' there corresponds a pair X, Y of compact Riemann surfaces with opposite orientations simultaneously uniformized by Γ', that is, $X \sqcup Y \simeq \Gamma'\backslash\Omega'$. We will continue to denote by P_F and P_{QF} the projective connections on $X \sqcup Y$ given by Fuchsian uniformizations of X and Y and Bers' simultaneous uniformization of X and Y respectively. Similarly to (1.1), we prove in Theorem 4.1 that there exists a smooth function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ such that

$$P_F - P_{QF} = \frac{1}{2}\,\partial S. \tag{1.10}$$

The function S is a classical Liouville action for the quasi-Fuchsian group Γ – the critical value of the Liouville action functional S on $\mathcal{CM}(X \sqcup Y)$. Its construction uses double homology and cohomology complexes naturally associated with the Γ-action on Ω. Namely, the homology double complex $\mathsf{K}_{\bullet,\bullet}$ is defined as a tensor product over the integral group ring $\mathbb{Z}\Gamma$ of the standard singular chain complex of Ω and the canonical bar-resolution complex for Γ, and the cohomology double complex $\mathsf{C}^{\bullet,\bullet}$ is bar-de Rham complex on Ω. The cohomology construction starts with the 2-form $\omega[\phi] \in \mathsf{C}^{2,0}$, where ϕ satisfies (1.7), and introduces $\theta[\phi] \in \mathsf{C}^{1,1}$ and $u \in \mathsf{C}^{1,2}$ by

$$\theta_{\gamma^{-1}}[\phi] = \left(\phi - \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right),$$

and

$$\begin{aligned} u_{\gamma_1^{-1},\gamma_2^{-1}} = &-\frac{1}{2}\log|\gamma_1'|^2\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1'\,dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}\,d\bar{z}\right) \\ &+\frac{1}{2}\log|\gamma_2'\circ\gamma_1|^2\left(\frac{\gamma_1''}{\gamma_1'}dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar{z}\right). \end{aligned}$$

Define $\Theta \in \mathsf{C}^{0,2}$ to be a group 2-cocycle satisfying $d\Theta = u$. The resulting cochain $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$ is a cocycle of degree 2 in the total complex $\operatorname{Tot}\mathsf{C}$. The corresponding homology construction starts with a fundamental domain $F \in \mathsf{K}_{2,0}$ for Γ in Ω and introduces chains $L \in \mathsf{K}_{1,1}$ and $V \in \mathsf{K}_{0,2}$ such that $\Sigma = F + L - V$ is a cycle of degree 2 in the total homology complex $\operatorname{Tot}\mathsf{K}$. The Liouville action functional is given by the evaluation map,

$$S[\phi] = \frac{i}{2}\,\langle\Psi[\phi], \Sigma\rangle, \tag{1.11}$$

where $\langle\ ,\ \rangle$ is the natural pairing between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$.

In the case when Γ is a Fuchsian group, the Liouville action functional on $X \simeq \Gamma\backslash\mathbb{U}$, similar to (1.8), can be written explicitly as follows:

$$\begin{aligned} S[\phi] =& \frac{i}{2}\iint\limits_F \omega[\phi] + \frac{i}{2}\sum_{k=1}^{g}\left(\int_{a_k}\theta_{\alpha_k}[\phi] - \int_{b_k}\theta_{\beta_k}[\phi]\right) \\ &+ \frac{i}{2}\sum_{k=1}^{g}\left(\Theta_{\alpha_k,\beta_k}(a_k(0)) - \Theta_{\beta_k,\alpha_k}(b_k(0)) + \Theta_{\gamma_k^{-1},\alpha_k\beta_k}(b_k(0))\right) \\ &- \frac{i}{2}\sum_{k=1}^{g}\Theta_{\gamma_g^{-1}\cdots\gamma_{k+1}^{-1},\gamma_k^{-1}}(b_g(0)), \end{aligned}$$

where

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + 4\pi i\varepsilon_{\gamma_1,\gamma_2}(2\log 2 + \log|c(\gamma_2)|^2),$$

$p \in \mathbb{R} \setminus \Gamma(\infty)$ and

$$\varepsilon_{\gamma_1,\gamma_2} = \begin{cases} 1 & \text{if } p < \gamma_2(\infty) < \gamma_1^{-1}p, \\ -1 & \text{if } p > \gamma_2(\infty) > \gamma_1^{-1}p, \\ 0 & \text{otherwise.} \end{cases}$$

Here a_k and b_k are the edges of the fundamental domain F for Γ in $\mathbb{U}$ (see Sect. 2.2.1) with initial points $a_k(0)$ and $b_k(0)$, α_k and β_k are corresponding generators of Γ and $\gamma_k = \alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}$. The action functional does not depend on the choice of the fundamental domain F for Γ, nor on the choice of $p \in \mathbb{R} \setminus \Gamma(\infty)$. Liouville action for quasi-Fuchsian group Γ is defined by a similar construction where both components of Ω are used (see Sect. 2.3.3).

Equation (1.10) is the global quasi-Fuchsian reciprocity. McMullen's quasi-Fuchsian reciprocity, as well as the equation $\partial(P_F - P_{QF}) = 0$, immediately follow from it. The classical action $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ is symmetric with respect to Riemann surfaces X and Y,

$$S(X, Y) = S(\bar{Y}, \bar{X}), \tag{1.12}$$

where $\bar{X}$ is the mirror image of X, and this property manifests the global quasi-Fuchsian reciprocity. Equation (1.9) now follows from (1.10) and (1.1). Its direct proof along the lines of [ZT87a, ZT87b] is given in Theorem 4.2. As an immediate corollary of (1.9) and (1.10), we obtain that the function $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$.

Our second result is a precise relation between two and three-dimensional constructions which establishes the holography principle for the quasi-Fuchsian case. Let $\mathbb{U}^3 = \{Z = (x, y, t) \in \mathbb{R}^3 \,|\, t > 0\}$ be hyperbolic 3-space. The quasi-Fuchsian group Γ acts discontinuously on $\mathbb{U}^3 \cup \Omega$ and the quotient $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is a hyperbolic 3-manifold with boundary $\Gamma\backslash\Omega \simeq X \sqcup Y$. According to the holography principle (see, e.g., [MM02] for a mathematically oriented exposition), the regularized hyperbolic volume of M – *on-shell Einstein-Hilbert action with a cosmological term*, is related to the Liouville action functional $S[\phi]$.

In the case when Γ is a classical Schottky group, i.e., when it has a fundamental domain bounded by Euclidean circles, the holography principle was established by K. Krasnov in [Kra00]. Namely, let $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ be the corresponding hyperbolic 3-manifold (realized using the Ford fundamental region) with boundary $X \simeq \Gamma\backslash\Omega$ – a compact Riemann surface of genus $g > 1$. For every $ds^2 = e^{\phi}|dz|^2 \in \mathcal{CM}(X)$ consider the family $\mathcal{H}_\varepsilon$ of surfaces given by the equation $f(Z) = te^{\phi(z)/2} = \varepsilon > 0$, where $z = x + iy$, and let $M_\varepsilon = M \cap \mathcal{H}_\varepsilon$. Denote by $V_\varepsilon[\phi]$ the hyperbolic volume of M_ε, by $A_\varepsilon[\phi]$ – the area of the boundary of M_ε in the metric on $\mathcal{H}_\varepsilon$ induced by the hyperbolic metric on $\mathbb{U}^3$, and by $A[\phi]$ – the area of X in the metric ds^2. In [Kra00] K. Krasnov obtained the following formula:

$$\lim_{\varepsilon\to 0}\left(V_\varepsilon[\phi] - \frac{1}{2}A_\varepsilon[\phi] + (2g-2)\pi\log\varepsilon\right) = -\frac{1}{4}\left(S[\phi] - A[\phi]\right). \tag{1.13}$$

It relates three-dimensional data – the regularized volume of M, to the two-dimensional data – the Liouville action functional $S[\phi]$, thus establishing the holography principle. Note that the metric ds^2 on the boundary of M appears entirely through regularization by means of the surfaces $\mathcal{H}_\varepsilon$, which are not Γ-invariant. As a result, arguments in [Kra00] work only for classical Schottky groups.

We extend homological algebra methods in [AT97] to the three-dimensional case when Γ is a quasi-Fuchsian group. Namely, we construct the Γ-invariant cut-off function f using a partition of unity for Γ, and prove in Theorem 5.1 that the on-shell regularized Einstein-Hilbert action functional

$$\mathcal{E}[\phi] = -4\lim_{\varepsilon\to 0}\left(V_\epsilon[\phi] - \frac{1}{2}A_\epsilon[\phi] + 2\pi(2g-2)\log\varepsilon\right),$$

is well-defined and satisfies the quasi-Fuchsian holography principle

$$\mathcal{E}[\phi] = S[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^{\phi} d^2z - 8\pi(2g-2)\log 2.$$

As an immediate corollary, we get another proof that the Liouville action functional $S[\phi]$ does not depend on the choice of fundamental domain F of Γ in Ω, provided it is the intersection of the fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ with Ω.

We also show that Γ-invariant cut-off surfaces $\mathcal{H}_\varepsilon$ can be chosen to be Epstein surfaces, which are naturally associated with the family of metrics $ds^2_\varepsilon = 4\varepsilon^{-2}e^{\phi}|dz|^2 \in \mathcal{CM}(X)$ by the inverse of the "hyperbolic Gauss map" [Eps84, Eps86] (see also [And98]). This construction also gives a geometric interpretation of the density $|\phi_z|^2 + e^{\phi}$ in terms of Epstein surfaces.

The Schottky and quasi-Fuchsian groups considered above are basically the only examples of geometrically finite, purely loxodromic Kleinian groups with finitely many components. Indeed, according to a theorem of Maskit [Mas88], every geometrically finite, purely loxodromic Kleinian group which has finitely many components in fact has at most two components. The one-component case corresponds to the Schottky groups and the two-component case – to the Fuchsian or quasi-Fuchsian groups and their $\mathbb{Z}_2$-extensions.

The third result of the paper is the generalization of the main results for quasi-Fuchsian groups – Theorems 4.1, 4.2 and 5.1, to Kleinian groups. Namely, we introduce the notion of a Kleinian group of Class A for which this generalization holds. By definition, a non-elementary, purely loxodromic, geometrically finite Kleinian group is of Class A if it has fundamental region R in $\mathbb{U}^3 \cup \Omega$ which is a finite three-dimensional CW-complex with no vertices in $\mathbb{U}^3$. The Schottky, Fuchsian, quasi-Fuchsian groups, and their free combinations are of Class A, and Class A is stable under quasiconformal deformations. We extend three-dimensional homological methods developed in Sect. 5 to the case of the Kleinian group Γ of Class A acting on $\mathbb{U}^3 \cup \Omega$. Namely, starting from the fundamental region R for Γ in $\mathbb{U}^3 \cup \Omega$, we construct a chain of degree 3 in total homology complex Tot K, whose boundary in Ω is a cycle Σ of degree 2 for the corresponding total homology complex of the region of discontinuity Ω. In Theorem 6.1 we establish the holography principle for Kleinian groups: we prove that the on-shell regularized Einstein-Hilbert action for the 3-manifold $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is well-defined and is related to the Liouville action functional for Γ, defined by the evaluation map (1.11). When Γ is a Schottky group, we get the functional (1.8) introduced in [ZT87b]. As in the quasi-Fuchsian case, the Liouville action functional does not depend on the choice of a fundamental domain F for Γ in Ω, as long as it is the intersection of a fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ with Ω. Denote by $\mathfrak{D}(\Gamma)$ the deformation space of a Kleinian group Γ. To every point $\Gamma' \in \mathfrak{D}(\Gamma)$ with the region of discontinuity Ω' there corresponds a disjoint union $X_1 \sqcup \cdots \sqcup X_n \simeq \Gamma'\backslash\Omega'$ of compact Riemann surfaces simultaneously uniformized by the Kleinian group Γ'. Conversely, by a theorem of Maskit [Mas88], for a given sequence of compact Riemann surfaces $X_1, \ldots, X_n$ there is a Kleinian group which simultaneously uniformizes them. Using the same notation, we denote by P_F the projective connection on $X_1 \sqcup \cdots \sqcup X_n$ given by the Fuchsian uniformization of these Riemann surfaces and by P_K – the projective connection given by their simultaneous uniformization by a Kleinian group ($P_K = P_{QF}$ for the quasi-Fuchsian case). Let $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ be the classical Liouville action. Theorem 6.2 states that

$$P_F - P_K = \frac{1}{2}\,\partial S,$$

which is the ultimate generalization of (1.1). Similarly, Theorem 6.3 is the statement

$$\bar{\partial}(P_F - P_K) = -i\,\omega_{WP},$$

which implies that $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$. As another immediate corollary of Theorem 6.2 we get McMullen's Kleinian reciprocity – Theorem 6.4.

Finally, we observe that our methods and results, with appropriate modifications, can be generalized to the case when quasi-Fuchsian and Class A Kleinian groups have torsion and contain parabolic elements. Our method also works for the Bers' universal Teichmüller space $T(1)$ and the related infinite-dimensional Kähler manifold $\mathrm{Diff}_+(S^1)/\,\mathrm{M\ddot{o}b}(S^1)$. We plan to discuss these generalizations elsewhere.

The content of the paper is the following. In Sect. 2 we give construction of the Liouville action functional following the method in [AT97], which we review briefly in 2.1. In Sect. 2.2 we define and establish the main properties of the Liouville action functional in the model case when Γ is a Fuchsian group, and in Sect. 2.3 we consider the technically more involved quasi-Fuchsian case. In Sect. 3 we recall all necessary basic facts from deformation theory. In Sect. 4 we prove our first main result – Theorems 4.1 and 4.2. In Sect. 5 we prove the second main result – Theorem 5.1 on the quasi-Fuchsian holography. Finally in Sect. 6 we generalize these results for Kleinian groups of Class A: we define the Liouville action functional and prove Theorems 6.1, 6.2 and 6.3.

2. Liouville Action Functional

Let Γ be a normalized, marked, purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity Ω, so that $\Gamma\backslash\Omega \simeq X \sqcup Y$, where X and Y are compact Riemann surfaces of genus $g > 1$ with opposite orientations. Here we define the Liouville action functional S_Γ for the group Γ as a functional on the space of smooth conformal metrics on $X \sqcup Y$ with the property that its Euler-Lagrange equation is the Liouville equation on $X \sqcup Y$. Its definition is based on the homological algebra methods developed in [AT97].

2.1. Homology and cohomology set-up. Let Γ be a group acting properly on a smooth manifold M. To this data one canonically associates double homology and cohomology complexes (see, e.g., [AT97] and references therein).

Let $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(M)$ be the standard singular chain complex of M with the differential ∂'. The group action on M induces a left Γ-action on $\mathsf{S}_\bullet$ by translating the chains and $\mathsf{S}_\bullet$ becomes a complex of left Γ-modules. Since the action of Γ on M is proper, $\mathsf{S}_\bullet$ is a complex of free left $\mathbb{Z}\Gamma$-modules, where $\mathbb{Z}\Gamma$ is the integral group ring of the group Γ. The complex $\mathsf{S}_\bullet$ is endowed with a right $\mathbb{Z}\Gamma$-module structure in the standard fashion: $c \cdot \gamma = \gamma^{-1}(c)$.

Let $\mathsf{B}_\bullet \equiv \mathsf{B}_\bullet(\mathbb{Z}\Gamma)$ be the canonical "bar" resolution complex for Γ with differential ∂''. Each $\mathsf{B}_n(\mathbb{Z}\Gamma)$ is a free left Γ-module on generators $[\gamma_1|\dots|\gamma_n]$, with the differential $\partial'' : \mathsf{B}_n \longrightarrow \mathsf{B}_{n-1}$ given by

$$\begin{aligned}\partial''[\gamma_1|\dots|\gamma_n] = \gamma_1[\gamma_2|\dots|\gamma_n] + \sum_{k=1}^{n-1}(-1)^k[\gamma_1|\dots|\gamma_k\gamma_{k+1}|\dots|\gamma_n] \\ + (-1)^n[\gamma_1|\dots|\gamma_{n-1}], \quad n > 1, \\ \partial''[\gamma] = \gamma\,[\;] - [\;], \quad n = 1,\end{aligned}$$

where $[\gamma_1|\dots|\gamma_n]$ is zero if some γ_i equals to the unit element id in Γ. Here $\mathsf{B}_0(\mathbb{Z}\Gamma)$ is a $\mathbb{Z}\Gamma$-module on one generator $[\;]$ and it can be identified with $\mathbb{Z}\Gamma$ under the isomorphism that sends $[\;]$ to 1; by definition, $\partial''[\;] = 0$.

The double homology complex $\mathsf{K}_{\bullet,\bullet}$ is defined as $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$, where the tensor product over $\mathbb{Z}\Gamma$ uses the right Γ-module structure on $\mathsf{S}_\bullet$. The associated total complex Tot K is equipped with the total differential $\partial = \partial' + (-1)^p\partial''$ on $\mathsf{K}_{p,q}$, and the complex $\mathsf{S}_\bullet$ is identified with $\mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_0$ by the isomorphism $c \mapsto c \otimes [\;]$.

The corresponding double complex in cohomology is defined as follows. Denote by $\mathsf{A}^\bullet \equiv \mathsf{A}^\bullet_{\mathbb{C}}(M)$ the complexified de Rham complex on M. Each A^n is a left Γ-module with the pull-back action of Γ, i.e., $\gamma \cdot \varpi = (\gamma^{-1})^*\varpi$ for $\varpi \in \mathsf{A}^\bullet$ and $\gamma \in \Gamma$. Define the double complex $\mathsf{C}^{p,q} = \mathrm{Hom}_{\mathbb{C}}(\mathsf{B}_q, \mathsf{A}^p)$ with differentials d, the usual de Rham differential, and $\delta = (\partial'')^*$, the group coboundary. Specifically, for $\varpi \in \mathsf{C}^{p,q}$,

$$\begin{aligned}(\delta\varpi)_{\gamma_1,\cdots,\gamma_{q+1}} = \gamma_1 \cdot \varpi_{\gamma_2,\cdots,\gamma_{q+1}} + \sum_{k=1}^{q}(-1)^k\varpi_{\gamma_1,\cdots,\gamma_k\gamma_{k+1},\cdots,\gamma_{q+1}} \\ + (-1)^{q+1}\varpi_{\gamma_1,\cdots,\gamma_q}.\end{aligned}$$

We write the total differential on $\mathsf{C}^{p,q}$ as $D = d + (-1)^p\delta$.

There is a natural pairing between $\mathsf{C}^{p,q}$ and $\mathsf{K}_{p,q}$ which assigns to the pair $(\varpi, c \otimes [\gamma_1|\dots|\gamma_q])$ the evaluation of the p-form $\varpi_{\gamma_1,\cdots,\gamma_q}$ over the p-cycle c,

$$\langle \varpi, c \otimes [\gamma_1|\dots|\gamma_q]\rangle = \int_c \varpi_{\gamma_1,\cdots,\gamma_q}.$$

By definition,

$$\langle \delta\varpi, c\rangle = \langle \varpi, \partial'' c\rangle,$$

so that using Stokes' theorem we get

$$\langle D\varpi, c\rangle = \langle \varpi, \partial c\rangle.$$

This pairing defines a non-degenerate pairing between corresponding cohomology and homology groups $H^\bullet(\mathrm{Tot}\,\mathsf{C})$ and $H_\bullet(\mathrm{Tot}\,\mathsf{K})$, which we continue to denote by $\langle\ ,\ \rangle$. In particular, if Φ is a cocycle in $(\mathrm{Tot}\,\mathsf{C})^n$ and C is a cycle in $(\mathrm{Tot}\,\mathsf{K})_n$, then the pairing $\langle \Phi, C\rangle$ depends only on cohomology classes $[\Phi]$ and $[C]$ and not on their representatives.

It is this property that will allow us to define the Liouville action functional by constructing the corresponding cocycle Ψ and cycle Σ. Specifically, we consider the following two cases.

1. Γ is purely hyperbolic Fuchsian group of genus $g > 1$ and $M = \mathbb{U}$ – the upper half-plane of the complex plane $\mathbb{C}$. In this case, since $\mathbb{U}$ is acyclic, we have [AT97]

$$H_\bullet(X, \mathbb{Z}) \cong H_\bullet(\Gamma, \mathbb{Z}) \cong H_\bullet(\mathrm{Tot}\,\mathsf{K})\,,$$

where the three homologies are: the singular homology of $X \simeq \Gamma\backslash\mathbb{U}$, a compact Riemann surface of genus $g > 1$, the group homology of Γ, and the homology of the complex $\mathrm{Tot}\,\mathsf{K}$ with respect to the total differential ∂. Similarly, for $M = \mathbb{L}$ – the lower half-plane of the complex plane $\mathbb{C}$, we have

$$H_\bullet(\bar{X}, \mathbb{Z}) \cong H_\bullet(\Gamma, \mathbb{Z}) \cong H_\bullet(\mathrm{Tot}\,\mathsf{K})\,,$$

where $\bar{X} \simeq \Gamma\backslash\mathbb{L}$ is the mirror image of X – a complex-conjugate of the Riemann surface X.

2. Γ is purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity Ω consisting of two simply-connected components Ω_1 and Ω_2 separated by a quasi-circle $\mathcal{C}$. The same isomorphisms hold, where $X \simeq \Gamma\backslash\Omega_1$ and $\bar{X}$ is replaced by $Y \simeq \Gamma\backslash\Omega_2$.

2.2. The Fuchsian case. Let Γ be a marked, normalized, purely hyperbolic Fuchsian group of genus $g > 1$, let $X \simeq \Gamma\backslash\mathbb{U}$ be the corresponding marked compact Riemann surface of genus g, and let $\bar{X} \simeq \Gamma\backslash\mathbb{L}$ be its mirror image. In this case it is possible to define Liouville action functionals on Riemann surfaces X and $\bar{X}$ separately. The definition will be based on the following specialization of the general construction in Sect. 2.1.

2.2.1. Homology computation. Here is a representation of the fundamental class $[X]$ of the Riemann surface X in $H_2(X,\mathbb{Z})$ as a cycle Σ of total degree 2 in the homology complex $\operatorname{Tot}\mathsf{K}$ [AT97].

Recall that the marking of Γ is given by a system of $2g$ standard generators $\alpha_1,\dots,\alpha_g$, $\beta_1,\dots,\beta_g$ satisfying the single relation

$$\gamma_1\cdots\gamma_g=\mathrm{id},$$

where $\gamma_k=[\alpha_k,\beta_k]=\alpha_k\beta_k\alpha_k^{-1}\beta_k^{-1}$. The marked group Γ is normalized, if the attracting and repelling fixed points of α_1 are, respectively, 0 and ∞, and the attracting fixed point of β_1 is 1. Every marked Fuchsian group Γ is conjugated in $\mathrm{PSL}(2,\mathbb{R})$ to a normalized marked Fuchsian group. For a given marking there is a standard choice of the fundamental domain $F\subset\mathbb{U}$ for Γ as a closed non-Euclidean polygon with $4g$ edges labeled by a_k, a'_k, b'_k, b_k satisfying $\alpha_k(a'_k)=a_k,\ \beta_k(b'_k)=b_k,\ k=1,2,\dots,g$ (see Fig. 1). The orientation of the edges is chosen such that

$$\partial' F=\sum_{k=1}^{g}(a_k+b'_k-a'_k-b_k).$$

Set $\partial' a_k=a_k(1)-a_k(0),\ \partial' b_k=b_k(1)-b_k(0)$, so that $a_k(0)=b_{k-1}(0)$. The relations between the vertices of F and the generators of Γ are the following: $\alpha_k^{-1}(a_k(0))=b_k(1),\ \beta_k^{-1}(b_k(0))=a_k(1),\ \gamma_k(b_k(0))=b_{k-1}(0)$, where $b_0(0)=b_g(0)$.

According to the isomorphism $\mathsf{S}_\bullet\simeq\mathsf{K}_{\bullet,0}$, the fundamental domain F is identified with $F\otimes[\ ]\in\mathsf{K}_{2,0}$. We have $\partial'' F=0$ and, as it follows from the previous formula,

$$\partial' F=\sum_{k=1}^{g}\left(\beta_k^{-1}(b_k)-b_k-\alpha_k^{-1}(a_k)+a_k\right)=\partial'' L,$$

where $L\in\mathsf{K}_{1,1}$ is given by

$$L=\sum_{k=1}^{g}\left(b_k\otimes[\beta_k]-a_k\otimes[\alpha_k]\right). \tag{2.1}$$

There exists $V\in\mathsf{K}_{0,2}$ such that $\partial'' V=\partial' L$. A straightforward computation gives the following explicit expression:

$$V=\sum_{k=1}^{g}\left(a_k(0)\otimes[\alpha_k|\beta_k]-b_k(0)\otimes[\beta_k|\alpha_k]+b_k(0)\otimes\left[\gamma_k^{-1}|\alpha_k\beta_k\right]\right)$$
$$-\sum_{k=1}^{g-1}b_g(0)\otimes\left[\gamma_g^{-1}\dots\gamma_{k+1}^{-1}|\gamma_k^{-1}\right]. \tag{2.2}$$

Using $\partial'' F=0$, $\partial' F=\partial'' L$, $\partial'' V=\partial' L$, and $\partial' V=0$, we obtain that the element $\Sigma=F+L-V$ of total degree 2 is a cycle in $\operatorname{Tot}\mathsf{K}$, that is $\partial\Sigma=0$. The cycle $\Sigma\in(\operatorname{Tot}\mathsf{K})_2$ represents the fundamental class $[X]$. It is proved in [AT97] that the corresponding homology class $[\Sigma]$ in $H_\bullet(\operatorname{Tot}\mathsf{K})$ does not depend on the choice of the fundamental domain F for the group Γ.

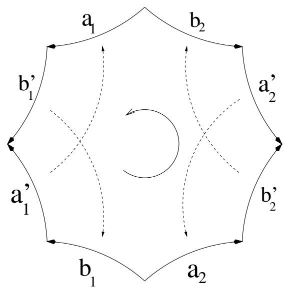

Fig. 1. Conventions for the fundamental domain F

2.2.2. Cohomology computation. The corresponding construction in cohomology is the following. Start with the space $\mathcal{CM}(X)$ of all conformal metrics on $X \simeq \Gamma\backslash\mathbb{U}$. Every $ds^2 \in \mathcal{CM}(X)$ can be represented as $ds^2 = e^{\phi}|dz|^2$, where $\phi \in C^\infty(\mathbb{U}, \mathbb{R})$ satisfies

$$\phi \circ \gamma + \log|\gamma'|^2 = \phi \quad \text{for all } \gamma \in \Gamma. \tag{2.3}$$

In what follows we will always identify $\mathcal{CM}(X)$ with the affine subspace of $C^\infty(\mathbb{U}, \mathbb{R})$ defined by (2.3).

The "bulk" 2-form ω for the Liouville action is given by

$$\omega[\phi] = \left(|\phi_z|^2 + e^{\phi}\right) dz \wedge d\bar{z}, \tag{2.4}$$

where $\phi \in \mathcal{CM}(X)$. Considering it as an element in $\mathsf{C}^{2,0}$ and using (2.3) we get

$$\delta\omega[\phi] = d\theta[\phi],$$

where $\theta[\phi] \in \mathsf{C}^{1,1}$ is given explicitly by

$$\theta_{\gamma^{-1}}[\phi] = \left(\phi - \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right). \tag{2.5}$$

Next, set

$$u = \delta\theta[\phi] \in \mathsf{C}^{1,2}.$$

From the definition of θ and $\delta^2 = 0$ it follows that the 1-form u is closed. An explicit calculation gives

$$\begin{aligned} u_{\gamma_1^{-1},\gamma_2^{-1}} = &-\frac{1}{2}\log|\gamma_1'|^2\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1'\,dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}\,d\bar{z}\right) \\ &+\frac{1}{2}\log|\gamma_2'\circ\gamma_1|^2\left(\frac{\gamma_1''}{\gamma_1'}dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}d\bar{z}\right), \end{aligned} \tag{2.6}$$

and shows that u does not depend on $\phi \in \mathcal{CM}(X)$.

Remark 2.1. The explicit formulas above are valid in the general case, when the domain $\Omega \subset \hat{\mathbb{C}}$ is invariant under the action of a Kleinian group Γ. Namely, define the 2-form ω by formula (2.4), where ϕ satisfies (2.3) in Ω. Then the solution θ to the equation $\delta\omega[\phi] = d\theta[\phi]$ is given by the formula (2.5) and $u = \delta\theta[\phi]$ – by (2.6).

There exists a cochain $\Theta \in \mathsf{C}^{0,2}$ satisfying

$$d\Theta = u \text{ and } \delta\Theta = 0.$$

Indeed, since the 1-form u is closed and $\mathbb{U}$ is simply-connected, Θ can be defined as a particular antiderivative of u satisfying $\delta\Theta = 0$. This can be done as follows. Consider the hyperbolic (Poincaré) metric on $\mathbb{U}$

$$e^{\phi_{hyp}(z)}|dz|^2 = \frac{|dz|^2}{y^2}, \quad z = x + iy \in \mathbb{U}.$$

This metric is $\mathrm{PSL}(2, \mathbb{R})$-invariant and its push-forward to X is a hyperbolic metric on X. Explicit computation yields

$$\omega[\phi_{hyp}] = 2e^{\phi_{hyp}}\, dz \wedge d\bar{z},$$

so that $\delta\omega[\phi_{hyp}] = 0$. Thus the 1-form $\theta[\phi_{hyp}]$ on $\mathbb{U}$ is closed and, therefore, is exact,

$$\theta[\phi_{hyp}] = dl,$$

for some $l \in \mathsf{C}^{0,1}$. Set

$$\Theta = \delta l. \tag{2.7}$$

It is now immediate that $\delta\Theta = 0$ and $\delta\theta[\phi] = u = d\Theta$ for all $\phi \in \mathcal{CM}(X)$. Thus $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$ is a 2-cocycle in the cohomology complex $\mathrm{Tot}\,\mathsf{C}$, that is, $D\Psi[\phi] = 0$.

Remark 2.2. For every $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ define the 1-form $\theta_\gamma\left[\phi_{hyp}\right]$ by the same formula (2.5),

$$\theta_{\gamma^{-1}}[\phi_{hyp}] = -\left(2\log y + \frac{1}{2}\log|\gamma'|^2\right)\left(\frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right). \tag{2.8}$$

Since for every $\gamma \in \mathrm{PSL}(2, \mathbb{R})$,

$$(\delta \log y)_{\gamma^{-1}} = \log(y \circ \gamma) - \log y = \frac{1}{2}\log|\gamma'|^2,$$

the 1-form $u = \delta\theta[\phi]$ is still given by (2.6) and is a $\mathsf{A}^1(\mathbb{U})$-valued group 2-cocycle for $\mathrm{PSL}(2, \mathbb{R})$, that is, $(\delta u)_{\gamma_1,\gamma_2,\gamma_3} = 0$ for all $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{PSL}(2, \mathbb{R})$. Also 0-form Θ given by (2.7) satisfies $d\Theta = u$ and is a $\mathsf{A}^0(\mathbb{U})$-valued group 2-cocycle for $\mathrm{PSL}(2, \mathbb{R})$.

2.2.3. The action functional. The evaluation map $\langle \Psi[\phi], \Sigma \rangle$ does not depend on the choice of the fundamental domain F for Γ [AT97]. It also does not depend on a particular choice of antiderivative l, since by the Stokes' theorem

$$\langle \Theta, V \rangle = \langle \delta l, V \rangle = \langle l, \partial'' V \rangle = \langle l, \partial' L \rangle = \langle \theta[\phi_{hyp}], L \rangle. \tag{2.9}$$

This justifies the following definition.

Definition 2.1. *The Liouville action functional $S[\,\cdot\,; X] : \mathcal{CM}(X) \to \mathbb{R}$ is defined by the evaluation map*

$$S[\phi; X] = \frac{i}{2} \langle \Psi[\phi], \Sigma \rangle \,, \;\; \phi \in \mathcal{CM}(X).$$

For brevity, set $S[\phi] = S[\phi; X]$. The following lemma shows that the difference of any two values of the functional S is given by the bulk term only.

Lemma 2.1. *For all $\phi \in \mathcal{CM}(X)$ and $\sigma \in C^{\infty}(X, \mathbb{R})$,*

$$S[\phi + \sigma] - S[\phi] = \iint\limits_F \left(|\sigma_z|^2 + \left(e^{\sigma} + K\,\sigma - 1 \right) e^{\phi} \right) d^2 z,$$

where $d^2 z = dx \wedge dy$ is the Lebesgue measure and $K = -2e^{-\phi}\phi_{z\bar{z}}$ is the Gaussian curvature of the metric $e^{\phi}|dz|^2$.

Proof. We have

$$\omega[\phi + \sigma] - \omega[\phi] = \omega[\phi; \sigma] + d\tilde{\theta},$$

where

$$\omega[\phi; \sigma] = \left(|\sigma_z|^2 + \left(e^{\sigma} + K\,\sigma - 1 \right) e^{\phi} \right) dz \wedge d\bar{z},$$

and

$$\tilde{\theta} = \sigma \left(\phi_{\bar{z}} d\bar{z} - \phi_z dz \right).$$

Since

$$\delta\tilde{\theta}_{\gamma^{-1}} = \sigma \left(\frac{\gamma''}{\gamma'} dz - \frac{\overline{\gamma''}}{\overline{\gamma'}} d\bar{z} \right) = \theta[\phi + \sigma] - \theta[\phi],$$

the assertion of the lemma follows from the Stokes' theorem. □

Corollary 2.1. *The Euler-Lagrange equation for the functional S is the Liouville equation, the critical point of S – the hyperbolic metric ϕ_{hyp}, is non-degenerate, and the classical action – the critical value of S, is twice the hyperbolic area of X, that is, $4\pi(2g - 2)$.*

Proof. As it follows from Lemma 2.1,

$$\left.\frac{dS[\phi+t\sigma]}{dt}\right|_{t=0}=\iint\limits_F (K+1)\,\sigma e^{\phi}d^2z,$$

so that the Euler-Lagrange equation is the Liouville equation $K=-1$. Since

$$\left.\frac{d^2S[\phi_{hyp}+t\sigma]}{dt^2}\right|_{t=0}=\iint\limits_F \left(2|\sigma_z|^2+\sigma^2e^{\phi_{hyp}}\right)d^2z>0 \quad\text{if}\quad \sigma\neq 0,$$

the critical point ϕ_{hyp} is non-degenerate. Using (2.9) we get

$$S[\phi_{hyp}]=\frac{i}{2}\langle\Psi[\phi_{hyp}],\Sigma\rangle=\frac{i}{2}\langle\omega[\phi_{hyp}],F\rangle=2\iint\limits_F\frac{d^2z}{y^2}=4\pi(2g-2).$$

□

Remark 2.3. Let $\Delta[\phi]=-e^{-\phi}\partial_z\partial_{\bar z}$ be the Laplace operator of the metric $ds^2=e^{\phi}|dz|^2$ acting on functions on X, and let $\det\Delta[\phi]$ be its zeta-function regularized determinant (see, e.g., [OPS88] for details). Denote by $A[\phi]$ the area of X with respect to the metric ds^2 and set

$$\mathcal{I}[\phi]=\log\frac{\det\Delta[\phi]}{A[\phi]}.$$

Polyakov's "conformal anomaly" formula [Pol81] reads

$$\mathcal{I}[\phi+\sigma]-\mathcal{I}[\phi]=-\frac{1}{12\pi}\iint\limits_F\left(|\sigma_z|^2+K\sigma\, e^{\phi}\right)d^2z,$$

where $\sigma\in C^{\infty}(X,\mathbb{R})$ (see [OPS88] for rigorous proof). Comparing it with Lemma 2.1 we get

$$\mathcal{I}[\phi+\sigma]+\frac{1}{12\pi}\check{S}[\phi+\sigma]=\mathcal{I}[\phi]+\frac{1}{12\pi}\check{S}[\phi],$$

where $\check{S}[\phi]=S[\phi]-A[\phi]$.

Lemma 2.1, Corollary 2.1 (without the assertion on classical action) and Remark 2.3 remain valid if Θ is replaced by $\Theta+c$, where c is an arbitrary group 2-cocycle with values in $\mathbb{C}$. The choice (2.7), or rather its analog for the quasi-Fuchsian case, will be important in Sect. 4, where we consider classical action for families of Riemann surfaces. For this purpose, we present an explicit formula for Θ as a particular antiderivative of the 1-form u.

Let $p\in\overline{\mathbb{U}}$ be an arbitrary point on the closure of $\mathbb{U}$ in $\mathbb{C}$ (nothing will depend on the choice of p). Set

$$l_\gamma(z)=\int_p^z\theta_\gamma[\phi_{hyp}]\text{ for all }\gamma\in\Gamma, \tag{2.10}$$

where the path of integration P connects points p and z and, possibly except p, lies entirely in $\mathbb{U}$. If $p \in \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, it is assumed that P is smooth and is not tangent to $\mathbb{R}_\infty$ at p. Such paths are called admissible. A 1-form ϑ on $\mathbb{U}$ is called integrable along the admissible path P with the endpoint $p \in \mathbb{R}_\infty$, if the limit of $\int_{p'}^{z} \vartheta$, as $p' \to p$ along P, exists. Similarly, a path P is called Γ-closed if its endpoints are p and γp for some $\gamma \in \Gamma$, and $P \setminus \{p, \gamma p\} \subset \mathbb{U}$. A Γ-closed path P with endpoints p and γp, $p \in \mathbb{R}_\infty$, is called admissible if it is not tangent to $\mathbb{R}_\infty$ at p and there exists $p' \in P$ such that the translate by γ of the part of P between the points p' and p belongs to P. A 1-form ϑ is integrable along the Γ-closed admissible path P, if the limit of $\int_{p'}^{\gamma p'} \vartheta$, as $p' \to p$ along P, exists.

Let

$$W = \sum_{k=1}^{g} \Big(P_{k-1} \otimes [\alpha_k|\beta_k] - P_k \otimes [\beta_k|\alpha_k] + P_k \otimes \big[\gamma_k^{-1}|\alpha_k\beta_k\big] \Big) - \sum_{k=1}^{g-1} P_g \otimes \big[\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}\big] \in \mathsf{K}_{1,2}, \tag{2.11}$$

where P_k is any admissible path from p to $b_k(0)$, $k = 1, \dots, g$, and $P_g = P_0$. Since $P_k(1) = b_k(0) = a_{k+1}(0)$, we have

$$\partial' W = V - U,$$

where

$$U = \sum_{k=1}^{g} \Big(p \otimes [\alpha_k|\beta_k] - p \otimes [\beta_k|\alpha_k] + p \otimes \big[\gamma_k^{-1}|\alpha_k\beta_k\big] \Big) - \sum_{k=1}^{g-1} p \otimes \big[\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}\big] \in \mathsf{K}_{1,2}. \tag{2.12}$$

We have the following statement.

Lemma 2.2. *Let $\vartheta \in \mathsf{C}^{1,1}$ be a closed 1-form on $\mathbb{U}$ and $p \in \overline{\mathbb{U}}$. In case $p \in \mathbb{R}_\infty$ suppose that $\delta\vartheta$ is integrable along any admissible path with endpoints in $\Gamma \cdot p$ and ϑ is integrable along any Γ-closed admissible path with endpoints in $\Gamma \cdot p$. Then*

$$\langle \vartheta, L \rangle = \langle \delta\vartheta, W \rangle + \sum_{k=1}^{g} \left(\int_p^{\alpha_k^{-1} p} \vartheta_{\beta_k} - \int_p^{\beta_k^{-1} p} \vartheta_{\alpha_k} + \int_p^{\gamma_k p} \vartheta_{\alpha_k\beta_k} - \int_p^{\gamma_{k+1}\dots\gamma_g p} \vartheta_{\gamma_k^{-1}} \right),$$

where paths of integration are admissible if $p \in \mathbb{R}_\infty$.

Proof. Since ϑ_γ is closed and $\mathbb{U}$ is simply-connected, we can define function l_γ on $\mathbb{U}$ by

$$l_\gamma(z) = \int_p^z \vartheta_\gamma,$$

where $p \in \mathbb{U}$. We have, using Stokes' theorem and $d(\delta l) = \delta(dl) = \delta\vartheta$,

$$\begin{aligned}\langle\vartheta, L\rangle &= \langle dl, L\rangle = \langle l, \partial' L\rangle = \langle l, \partial'' V\rangle = \langle \delta l, V\rangle \\ &= \langle \delta l, \partial' W\rangle + \langle \delta l, U\rangle = \langle d(\delta l), W\rangle + \langle \delta l, U\rangle \\ &= \langle \delta\vartheta, W\rangle \;+\; \langle \delta l, U\rangle.\end{aligned}$$

Since

$$(\delta l)_{\gamma_1,\gamma_2}(p) = \int_p^{\gamma_1^{-1}p} \vartheta_{\gamma_2},$$

we get the statement of the lemma if $p \in \mathbb{U}$. In case $p \in \mathbb{R}_\infty$, replace p by $p' \in \mathbb{U}$. Conditions of the lemma guarantee the convergence of integrals as $p' \to p$ along corresponding paths. □

Remark 2.4. Expression $\langle \delta l, U\rangle$, which appears in the statement of the lemma, does not depend on the choice of a particular antiderivative of the closed 1-form ϑ. The same statement holds if we only assume that the 1-form $\delta\vartheta$ is integrable along admissible paths with endpoints in $\Gamma \cdot p$, and the 1-form ϑ has an antiderivative l (not necessarily vanishing at p) such that the limit of $(\delta l)_{\gamma_1,\gamma_2}(p')$, as $p' \to p$ along admissible paths, exists.

Lemma 2.3. *We have*

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + \eta(p)_{\gamma_1,\gamma_2}, \tag{2.13}$$

where $p \in \mathbb{R} \setminus \Gamma(\infty)$ and integration goes along admissible paths. The integration constants $\eta \in \mathsf{C}^{0,2}$ are given by

$$\eta(p)_{\gamma_1,\gamma_2} = 4\pi i\varepsilon(p)_{\gamma_1,\gamma_2}(2\log 2 + \log|c(\gamma_2)|^2), \tag{2.14}$$

and

$$\varepsilon(p)_{\gamma_1,\gamma_2} = \begin{cases} 1 & \text{if } p < \gamma_2(\infty) < \gamma_1^{-1}p, \\ -1 & \text{if } p > \gamma_2(\infty) > \gamma_1^{-1}p, \\ 0 & \text{otherwise}. \end{cases}$$

Here for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ we set $c(\gamma) = c$.

Proof. Since

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} + \int_p^{\gamma_1^{-1}p} \theta_{\gamma_2}[\phi_{hyp}],$$

it is sufficient to verify that

$$\frac{1}{2\pi i}\int_p^{\gamma_1 p} \theta_{\gamma_2^{-1}}[\phi_{hyp}] = \begin{cases} 4\log 2 + 2\log|c(\gamma_2)|^2 & \text{if } p < \gamma_2^{-1}(\infty) < \gamma_1 p, \\ -4\log 2 - 2\log|c(\gamma_2)|^2 & \text{if } p > \gamma_2^{-1}(\infty) > \gamma_1 p, \\ 0 & \text{otherwise}. \end{cases}$$

From (2.8) it follows that $\theta_{\gamma^{-1}}[\phi_{hyp}]$ is a closed 1-form on $\mathbb{U}$, integrable along admissible paths with $p \in \mathbb{R} \setminus \{\gamma^{-1}(\infty)\}$. Denote by $\theta^{(\varepsilon)}_{\gamma^{-1}}$ its restriction on the line $y = \varepsilon > 0$, $z = x + iy$. When $x \neq \gamma_2^{-1}(\infty)$, we obviously have

$$\lim_{\varepsilon \to 0} \theta^{(\varepsilon)}_{\gamma_2^{-1}} = 0,$$

uniformly in x on compact subsets of $\mathbb{R} \setminus \{\gamma_2^{-1}(\infty)\}$.

If $\gamma_2^{-1}(\infty)$ does not lie between points p and $\gamma_1 p$ on $\mathbb{R}$, we can approximate the path of integration by the interval on the line $y = \varepsilon$, which tends to 0 as $\varepsilon \to 0$. If $\gamma_2^{-1}(\infty)$ lies between points p and $\gamma_1 p$, we have to go around the point $\gamma_2^{-1}(\infty)$ via a small half-circle, so that

$$\int_p^{\gamma_1 p} \theta_{\gamma_2^{-1}}[\phi_{hyp}] = \lim_{r \to 0} \int_{C_r} \theta_{\gamma_2^{-1}}[\phi_{hyp}],$$

where C_r is the upper-half of the circle of radius r with center at $\gamma_2^{-1}(\infty)$, oriented clockwise if $p < \gamma_2^{-1}(\infty) < \gamma_1 p$. Evaluating the limit using the elementary formula

$$\int_0^{\pi} \log \sin t \, dt = -\pi \log 2,$$

and the Cauchy theorem, we get the formula. □

Corollary 2.2. *The Liouville action functional has the following explicit representation:*

$$S[\phi] = \frac{i}{2} \left(\langle \omega[\phi], F \rangle - \langle \theta[\phi], L \rangle + \langle u, W \rangle + \langle \eta, V \rangle \right).$$

Remark 2.5. Since $\langle \Theta, V \rangle = \langle u, W \rangle + \langle \eta, V \rangle$, it immediately follows from (2.9) that the Liouville action functional does not depend on the choice of point $p \in \mathbb{R} \setminus \Gamma(\infty)$ (actually it is sufficient to assume that $p \neq \gamma_1(\infty), (\gamma_1\gamma_2)(\infty)$ for all $\gamma_1, \gamma_2 \in \Gamma$ such that $V_{\gamma_1,\gamma_2} \neq 0$). This can also be proved by direct computation using Remark 2.2. Namely, let $p' \in \mathbb{R}_\infty$ be another choice, $p' = \sigma^{-1} p \in \mathbb{R}_\infty$ for some $\sigma \in \mathrm{PSL}(2, \mathbb{R})$. Setting $z = p$ in the equation $(\delta\Theta)_{\sigma,\gamma_1,\gamma_2} = 0$ and using $(\delta u)_{\sigma,\gamma_1,\gamma_2} = 0$, where $\gamma_1, \gamma_2 \in \Gamma$, we get

$$\int_p^{\sigma^{-1} p} u_{\gamma_1,\gamma_2} = -(\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.15}$$

where all paths of integration are admissible. Using

$$\eta(p)_{\sigma\gamma_1,\gamma_2} = \eta(\sigma^{-1} p)_{\gamma_1,\gamma_2} + \eta(p)_{\sigma,\gamma_2},$$

we get from (2.15) that

$$\int_p^z u_{\gamma_1,\gamma_2} + \eta(p)_{\gamma_1,\gamma_2} = \int_{p'}^z u_{\gamma_1,\gamma_2} + \eta(p')_{\gamma_1,\gamma_2} + (\delta\eta_\sigma)_{\gamma_1,\gamma_2},$$

where $(\eta_\sigma)_\gamma = \eta(p)_{\sigma,\gamma}$ is a constant group 1-cochain. The statement now follows from

$$\langle \delta\eta_\sigma, V \rangle = \langle \eta_\sigma, \partial'' V \rangle = \langle \eta_\sigma, \partial' L \rangle = \langle d\eta_\sigma, L \rangle = 0.$$

Another consequence of Lemmas 2.2 and 2.3 is the following.

Corollary 2.3. *Set*

$$\varkappa_{\gamma^{-1}} = \frac{\gamma''}{\gamma'}dz - \frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z} \in \mathsf{C}^{1,1}.$$

Then

$$\langle \varkappa, L\rangle = 4\pi i \langle \varepsilon, V\rangle = 4\pi i\, \chi(X),$$

where $\chi(X) = 2 - 2g$ is the Euler characteristic of Riemann surface $X \simeq \Gamma\backslash\mathbb{U}$.

Proof. Since $\delta\varkappa = 0$, the first equation immediately follows from the proofs of Lemmas 2.2 and 2.3. To prove the second equation, observe that

$$\varkappa = \delta\varkappa_1, \quad \text{where} \quad \varkappa_1 = -\phi_z dz + \phi_{\bar{z}} d\bar{z} \quad \text{and} \quad d\varkappa_1 = 2\phi_{z\bar{z}}\, dz \wedge d\bar{z}.$$

Therefore

$$\langle \varkappa, L\rangle = \langle \delta\varkappa_1, L\rangle = \langle \varkappa_1, \partial'' L\rangle = \langle \varkappa_1, \partial' F\rangle = \langle d\varkappa_1, F\rangle.$$

The Gaussian curvature of the metric $ds^2 = e^{\phi}|dz|^2$ is $K = -2e^{-\phi}\phi_{z\bar{z}}$, so by Gauss-Bonnet we get

$$\langle d\varkappa_1, F\rangle = 2\iint\limits_F \phi_{z\bar{z}}\, dz \wedge d\bar{z} = 2i \iint\limits_{\Gamma\backslash\mathbb{U}} K e^{\phi} d^2 z = 4\pi i\, \chi(X).$$

□

Using this corollary, we can "absorb" the integration constants η by shifting $\theta[\phi] \in \mathsf{C}^{1,1}$ by a multiple of closed 1-form $\varkappa$. Indeed, the 1-form $\theta[\phi]$ satisfies the equation $\delta\omega[\phi] = d\theta[\phi]$ and is defined up to addition of a closed 1-form. Set

$$\check{\theta}_\gamma[\phi] = \theta_\gamma[\phi] - (2\log 2 + \log|c(\gamma)|^2)\varkappa_\gamma, \tag{2.16}$$

and define $\check{u} = \delta\check{\theta}[\phi]$. Explicitly,

$$\begin{aligned} \check{u}_{\gamma_1^{-1},\gamma_2^{-1}} = u_{\gamma_1^{-1},\gamma_2^{-1}} &- \log\frac{|c(\gamma_2)|^2}{|c(\gamma_2\gamma_1)|^2}\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\,\gamma_1' dz - \frac{\overline{\gamma_2''}}{\overline{\gamma_2'}}\circ\gamma_1\,\overline{\gamma_1'}d\bar{z}\right) \\ &+ \log\frac{|c(\gamma_2\gamma_1)|^2}{|c(\gamma_1)|^2}\left(\frac{\gamma_1''}{\gamma_1'}\,dz - \frac{\overline{\gamma_1''}}{\overline{\gamma_1'}}\,d\bar{z}\right), \end{aligned} \tag{2.17}$$

where u is given by (2.6). As it follows from Lemma 2.2 and Corollary 2.3,

$$S[\phi] = \frac{i}{2}\Big(\langle\omega[\phi], F\rangle - \langle\check{\theta}[\phi], L\rangle + \langle\check{u}, W\rangle\Big). \tag{2.18}$$

The Liouville action functional for the mirror image $\bar{X}$ is defined similarly. Namely, for every chain c in the upper half-plane $\mathbb{U}$ denote by $\bar{c}$ its mirror image in the lower half-plane $\mathbb{L}$; the chain $\bar{c}$ has an opposite orientation to c. Set $\bar{\Sigma} = \bar{F} + \bar{L} - \bar{V}$, so that $\partial\bar{\Sigma} = 0$. For $\phi \in \mathcal{CM}(\bar{X})$, considered as a smooth real-valued function on $\mathbb{L}$ satisfying

(2.3), define $\omega[\phi] \in \mathsf{C}^{2,0}$, $\theta[\phi] \in \mathsf{C}^{1,1}$ and $\Theta \in \mathsf{C}^{0,2}$ by the same formulas (2.4), (2.5) and (2.7). Lemma 2.3 has an obvious analog for the lower half-plane $\mathbb{L}$, the analog of formula (2.13) for $z \in \mathbb{L}$ is

$$\Theta_{\gamma_1,\gamma_2}(z) = \int_p^z u_{\gamma_1,\gamma_2} - \eta(p)_{\gamma_1,\gamma_2}, \tag{2.19}$$

where the negative sign comes from the opposite orientation.

Remark 2.6. Similarly to (2.15) we get

$$\int_p^{\sigma^{-1}p} u_{\gamma_1,\gamma_2} = (\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.20}$$

where the path of integration, except the endpoints, lies in $\mathbb{L}$. From (2.15) and (2.20) we obtain

$$\int_C u_{\gamma_1,\gamma_2} = -2(\delta\eta(p))_{\sigma,\gamma_1,\gamma_2}, \tag{2.21}$$

where the path of integration C is a loop that starts at p, goes to $\sigma^{-1}p$ inside $\mathbb{U}$, continues inside $\mathbb{L}$ and ends at p. Note that formula (2.21) can also be verified directly using Stokes' theorem. Indeed, the 1-form u_{γ_1,γ_2} is closed and regular everywhere except points $\gamma_1(\infty)$ and $(\gamma_1\gamma_2)(\infty)$. Integrating over small circles around these points if they lie inside C and using (2.14), we get the result.

Set $\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta$, so that $D\Psi[\phi] = 0$. The Liouville action functional for $\bar{X}$ is defined by

$$[\phi; \bar{X}] = -\frac{i}{2}\langle\Psi[\phi], \bar{\Sigma}\rangle.$$

Using an analog of Lemma 2.2 in the lower half-plane $\mathbb{L}$ and

$$\langle\eta, \bar{V}\rangle = \langle\eta, V\rangle,$$

we obtain

$$S[\phi; \bar{X}] = -\frac{i}{2}\left(\langle\omega[\theta], \bar{F}\rangle - \langle\theta[\phi], \bar{L}\rangle + \langle u, \bar{W}\rangle - \langle\eta, V\rangle\right).$$

Finally, we have the following definition.

Definition 2.2. *The Liouville action functional $S_\Gamma : \mathcal{CM}(X \sqcup \bar{X}) \to \mathbb{R}$ for the Fuchsian group Γ acting on $\mathbb{U} \cup \mathbb{L}$ is defined by*

$$\begin{aligned} S_\Gamma[\phi] &= S[\phi; X] + S[\phi; \bar{X}] = \frac{i}{2}\langle\Psi[\phi], \Sigma - \bar{\Sigma}\rangle \\ &= \frac{i}{2}\left(\langle\omega[\phi], F - \bar{F}\rangle - \langle\theta[\phi], L - \bar{L}\rangle + \langle u, W - \bar{W}\rangle + 2\langle\eta, V\rangle\right), \end{aligned}$$

where $\phi \in \mathcal{CM}(X \sqcup \bar{X})$.

The functional S_Γ satisfies an obvious analog of Lemma 2.1. Its Euler-Lagrange equation is the Liouville equation, so that its single non-degenerate critical point is the hyperbolic metric on $\mathbb{U} \cup \mathbb{L}$. The corresponding classical action is $8\pi(2g-2)$ – twice the hyperbolic area of $X \sqcup \bar{X}$. Similarly to (2.18) we have

$$S_\Gamma[\phi] = \frac{i}{2}\Big(\langle\omega[\phi], F - \bar{F}\rangle - \langle\check{\theta}[\phi], L - \bar{L}\rangle + \langle\check{u}, W - \bar{W}\rangle\Big). \tag{2.22}$$

Remark 2.7. In the definition of S_Γ it is not necessary to choose a fundamental domain for Γ in $\mathbb{L}$ to be the mirror image of the fundamental domain in $\mathbb{U}$ since the corresponding homology class $[\Sigma - \bar{\Sigma}]$ does not depend on the choice of the fundamental domain of Γ in $\mathbb{U} \cup \mathbb{L}$.

2.3. The quasi-Fuchsian case. Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$. Its region of discontinuity Ω has two invariant components Ω_1 and Ω_2 separated by a quasi-circle $\mathcal{C}$. By definition, there exists a quasiconformal homeomorphism J_1 of $\hat{\mathbb{C}}$ with the following properties:

QF1 The mapping J_1 is holomorphic on $\mathbb{U}$ and $J_1(\mathbb{U}) = \Omega_1$, $J_1(\mathbb{L}) = \Omega_2$, and $J_1(\mathbb{R}_\infty) = \mathcal{C}$.
QF2 The mapping J_1 fixes 0, 1 and ∞.
QF3 The group $\tilde{\Gamma} = J_1^{-1} \circ \Gamma \circ J_1$ is Fuchsian.

Due to the normalization, any two maps satisfying **QF1–QF3** agree on $\mathbb{U}$, so that the group $\tilde{\Gamma}$ is independent of the choice of the map J_1. Setting $X \simeq \tilde{\Gamma}\backslash\mathbb{U}$, we get $\tilde{\Gamma}\backslash\mathbb{U}\cup\mathbb{L} \simeq X \sqcup \bar{X}$ and $\Gamma\backslash\Omega \simeq X \sqcup Y$, where X and Y are marked compact Riemann surfaces of genus $g > 1$ with opposite orientations. Conversely, according to Bers' simultaneous uniformization theorem [Ber60], for any pair of marked compact Riemann surfaces X and Y of genus $g > 1$ with opposite orientations there exists a unique, up to a conjugation in $\mathrm{PSL}(2, \mathbb{C})$, quasi-Fuchsian group Γ such that $\Gamma\backslash\Omega \simeq X \sqcup Y$.

Remark 2.8. It is customary (see, e.g., [Ahl87]) to define quasi-Fuchsian groups by requiring that the map J_1 is holomorphic in the lower half-plane $\mathbb{L}$. We will see in Sect. 4 that the above definition is somewhat more convenient.

Let μ be the Beltrami coefficient for the quasiconformal map J_1,

$$\mu = \frac{(J_1)_{\bar{z}}}{(J_1)_z},$$

that is, $J_1 = f^\mu$ – the unique, normalized solution of the Beltrami equation on $\hat{\mathbb{C}}$ with Beltrami coefficient μ. Obviously, $\mu = 0$ on $\mathbb{U}$. Define another Beltrami coefficient $\hat{\mu}$ by

$$\hat{\mu}(z) = \begin{cases} \overline{\mu(\bar{z})} & \text{if } z \in \mathbb{U}, \\ \mu(z) & \text{if } z \in \mathbb{L}. \end{cases}$$

Since $\hat{\mu}$ is a symmetric, normalized solution $f^{\hat{\mu}}$ of the Beltrami equation,

$$f^{\hat{\mu}}_{\bar{z}}(z) = \hat{\mu}(z) f^{\hat{\mu}}_z(z)$$

is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ which preserves $\mathbb{U}$ and $\mathbb{L}$. The quasiconformal map $J_2 = J_1 \circ (f^{\hat{\mu}})^{-1}$ is then conformal on the lower half-plane $\mathbb{L}$ and has properties similar to **QF1–QF3**. In particular, $J_2^{-1} \circ \Gamma \circ J_2 = \hat{\Gamma} = f^{\hat{\mu}} \circ \tilde{\Gamma} \circ (f^{\hat{\mu}})^{-1}$ is a Fuchsian group and $\hat{\Gamma}\backslash\mathbb{L} \simeq Y$. Thus for a given Γ the restriction of the map J_2 to $\mathbb{L}$ does not depend on the choice of J_2 (and hence of J_1). These properties can be summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} & \xrightarrow{J_1 = f^{\mu}} & \Omega_1 \cup \mathcal{C} \cup \Omega_2 \\ \big\downarrow f^{\hat{\mu}} & & \big\uparrow J_2 \\ \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} & \xrightarrow{=} & \mathbb{U} \cup \mathbb{R}_\infty \cup \mathbb{L} \end{array}$$

where maps J_1, J_2 and $f^{\hat{\mu}}$ intertwine corresponding pairs of groups Γ, $\tilde{\Gamma}$ and $\hat{\Gamma}$.

2.3.1. Homology construction. The map J_1 induces a chain map between double complexes $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ for the pairs $\mathbb{U} \cup \mathbb{L}$, $\tilde{\Gamma}$ and Ω, Γ, by pushing forward chains $S_\bullet(\mathbb{U} \cup \mathbb{L}) \ni c \mapsto J_1(c) \in S_\bullet(\Omega)$ and group elements $\tilde{\Gamma} \ni \gamma \mapsto J_1 \circ \gamma \circ J_1^{-1} \in \Gamma$. We will continue to denote this chain map by J_1. Obviously, the chain map J_1 induces an isomorphism between homology groups of corresponding total complexes Tot K.

Let $\Sigma = F + L - V$ be total cycle of degree 2 representing the fundamental class of X in the total homology complex for the pair $\mathbb{U}$, $\tilde{\Gamma}$, constructed in the previous section, and let $\Sigma' = F' + L' - V'$ be the corresponding cycle for $\bar{X}$. The total cycle $\Sigma(\Gamma)$ of degree 2 representing the fundamental class of $X \sqcup Y$ in the total complex for the pair Ω, Γ can be realized as a push-forward of the total cycle $\Sigma(\tilde{\Gamma}) = \Sigma - \Sigma'$ by J_1,

$$\Sigma(\Gamma) = J_1(\Sigma(\tilde{\Gamma})) = J_1(\Sigma) - J_1(\Sigma').$$

We will denote push-forwards by J_1 of the chains F, L, V in $\mathbb{U}$ by F_1, L_1, V_1, and push-forwards of the corresponding chains F', L', V' in $\mathbb{L}$ – by F_2, L_2, V_2, where indices 1 and 2 refer, respectively, to domains Ω_1 and Ω_2.

The definition of chains W_i is more subtle. Namely, the quasi-circle $\mathcal{C}$ is not generally smooth or even rectifiable, so that an arbitrary path from an interior point of Ω_i to $p \in \mathcal{C}$ inside Ω_i is not rectifiable either. Thus if we define W_1 as a push-forward by J_1 of W constructed using arbitrary admissible paths in $\mathbb{U}$, the paths in W_1 in general will no longer be rectifiable. The same applies to the push-forward by J_1 of the corresponding chain in $\mathbb{L}$. However, the definition of $\langle u, W_1\rangle$ uses integration of the 1-form u_{γ_1,γ_2} along the paths in W_1, and these paths should be rectifiable in order that $\langle u, W_1\rangle$ is well-defined. The invariant construction of such paths in Ω_i is based on the following elegant observation communicated to us by M. Lyubich.

Since the quasi-Fuchsian group Γ is normalized, it follows from **QF2** that the Fuchsian group $\tilde{\Gamma} = J_1^{-1} \circ \Gamma \circ J_1$ is also normalized and $\tilde{\alpha}_1 \in \tilde{\Gamma}$ is a dilation $\tilde{\alpha}_1 z = \tilde{\lambda} z$ with the axis $i\mathbb{R}_{\geq 0}$ and $0 < \tilde{\lambda} < 1$. Corresponding loxodromic element $\alpha_1 = J_1 \circ \tilde{\alpha_1} \circ J_1^{-1} \in \Gamma$ is also a dilation $\alpha_1 z = \lambda z$, where $0 < |\lambda| < 1$. Choose $\tilde{z}_0 \in i\mathbb{R}_{\geq 0}$ and denote by $\tilde{I} = [\tilde{z}_0, 0]$ the interval on $i\mathbb{R}_{\geq 0}$ with endpoints $\tilde{z}_0$ and 0 – the attracting fixed point of $\tilde{\alpha}_1$. Set $z_0 = J_1(\tilde{z}_0)$ and $I = J_1(\tilde{I})$. The path I connects points $z_0 \in \Omega_1$ and $0 = J_1(0) \in \mathcal{C}$ inside Ω_1, is smooth everywhere except the endpoint 0, and is rectifiable. Indeed, set $\tilde{I}_0 = [\tilde{z}_0, \tilde{\lambda}\tilde{z}_0] \subset i\mathbb{R}_{>0}$ and cover the interval $\tilde{I}$ by subintervals $\tilde{I}_n$ defined by $\tilde{I}_{n+1} = \tilde{\alpha}_1(\tilde{I}_n)$, $n = 0, 1, \ldots, \infty$. Corresponding paths $I_n = J_1(\tilde{I}_n)$ cover the path I, and due to the property $I_{n+1} = \alpha_1(I_n)$, which follows from **QF3**, we have

$$I = \bigcup_{n=0}^{\infty} \alpha_1^n(I_0).$$

Thus

$$l(I) = \sum_{n=0}^{\infty} |\lambda^n| l(I_0) = \frac{l(I_0)}{1 - |\lambda|} < \infty,$$

where $l(P)$ denotes the Euclidean length of a smooth path P.

The same construction works for every $p \in \mathcal{C} \setminus \{\infty\}$ which is a fixed point of an element in Γ, and we define Γ-contracting paths in Ω_1 at p as follows.

Definition 2.3. *Path P connecting points $z \in \Omega_1$ and $p \in \mathcal{C} \setminus \{\infty\}$ inside Ω_1 is called Γ-contracting in Ω_1 at p, if the following conditions are satisfied.*

C1 *Paths P is smooth except at the point p.*
C2 *The point p is a fixed point for Γ.*
C3 *There exists $p' \in P$ and an arc P_0 on the path P such that the iterates $\gamma^n(P_0)$, $n \in \mathbb{N}$, where $\gamma \in \Gamma$ has p as the attracting fixed point, entirely cover the part of P from the point p' to the point p.*

As in Sect. 2.2, we define Γ-closed paths and Γ-closed contracting paths in Ω_1 at p. The definition of Γ-contracting paths in Ω_2 is analogous. Finally, we define Γ-contracting paths in Ω as follows.

Definition 2.4. *Path P is called Γ-contracting in Ω, if $P = P_1 \cup P_2$, where $P_1 \cap P_2 = p \in \mathcal{C}$, and $P_1 \setminus \{p\} \subset \Omega_1$ and $P_2 \setminus \{p\} \subset \Omega_2$ are Γ-contracting paths at p in the sense of the previous definition.*

Γ-contracting paths are rectifiable.

Lemma 2.4. *Let Γ and Γ' be two marked normalized quasi-Fuchsian groups with regions of discontinuity Ω and Ω', and let f be a normalized quasiconformal homeomorphism of $\mathbb{C}$ which intertwines Γ and Γ' and is smooth in Ω. Then the push-forward by f of a Γ-contracting path in Ω is a Γ'-contracting path in Ω'.*

Proof. Obvious: if p is the attracting fixed point for $\gamma \in \Gamma$, then $p' = f(p)$ is the attracting fixed point for $\gamma' = f \circ \gamma \circ f^{-1} \in \Gamma'$. □

Now define a chain W for the Fuchsian group $\tilde{\Gamma}$ by first connecting points $P_1(1), \ldots, P_g(1)$ to some point $\tilde{z}_0 \in i\mathbb{R}_{>0}$ by smooth paths inside $\mathbb{U}$ and then connecting this point to 0 by $\tilde{I}$. The chain W' in $\mathbb{L}$ is defined similarly. Setting $W_1 = J_1(W)$ and $W_2 = J_1(W')$, we see that the chain $W_1 - W_2$ in Ω consists of Γ-contracting paths in Ω at 0. Connecting $P_1(1), \ldots, P_g(1)$ to 0 by arbitrary Γ-contracting paths at 0 results in 1-chains which are homotopic to the 1-chains W_1 and W_2 in components Ω_1 and Ω_2 respectively. Finally, we define chain $U_1 = U_2$ as push-forward by J_1 of the corresponding chain $U = U'$ with $p = 0$.

2.3.2. Cohomology construction. Let $\mathcal{CM}(X \sqcup Y)$ be the space of all conformal metrics $ds^2 = e^{\phi}|dz|^2$ on $X \sqcup Y$, which we will always identify with the affine space of smooth real-valued functions ϕ on Ω satisfying (2.3). For $\phi \in \mathcal{CM}(X \sqcup Y)$ we define cochains $\omega[\phi]$, $\theta[\phi]$, u, η and Θ in the total cohomology complex Tot C for the pair $\Omega,\ \Gamma$ by the same formulas (2.4), (2.5), (2.6), (2.14) and (2.13), (2.19) as in the Fuchsian case, where $p = 0 \in \mathcal{C}$, integration goes over Γ-contracting paths at 0, and $\tilde{\gamma} \in \tilde{\Gamma}$ are replaced by $\gamma = J_1 \circ \tilde{\gamma} \circ J_1^{-1} \in \Gamma$. The ordering of points on $\mathcal{C}$ used in the definition (2.14) of the constants of integration η_{γ_1,γ_2} is defined by the orientation of $\mathcal{C}$.

Remark 2.9. Since 1-form u is closed and regular in $\Omega_1 \cup \Omega_2$, it follows from Stokes' theorem that in the definition (2.13) and (2.19) of the cochain $\Theta \in \mathsf{C}^{0,2}$ we can use any rectifiable path from z to 0 inside Ω_1 and Ω_2 respectively.

As opposed to the Fuchsian case, we can no longer guarantee that the cochain $\omega[\phi] - \theta[\phi] - \Theta$ is a 2-cocycle in the total cohomology complex Tot C. Indeed, we have, using $\delta u = 0$,

$$(\delta\Theta)_{\gamma_1,\gamma_2,\gamma_3}(z) = \begin{cases} \int_{P_1} u_{\gamma_2,\gamma_3} + (\delta\eta)_{\gamma_1,\gamma_2,\gamma_3} = (d_1)_{\gamma_1,\gamma_2,\gamma_3} & \text{if } z \in \Omega_1, \\ \int_{P_2} u_{\gamma_2,\gamma_3} - (\delta\eta)_{\gamma_1,\gamma_2,\gamma_3} = (d_2)_{\gamma_1,\gamma_2,\gamma_3} & \text{if } z \in \Omega_2, \end{cases} \tag{2.23}$$

where paths of integration P_1 and P_2 are Γ-closed contracting paths connecting points 0 and $\gamma_1^{-1}(0)$ inside Ω_1 and Ω_2 respectively. Since the analog of Lemma 2.3 does not hold in the quasi-Fuchsian case, we can not conclude that $d_1 = d_2 = 0$. However, $d_1, d_2 \in \mathsf{C}^{0,3}$ are z-independent group 3-cocycles and

$$(d_1 - d_2)_{\gamma_1,\gamma_2,\gamma_3} = \int_C u_{\gamma_2,\gamma_3} + 2(\delta\eta)_{\gamma_1,\gamma_2,\gamma_3}, \tag{2.24}$$

where $C = P_1 - P_2$ is a loop that starts at 0, goes to $\gamma_1^{-1}(0)$ inside Ω_1, continues inside Ω_2 and ends at 0. In the Fuchsian case we have Eq. (2.21), which can be derived using the Stokes' theorem (see Remark 2.6). The same derivation repeats verbatim for the quasi-Fuchsian case, and we get

$$\int_C u_{\gamma_2,\gamma_3} = -2(\delta\eta)_{\gamma_1,\gamma_2,\gamma_3},$$

so that $d_1 = d_2$. Since $H^3(\Gamma, \mathbb{C}) = 0$, there exists a constant 2-cochain κ such that $\delta\kappa = -d_1 = -d_2$. Then $\Theta + \kappa$ is a group 2-cocycle, that is, $\delta(\Theta + \kappa) = 0$. As the result, we obtain that

$$\Psi[\phi] = \omega[\phi] - \theta[\phi] - \Theta - \kappa \in (\operatorname{Tot} \mathsf{C})^2$$

is a 2-cocycle in total cohomology complex Tot C for the pair $\Omega,\ \Gamma$, that is, $D\Psi[\phi] = 0$.

Remark 2.10. The map J_1 induces a cochain map between double cohomology complexes Tot C for the pairs $\mathbb{U} \cup \mathbb{L}, \tilde{\Gamma}$ and $\Omega,\ \Gamma$, by pulling back cochains and group elements,

$$(J_1 \cdot \varpi)_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_q} = J_1^* \varpi_{\gamma_1,\ldots,\gamma_q} \in \mathsf{C}^{p,q}(\mathbb{U} \cup \mathbb{L}),$$

where $\varpi \in \mathsf{C}^{p,q}(\Omega)$ and $\tilde{\gamma} = J_1^{-1} \circ \gamma \circ J_1$. This cochain map induces an isomorphism of the cohomology groups of corresponding total complexes $\operatorname{Tot}\mathsf{C}$. The map J_1 also induces a natural isomorphism between the affine spaces $\mathcal{CM}(X \sqcup Y)$ and $\mathcal{CM}(X \sqcup \bar{X})$,

$$J_1 \cdot \phi = \phi \circ J_1 + \log |(J_1)_z|^2 \in \mathcal{CM}(X \sqcup \bar{X}),$$

where $\phi \in \mathcal{CM}(X \sqcup Y)$. However,

$$|(J_1 \cdot \phi)_z|^2 \, dz \wedge d\bar{z} \neq J_1^* \left(|\phi_z|^2 \, dz \wedge d\bar{z} \right),$$

and cochains $\omega[\phi]$, $\theta[\phi]$, u and Θ for the pair $\Omega, \ \Gamma$ are not pull-backs of cochains for the pair $\mathbb{U} \cup \mathbb{L}, \ \tilde{\Gamma}$ corresponding to $J_1 \cdot \phi \in \mathcal{CM}(X \sqcup \bar{X})$.

2.3.3. The Liouville action functional. Discussion in the previous section justifies the following definition.

Definition 2.5. *The Liouville action functional $S_\Gamma : \mathcal{CM}(X \sqcup Y) \to \mathbb{R}$ for the quasi-Fuchsian group Γ is defined by*

$$\begin{aligned} S_\Gamma[\phi] =& \frac{i}{2} \langle \Psi[\phi], \Sigma(\Gamma) \rangle = \frac{i}{2} \langle \Psi[\phi], \Sigma_1 - \Sigma_2 \rangle \\ =& \frac{i}{2} \left(\langle \omega[\phi], F_1 - F_2 \rangle - \langle \theta[\phi], L_1 - L_2 \rangle + \langle \Theta + \kappa, V_1 - V_2 \rangle \right), \end{aligned}$$

where $\phi \in \mathcal{CM}(X \sqcup Y)$.

Remark 2.11. Since $\Psi[\phi]$ is a total 2-cocycle, the Liouville action functional S_Γ does not depend on the choice of fundamental domain for Γ in Ω, i.e on the choice of fundamental domains F_1 and F_2 for Γ in Ω_1 and Ω_2. In particular, if Σ_1 and Σ_2 are push-forwards by the map J_1 of the total cycle Σ and its mirror image $\bar{\Sigma}$, then $\langle \kappa, V_1 - V_2 \rangle = 0$ and we have

$$S_\Gamma[\phi] = \frac{i}{2} \left(\langle \omega[\phi], F_1 - F_2 \rangle - \langle \theta[\phi], L_1 - L_2 \rangle + \langle u, W_1 - W_2 \rangle + 2\langle \eta, V_1 \rangle \right). \tag{2.25}$$

In general, the constant group 2-cocycle κ drops out from the definition for any choice of fundamental domains F_1 and F_2 which is associated with the same marking of Γ, i.e., when the same choice of standard generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ is used both in Ω_1 and in Ω_2. Indeed, in this case V_1 and V_2 have the same $\mathsf{B}_2(\mathbb{Z}\Gamma)$-structure and $\langle \kappa, V_1 - V_2 \rangle = 0$. Moreover, since the 1-form u is closed and regular in $\Omega_1 \cup \Omega_2$, we can use arbitrary rectifiable paths with endpoint 0 inside Ω_1 and Ω_2 in the definition of chains W_1 and W_2 respectively.

Remark 2.12. We can also define chains W_1 and W_2 by using Γ-contracting paths at any Γ-fixed point $p \in \mathcal{C} \setminus \{\infty\}$. As in Remark 2.5 it is easy to show that

$$\langle \Theta, V_1 - V_2 \rangle = \langle u, W_1 - W_2 \rangle + 2\langle \eta, V_1 \rangle$$

does not depend on the choice of a fixed point p.

As in the Fuchsian case, the Euler-Lagrange equation for the functional S_Γ is the Liouville equation and the hyperbolic metric $e^{\phi_{hyp}}|dz|^2$ on Ω is its single non-degenerate critical point. It is explicitly given by

$$e^{\phi_{hyp}(z)} = \frac{|(J_i^{-1})'(z)|^2}{(\operatorname{Im} J_i^{-1}(z))^2} \quad \text{if} \quad z \in \Omega_i, \quad i = 1, 2. \tag{2.26}$$

Remark 2.13. Corresponding classical action $S_\Gamma[\phi_{hyp}]$ is no longer twice the hyperbolic area of $X \sqcup Y$, as it was in the Fuchsian case, but rather non-trivially depends on Γ. This is due to the fact that in the quasi-Fuchsian case the $(1,1)$-form $\omega[\phi_{hyp}]$ on Ω is not a $(1,1)$-tensor for Γ, as it was in the Fuchsian case.

Similarly to (2.22) we have

$$S_\Gamma[\phi] = \frac{i}{2}\Big(\langle\omega[\phi], F_1 - F_2\rangle - \langle\check{\theta}[\phi], L_1 - L_2\rangle + \langle\check{u}, W_1 - W_2\rangle\Big), \tag{2.27}$$

where F_1 and F_2 are fundamental domains for the marked group Γ in Ω_1 and Ω_2 respectively.

3. Deformation Theory

3.1. The deformation space. Here we collect the basic facts from deformation theory of Kleinian groups (see, e.g., [Ahl87, Ber70, Ber71, Kra72b]). Let Γ be a non-elementary, finitely generated purely loxodromic Kleinian group, let Ω be its region of discontinuity, and let $\Lambda = \hat{\mathbb{C}} \setminus \Omega$ be its limit set. The deformation space $\mathfrak{D}(\Gamma)$ is defined as follows. Let $\mathcal{A}^{-1,1}(\Gamma)$ be the space of Beltrami differentials for Γ – the Banach space of $\mu \in L^\infty(\mathbb{C})$ satisfying

$$\mu(\gamma(z))\,\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z) \text{ for all } \gamma \in \Gamma,$$

and

$$\mu|_\Lambda = 0.$$

Denote by $\mathcal{B}^{-1,1}(\Gamma)$ the open unit ball in $\mathcal{A}^{-1,1}(\Gamma)$ with respect to the $\|\cdot\|_\infty$ norm,

$$\|\mu\|_\infty = \sup_{z\in\mathbb{C}} |\mu(z)| < 1.$$

For each Beltrami coefficient $\mu \in \mathcal{B}^{-1,1}(\Gamma)$ there exists a unique homeomorphism $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ satisfying the Beltrami equation

$$f^\mu_{\bar{z}} = \mu f^\mu_z$$

and fixing the points 0, 1 and ∞. Set $\Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ and define

$$\mathfrak{D}(\Gamma) = \mathcal{B}^{-1,1}(\Gamma)/\sim,$$

where $\mu \sim \nu$ if and only if $f^\mu = f^\nu$ on Λ, which is equivalent to the condition $f^\mu \circ \gamma \circ (f^\mu)^{-1} = f^\nu \circ \gamma \circ (f^\nu)^{-1}$ for all $\gamma \in \Gamma$.

Similarly, if Δ is a union of invariant components of Γ, the deformation space $\mathfrak{D}(\Gamma, \Delta)$ is defined using Beltrami coefficients supported on Δ.

By Ahlfors finiteness theorem Ω has finitely many non-equivalent components $\Omega_1, \dots, \Omega_n$. Let Γ_i be the stabilizer subgroup of the component Ω_i, $\Gamma_i = \{\gamma \in \Gamma \mid \gamma(\Omega_i) = \Omega_i\}$ and let $X_i \simeq \Gamma_i \backslash \Omega_i$ be the corresponding compact Riemann surface of genus $g_i > 1, i = 1, \dots, n$. The decomposition

$$\Gamma \backslash \Omega = \Gamma_1 \backslash \Omega_1 \sqcup \cdots \sqcup \Gamma_n \backslash \Omega_n$$

establishes the isomorphism [Kra72b]

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma_1, \Omega_1) \times \cdots \times \mathfrak{D}(\Gamma_n, \Omega_n).$$

Remark 3.1. When Γ is a purely hyperbolic Fuchsian group of genus $g > 1$, $\mathfrak{D}(\Gamma, \mathbb{U}) = \mathfrak{T}(\Gamma)$ – the Teichmüller space of Γ. Every conformal bijection $\Gamma \backslash \mathbb{U} \to X$ establishes an isomorphism between $\mathfrak{T}(\Gamma)$ and $\mathfrak{T}(X)$, the Teichmüller space of marked Riemann surface X. Similarly, $\mathfrak{D}(\Gamma, \mathbb{L}) = \bar{\mathfrak{T}}(\Gamma)$, the mirror image of $\mathfrak{T}(\Gamma)$ – the complex manifold which is complex conjugate to $\mathfrak{T}(\Gamma)$. Correspondingly, $\Gamma \backslash \mathbb{L} \to \bar{X}$ establishes the isomorphism $\bar{\mathfrak{T}}(\Gamma) \simeq \mathfrak{T}(\bar{X})$, so that

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{T}(X) \times \mathfrak{T}(\bar{X}).$$

The deformation space $\mathfrak{D}(\Gamma)$ is "twice larger" than the Teichmüller space $\mathfrak{T}(\Gamma)$ because its definition uses all Beltrami coefficients μ for Γ, and not only those satisfying the reflection property $\mu(\bar{z}) = \overline{\mu(z)}$, used in the definition of $\mathfrak{T}(\Gamma)$.

The deformation space $\mathfrak{D}(\Gamma)$ has a natural structure of a complex manifold, explicitly described as follows (see, e.g., [Ahl87]). Let $\mathcal{H}^{-1,1}(\Gamma)$ be the Hilbert space of Beltrami differentials for Γ with the following scalar product:

$$(\mu_1, \mu_2) = \iint\limits_{\Gamma \backslash \Omega} \mu_1 \bar{\mu}_2 \rho = \iint\limits_{\Gamma \backslash \Omega} \mu_1(z) \overline{\mu_2(z)} \rho(z)\, d^2 z, \tag{3.1}$$

where $\mu_1, \mu_2 \in \mathcal{H}^{-1,1}(\Gamma)$ and $\rho = e^{\phi_{hyp}}$ is the density of the hyperbolic metric on $\Gamma \backslash \Omega$. Denote by $\Omega^{-1,1}(\Gamma)$ the finite-dimensional subspace of harmonic Beltrami differentials with respect to the hyperbolic metric. It consists of $\mu \in \mathcal{H}^{-1,1}(\Gamma)$ satisfying

$$\partial_z(\rho \mu) = 0.$$

The complex vector space $\Omega^{-1,1}(\Gamma)$ is identified with the holomorphic tangent space to $\mathfrak{D}(\Gamma)$ at the origin. Choose a basis $\mu_1, \dots, \mu_d$ for $\Omega^{-1,1}(\Gamma)$, let $\mu = \varepsilon_1 \mu_1 + \cdots + \varepsilon_d \mu_d$, and let f^μ be the normalized solution of the Beltrami equation. Then the correspondence $(\varepsilon_1, \dots, \varepsilon_d) \mapsto \Gamma^\mu = f^\mu \circ \Gamma \circ (f^\mu)^{-1}$ defines complex coordinates in a neighborhood of the origin in $\mathfrak{D}(\Gamma)$, called Bers coordinates. The holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at the origin can be naturally identified with the vector space $\Omega^{2,0}(\Gamma)$ of holomorphic quadratic differentials – holomorphic functions q on Ω satisfying

$$q(\gamma z) \gamma'(z)^2 = q(z) \text{ for all } \gamma \in \Gamma.$$

The pairing between holomorphic cotangent and tangent spaces to $\mathfrak{D}(\Gamma)$ at the origin is given by

$$q(\mu) = \iint_{\Gamma\backslash\Omega} q\mu = \iint_{\Gamma\backslash\Omega} q(z)\mu(z)\,d^2z.$$

There is a natural isomorphism Φ^μ between the deformation spaces $\mathfrak{D}(\Gamma)$ and $\mathfrak{D}(\Gamma^\mu)$, which maps $\Gamma^\nu \in \mathfrak{D}(\Gamma)$ to $(\Gamma^\mu)^\lambda \in \mathfrak{D}(\Gamma^\mu)$, where, in accordance with $f^\nu = f^\lambda \circ f^\mu$,

$$\lambda = \left(\frac{\nu - \mu}{1 - \nu\bar{\mu}}\frac{f_z^\mu}{\bar{f}_{\bar{z}}^\mu}\right) \circ (f^\mu)^{-1}.$$

The isomorphism Φ^μ allows us to identify the holomorphic tangent space to $\mathfrak{D}(\Gamma)$ at Γ^μ with the complex vector space $\Omega^{-1,1}(\Gamma^\mu)$, and holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at Γ^μ with the complex vector space $\Omega^{2,0}(\Gamma^\mu)$. It also allows us to introduce the Bers coordinates in the neighborhood of Γ^μ in $\mathfrak{D}(\Gamma)$, and to show directly that these coordinates transform complex-analytically. For the de Rham differential d on $\mathfrak{D}(\Gamma)$ we denote by $d = \partial + \bar{\partial}$ the decomposition into $(1, 0)$ and $(0, 1)$ components.

The differential of the isomorphism $\Phi^\mu : \mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma^\mu)$ at $\nu = \mu$ is given by the linear map $D^\mu : \Omega^{-1,1}(\Gamma) \to \Omega^{-1,1}(\Gamma^\mu)$,

$$\nu \mapsto D^\mu\nu = P^\mu_{-1,1}\left[\left(\frac{\nu}{1 - |\mu|^2}\frac{f_z^\mu}{\bar{f}_{\bar{z}}^\mu}\right) \circ (f^\mu)^{-1}\right],$$

where $P^\mu_{-1,1}$ is orthogonal projection from $\mathcal{H}^{-1,1}(\Gamma^\mu)$ to $\Omega^{-1,1}(\Gamma^\mu)$. The map D^μ allows to extend a tangent vector ν at the origin of $\mathfrak{D}(\Gamma)$ to a local vector field $\partial/\partial\varepsilon_\nu$ on the coordinate neighborhood of the origin,

$$\left.\frac{\partial}{\partial\varepsilon_\nu}\right|_{\Gamma^\mu} = D^\mu\nu \in \Omega^{-1,1}(\Gamma^\mu).$$

The scalar product (3.1) in $\Omega^{-1,1}(\Gamma^\mu)$ defines a Hermitian metric on the deformation space $\mathfrak{D}(\Gamma)$. This metric is called the Weil-Petersson metric and it is Kähler . We denote its symplectic form by ω_{WP},

$$\omega_{WP}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right)\bigg|_{\Gamma^\lambda} = \frac{i}{2}\left(D^\lambda\mu, D^\lambda\nu\right), \quad \mu, \nu \in \Omega^{-1,1}(\Gamma).$$

3.2. Variational formulas. Here we collect necessary variational formulas. Let l and m be integers. A tensor of type (l, m) for Γ is a C^∞-function ω on Ω satisfying

$$\omega(\gamma z)\gamma'(z)^l\overline{\gamma'(z)}^m = \omega(z) \text{ for all } \gamma \in \Gamma.$$

Let ω^ε be a smooth family of tensors of type (l, m) for $\Gamma^{\varepsilon\mu}$, where $\mu \in \Omega^{-1,1}(\Gamma)$ and $\varepsilon \in \mathbb{C}$ is sufficiently small. Set

$$(f^{\varepsilon\mu})^*(\omega^\varepsilon) = \omega^\varepsilon \circ f^{\varepsilon\mu}(f_z^{\varepsilon\mu})^l(\bar{f}_{\bar{z}}^{\varepsilon\mu})^m,$$

which is a tensor of type (l, m) for Γ – a pull-back of the tensor ω^{ε} by $f^{\varepsilon\mu}$. The Lie derivatives of the family ω^{ε} along the vector fields $\partial/\partial\varepsilon_{\mu}$ and $\partial/\partial\bar{\varepsilon}_{\mu}$ are defined in the standard way,

$$L_{\mu}\omega = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}(f^{\varepsilon\mu})^{*}(\omega^{\varepsilon}) \text{ and } L_{\bar{\mu}}\omega = \frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}(f^{\varepsilon\mu})^{*}(\omega^{\varepsilon}).$$

When ω is a function on $\mathfrak{D}(\Gamma)$ – a tensor of type $(0, 0)$, Lie derivatives reduce to directional derivatives

$$L_{\mu}\omega = \partial\omega(\mu) \text{ and } L_{\bar{\mu}}\omega = \bar{\partial}\omega(\bar{\mu})$$

– the evaluation of 1-forms $\partial\omega$ and $\bar{\partial}\omega$ on tangent vectors μ and $\bar{\mu}$.

For the Lie derivatives of vector fields $\nu^{\epsilon\mu} = D^{\epsilon\mu}\nu$ we get [Wol86] that $L_{\mu}\nu = 0$ and $L_{\bar{\mu}}\nu$ is orthogonal to $\Omega^{-1,1}(\Gamma)$. In other words,

$$\left[\frac{\partial}{\partial\varepsilon_{\mu}}, \frac{\partial}{\partial\varepsilon_{\nu}}\right] = \left[\frac{\partial}{\partial\varepsilon_{\mu}}, \frac{\partial}{\partial\bar{\varepsilon}_{\nu}}\right] = 0$$

at the point Γ in $\mathfrak{D}(\Gamma)$.

For every $\Gamma^{\mu} \in \mathfrak{D}(\Gamma)$, the density ρ^{μ} of the hyperbolic metric on Ω^{μ} is a $(1, 1)$-tensor for Γ^{μ}. Lie derivatives of the smooth family of $(1, 1)$-tensors ρ parameterized by $\mathfrak{D}(\Gamma)$ are given by the following lemma of Ahlfors.

Lemma 3.1. *For every* $\mu \in \Omega^{-1,1}(\Gamma)$,

$$L_{\mu}\rho = L_{\bar{\mu}}\rho = 0.$$

Proof. Let $\Omega_1, \dots, \Omega_n$ be the maximal set of non-equivalent components of Ω and let $\Gamma_1, \dots, \Gamma_n$ be the corresponding stabilizer groups,

$$\Gamma\backslash\Omega = \Gamma_1\backslash\Omega_1 \sqcup \cdots \sqcup \Gamma_n\backslash\Omega_n \simeq X_1 \sqcup \cdots \sqcup X_n.$$

For every Ω_i denote by $J_i : \mathbb{U} \to \Omega_i$ the corresponding covering map and by $\tilde{\Gamma}_i$ – the Fuchsian model of group Γ_i, characterized by the condition $\tilde{\Gamma}_i\backslash\mathbb{U} \simeq \Gamma_i\backslash\Omega_i \simeq X_i$ (see, e.g., [Kra72b]).

Let $\mu \in \Omega^{-1,1}(\Gamma)$. For every component Ω_i the quasiconformal map $f^{\varepsilon\mu}$ gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{F^{\varepsilon\hat{\mu}_i}} & \mathbb{U} \\ \Big\downarrow{\scriptstyle J_i} & & \Big\downarrow{\scriptstyle J_i^{\varepsilon\mu}} \\ \Omega_i & \xrightarrow{f^{\varepsilon\mu}} & \Omega_i^{\varepsilon\mu} \end{array} \qquad (3.2)$$

where $F^{\varepsilon\hat{\mu}_i}$ is the normalized quasiconformal homeomorphism of $\mathbb{U}$ with Beltrami differential $\hat{\mu}_i = J_i^{*}\mu$ for the Fuchsian group $\tilde{\Gamma}_i$. Let $\hat{\rho}$ be the density of the hyperbolic metric on $\mathbb{U}$; it satisfies $\hat{\rho} = J_i^{*}\rho$, where ρ is the density of the hyperbolic metric on Ω_i. Therefore, the Beltrami differential $\hat{\mu}_i$ is harmonic with respect to the hyperbolic metric on $\mathbb{U}$. It follows from the commutativity of the diagram that

$$(f^{\varepsilon\mu})^{*}\rho^{\varepsilon\mu} = ((J_i^{\varepsilon\mu})^{-1} \circ f^{\varepsilon\mu})^{*}\hat{\rho} = (F^{\varepsilon\hat{\mu}_i} \circ J_i^{-1})^{*}\hat{\rho} = (J_i^{-1})^{*}(F^{\varepsilon\hat{\mu}_i})^{*}\hat{\rho}.$$

Now the assertion of the lemma reduces to

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}(F^{\varepsilon\hat{\mu}_i})^*\hat{\rho}=0,$$

which is the classical result of Ahlfors [Ahl61]. □

Set

$$\dot{f}=\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}f^{\varepsilon\mu},$$

then

$$\dot{f}(z)=-\frac{1}{\pi}\iint\limits_{\mathbb{C}}\frac{z(z-1)\mu(w)}{(w-z)w(w-1)}\,d^2w. \tag{3.3}$$

We have

$$\dot{f}_{\bar{z}}=\mu$$

and also

$$\frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0}f^{\varepsilon\mu}=0.$$

As it follows from Ahlfors lemma,

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\rho^{\varepsilon\mu}\circ f^{\varepsilon\mu}\,|f_z^{\varepsilon\mu}|^2\right)=0.$$

Using $\rho=e^{\phi_{hyp}}$ and the fact that $f^{\varepsilon\mu}$ depends holomorphically on ε, we get

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\phi_{hyp}^{\varepsilon\mu}\circ f^{\varepsilon\mu}\right)=-\dot{f}_z. \tag{3.4}$$

Differentiation with respect to z and $\bar{z}$ yields

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\left(\phi_{hyp}^{\varepsilon\mu}\right)_z\circ f^{\varepsilon\mu}f_z^{\varepsilon\mu}\right)=-\dot{f}_{zz}, \tag{3.5}$$

and

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\left(\phi_{hyp}^{\varepsilon\mu}\right)_{\bar{z}}\circ f^{\varepsilon\mu}\,\bar{f}_{\bar{z}}^{\varepsilon\mu}\right)=-\left((\phi_{hyp})_z\dot{f}_{\bar{z}}+\dot{f}_{z\bar{z}}\right). \tag{3.6}$$

For $\gamma\in\Gamma$ set $\gamma^{\varepsilon\mu}=f^{\varepsilon\mu}\circ\gamma\circ(f^{\varepsilon\mu})^{-1}\in\Gamma^{\varepsilon\mu}$. We have

$$(\gamma^{\varepsilon\mu})'\circ f^{\varepsilon\mu}f_z^{\varepsilon\mu}=f_z^{\varepsilon\mu}\circ\gamma\;\gamma',$$

and

$$\log|(\gamma^{\varepsilon\mu})'\circ f^{\varepsilon\mu}|^2+\log|f_z^{\varepsilon\mu}|^2=\log|f_z^{\varepsilon\mu}\circ\gamma|^2+\log|\gamma'|^2.$$

Therefore

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\log|(\gamma^{\varepsilon\mu})'\circ f^{\varepsilon\mu}|^2\right)=\dot{f}_z\circ\gamma-\dot{f}_z, \tag{3.7}$$

and, differentiating with respect to z,

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\left(\frac{(\gamma^{\varepsilon\mu})''}{(\gamma^{\varepsilon\mu})'}\circ f^{\varepsilon\mu}f_z^{\varepsilon\mu}\right)=\dot{f}_{zz}\circ\gamma\ \gamma'-\dot{f}_{zz}. \tag{3.8}$$

Denote by

$$\mathcal{S}(h)=\left(\frac{h_{zz}}{h_z}\right)_z-\frac{1}{2}\left(\frac{h_{zz}}{h_z}\right)^2=\frac{h_{zzz}}{h_z}-\frac{3}{2}\left(\frac{h_{zz}}{h_z}\right)^2$$

the Schwarzian derivative of the function h.

Lemma 3.2. *Set*

$$\dot{\gamma}=\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\gamma^{\varepsilon\mu},\ \gamma\in\Gamma.$$

Then for all $\gamma\in\Gamma$*,*

$$\dot{f}_z\circ\gamma-\dot{f}_z=\frac{\dot{f}\circ\gamma\gamma''}{(\gamma')^2}+\left(\frac{\dot{\gamma}}{\gamma'}\right)', \tag{i}$$

and is well-defined on the limit set Λ*. Also we have*

$$\dot{f}_{z\bar{z}}\circ\gamma\ \overline{\gamma'}-\dot{f}_{z\bar{z}}=\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}, \tag{ii}$$

$$\dot{f}_{zz}\circ\gamma\ \gamma'-\dot{f}_{zz}=\frac{1}{2}(\dot{f}_z\circ\gamma+\dot{f}_z)\frac{\gamma''}{\gamma'}-\frac{2\dot{c}}{cz+d},\ \textit{for all}\ \gamma\in\Gamma. \tag{iii}$$

Proof. To prove formula (i), consider the equation

$$\dot{f}\circ\gamma=\dot{\gamma}+\gamma'\dot{f}, \tag{3.9}$$

which follows from $\gamma^{\varepsilon\mu}\circ f^{\varepsilon\mu}=f^{\varepsilon\mu}\circ\gamma$. Differentiating with respect to z gives (i). Since $\dot{f}$ is a homeomorphism of $\hat{\mathbb{C}}$ and $\dot{\gamma}/\gamma'$ is a quadratic polynomial in z, formula (i) shows that $\dot{f}_z\circ\gamma-\dot{f}_z$ is well-defined on Λ.

The formula (ii) immediately follows from $\dot{f}_{\bar{z}}=\mu$ and

$$\mu\circ\gamma\frac{\overline{\gamma'}}{\gamma'}=\mu,\ \gamma\in\Gamma.$$

To derive formula (iii), twice differentiating (3.9) with respect to z we obtain

$$\dot{f}_z\circ\gamma\gamma'=\dot{\gamma}'+\gamma''\dot{f}+\gamma'\dot{f}_z,$$

and

$$\gamma'(\dot{f}_{zz}\circ\gamma\gamma'-\dot{f}_{zz})=\dot{\gamma}''+\gamma'''\dot{f}+2\gamma''\dot{f}_z-\dot{f}_z\circ\gamma\gamma''.$$

Since

$$\gamma''' = \frac{3}{2}\frac{(\gamma'')^2}{\gamma'},$$

as it follows from $\mathcal{S}(\gamma) = 0$, we can eliminate $\gamma''\dot{f}$ from the two formulas above and obtain

$$\dot{f}_{zz} \circ \gamma\ \gamma' - \dot{f}_{zz} = \frac{1}{2}(\dot{f}_z \circ \gamma + \dot{f}_z)\frac{\gamma''}{\gamma'} + \frac{\dot{\gamma}''}{\gamma'} - \frac{3}{2}\frac{\gamma''\dot{\gamma}'}{(\gamma')^2}.$$

Using $2c = -\gamma''/(\gamma')^{3/2}$, we see that the last two terms in this equation are equal to $-2\dot{c}/(cz+d)$, which proves the lemma. □

Finally, we present the following formulas by Ahlfors [Ahl61]. Let $F^{\varepsilon\hat{\mu}}$ be the quasiconformal homeomorphism of $\mathbb{U}$ with Beltrami differential $\hat{\mu}$ for the Fuchsian group Γ. If $\hat{\mu}$ is harmonic on $\mathbb{U}$ with respect to the hyperbolic metric, then

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} F^{\varepsilon\hat{\mu}}_{zzz}(z) = 0, \tag{3.10}$$

$$\frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} F^{\varepsilon\hat{\mu}}_{zzz}(z) = -\frac{1}{2}\hat{\rho}\overline{\hat{\mu}(z)}. \tag{3.11}$$

4. Variation of the Classical Action

4.1. Classical action. Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$ with region of discontinuity $\Omega = \Omega_1 \cup \Omega_2$, let $X \sqcup Y \simeq \Gamma\backslash\Omega$ be corresponding marked Riemann surfaces with opposite orientations and let

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma, \Omega_1) \times \mathfrak{D}(\Gamma, \Omega_2)$$

be the deformation space of Γ. Spaces $\mathfrak{D}(\Gamma, \Omega_1)$ and $\mathfrak{D}(\Gamma, \Omega_2)$ are isomorphic to the Teichmüller spaces $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ – they are their quasi-Fuchsian models which use Bers' simultaneous uniformization of the varying Riemann surface in $\mathfrak{T}(X)$ and fixed Y and, respectively, fixed X and the varying Riemann surface in $\mathfrak{T}(Y)$. Therefore,

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{T}(X) \times \mathfrak{T}(Y). \tag{4.1}$$

Denote by $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ the corresponding affine bundle of projective connections, modeled over the holomorphic cotangent bundle of $\mathfrak{D}(\Gamma)$. We have

$$\mathfrak{P}(\Gamma) \simeq \mathfrak{P}(X) \times \mathfrak{P}(Y). \tag{4.2}$$

For every $\Gamma^\mu \in \mathfrak{D}(\Gamma)$ denote by $S_{\Gamma^\mu} = S_{\Gamma^\mu}[\phi_{hyp}]$ the classical Liouville action. It follows from the results in Sect. 2.3.3 that S_{Γ^μ} gives rise to a well-defined real-valued function S on $\mathfrak{D}(\Gamma)$. Indeed, if $\mu \sim \nu$, then the corresponding total cycles $f^\mu(\Sigma(\Gamma))$ and $f^\nu(\Sigma(\Gamma))$ represent the same class in the total homology complex Tot K for the pair $\Omega^\mu,\ \Gamma^\mu$, so that

$$\langle\Psi\left[\phi_{hyp}\right], f^\mu(\Sigma(\Gamma))\rangle = \langle\Psi\left[\phi_{hyp}\right], f^\nu(\Sigma(\Gamma))\rangle.$$

Moreover, real-analytic dependence of solutions of the Beltrami equation on parameters ensures that the classical action S is a real-analytic function on $\mathfrak{D}(\Gamma)$.

To every $\Gamma' \in \mathfrak{D}(\Gamma)$ with the region of discontinuity Ω' there corresponds a pair of marked Riemann surfaces X' and Y' simultaneously uniformized by Γ', $X' \sqcup Y' \simeq \Gamma' \backslash \Omega'$. Set $S(X', Y') = S_{\Gamma'}$ and denote by S_Y and S_X restrictions of the function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$ onto $\mathfrak{T}(X)$ and $\mathfrak{T}(Y)$ respectively. Let ι be the complex conjugation and let $\bar{\Gamma} = \iota(\Gamma)$ be the quasi-Fuchsian group complex conjugated to Γ. The correspondence $\mu \mapsto \iota \circ \mu \circ \iota$ establishes the complex-analytic anti-isomorphism

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\bar{\Gamma}) \simeq \mathfrak{T}(\bar{Y}) \times \mathfrak{T}(\bar{X}).$$

The classical Liouville action has the symmetry property

$$S(X', Y') = S(\bar{Y}', \bar{X}'). \tag{4.3}$$

For every $\phi \in \mathcal{CM}(\Gamma \backslash \Omega)$ set

$$\vartheta[\phi] = 2\phi_{zz} - \phi_z^2.$$

It follows from the Liouville equation that $\vartheta = \vartheta[\phi_{hyp}] \in \Omega^{2,0}(\Gamma)$, i.e., is a holomorphic quadratic differential for Γ. It follows from (2.26) that

$$\vartheta(z) = \begin{cases} 2\mathcal{S}\left(J_1^{-1}\right)(z) \text{ if } z \in \Omega_1, \\ 2\mathcal{S}\left(J_2^{-1}\right)(z) \text{ if } z \in \Omega_2. \end{cases} \tag{4.4}$$

Define a $(1, 0)$-form $\boldsymbol{\vartheta}$ on the deformation space $\mathfrak{D}(\Gamma)$ by assigning to every $\Gamma' \in \mathfrak{D}(\Gamma)$ the corresponding $\vartheta[\phi'_{hyp}] \in \Omega^{2,0}(\Gamma')$ – a vector in the holomorphic cotangent space to $\mathfrak{D}(\Gamma)$ at Γ'.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let P_F and P_{QF} be Fuchsian and quasi-Fuchsian projective connections on $X' \sqcup Y' \simeq \Gamma' \backslash \Omega'$, defined by the coverings $\pi_F : \mathbb{U} \cup \mathbb{L} \to X' \sqcup Y'$ and $\pi_{QF} : \Omega_1 \cup \Omega_2 \to X' \sqcup Y'$ respectively. We will continue to denote corresponding sections of the affine bundle $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ by P_F and P_{QF} respectively. The difference $P_F - P_{QF}$ is a $(1, 0)$-form on $\mathfrak{D}(\Gamma)$.

Lemma 4.1. *On the deformation space $\mathfrak{D}(\Gamma)$,*

$$\boldsymbol{\vartheta} = 2(P_F - P_{QF}).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{U} \cup \mathbb{L} & \xrightarrow{J} & \Omega_1 \cup \Omega_2 \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_{QF}} \\ X \sqcup Y & \xrightarrow{=} & X \sqcup Y, \end{array}$$

where the covering map J is equal to the map J_1 on the component $\mathbb{U}$ and to the map J_2 on the component $\mathbb{L}$. As explained in the Introduction, $P_F = \mathcal{S}(\pi_F^{-1})$ and $P_{QF} = \mathcal{S}(\pi_{QF}^{-1})$, and it follows from the property **SD1** and commutativity of the diagram that

$$\left(\mathcal{S}\left(\pi_F^{-1}\right) - \mathcal{S}\left(\pi_{QF}^{-1}\right)\right) \circ \pi_{QF}\, (\pi'_{QF})^2 = \mathcal{S}\left(J^{-1}\right).$$

□

4.2. First variation. Here we compute the $(1,0)$-form ∂S on $\mathfrak{D}(\Gamma)$.

Theorem 4.1. *On the deformation space $\mathfrak{D}(\Gamma)$,*

$$\partial S = 2(P_F - P_{QF}).$$

Proof. It is sufficient to prove that for every $\mu \in \Omega^{-1,1}(\Gamma)$

$$L_\mu S = \boldsymbol{\vartheta}(\mu) = \iint\limits_{\Gamma\backslash\Omega} \vartheta\mu. \tag{4.5}$$

Indeed, using the isomorphism $\Phi^\nu : \mathfrak{D}(\Gamma) \to \mathfrak{D}(\Gamma^\nu)$, it is easy to see that the variation formula (4.5) is valid at every point $\Gamma^\nu \in \mathfrak{D}(\Gamma)$ if it is valid at the origin. The actual computation of $L_\mu S$ is quite similar to that in [ZT87b] for the case of Schottky groups, with the clarifying role of a homological algebra.

Let $\tilde{\Gamma}$ be the Fuchsian group corresponding to Γ and let $\Sigma = F + L - V$ be the corresponding total cycle of degree 2 representing the fundamental class of X in the total complex Tot K for the pair $\mathbb{U}, \tilde{\Gamma}$. As in Sect. 2.3.1, set $\Sigma(\Gamma) = J_1(\Sigma - \bar{\Sigma})$. The corresponding total cycle for the pair $\Omega^{\varepsilon\mu}$, $\Gamma^{\varepsilon\mu} = f^{\varepsilon\mu} \circ \Gamma \circ (f^{\varepsilon\mu})^{-1}$ can be chosen as $\Sigma(\Gamma^{\varepsilon\mu}) = f^{\varepsilon\mu}(\Sigma(\Gamma))$. According to Remark 2.11,

$$S_{\Gamma^{\varepsilon\mu}} = \frac{i}{2}\left\langle \Psi\left[\phi_{hyp}^{\varepsilon\mu}\right], f^{\varepsilon\mu}(\Sigma(\Gamma))\right\rangle.$$

Moreover, as it follows from Lemma 2.4, we can choose $\Gamma^{\varepsilon\mu}$-contracting at 0 paths of integration in the definition of $\Theta^{\varepsilon\mu}$ or, equivalently, paths in the definition of $W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu}$, to be the push-forwards by $f^{\varepsilon\mu}$ of the corresponding Γ-contracting at 0 paths. Denoting $\omega^{\varepsilon\mu} = \omega\left[\phi_{hyp}^{\varepsilon\mu}\right]$, $\check{\theta}^{\varepsilon\mu} = \check{\theta}\left[\phi_{hyp}^{\varepsilon\mu}\right]$, and using (2.27) we have

$$S_{\Gamma^{\varepsilon\mu}} = \frac{i}{2}\left(\langle\omega^{\varepsilon\mu}, F_1^{\varepsilon\mu} - F_2^{\varepsilon\mu}\rangle - \langle\check{\theta}^{\varepsilon\mu}, L_1^{\varepsilon\mu} - L_2^{\varepsilon\mu}\rangle + \langle\check{u}^{\varepsilon\mu}, W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu}\rangle\right).$$

Changing variables and formally differentiating under the integral sign in the term $\langle\check{u}^{\varepsilon\mu}, W_1^{\varepsilon\mu} - W_2^{\varepsilon\mu}\rangle$, we obtain

$$\begin{aligned} L_\mu S &= \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} S_{\Gamma^{\varepsilon\mu}} \\ &= \frac{i}{2}\left(\langle L_\mu\omega, F_1 - F_2\rangle - \langle L_\mu\check{\theta}, L_1 - L_2\rangle + \langle L_\mu\check{u}, W_1 - W_2\rangle\right). \end{aligned}$$

We will justify this formula at the end of the proof. Here we observe that though $\omega^{\varepsilon\mu}$, $\check{\theta}^{\varepsilon\mu}$ and $\check{u}^{\varepsilon\mu}$ are not tensors for $\Gamma^{\varepsilon\mu}$, they are differential forms on $\Omega^{\varepsilon\mu}$ so that their Lie derivatives are given by the same formulas as in Sect. 3.2.

Using Ahlfors lemma and formulas (3.4)–(3.6), we get

$$\begin{aligned} L_\mu\omega &= -\left(\left(\phi_{hyp}\right)_{\bar{z}} \dot{f}_{zz} + \left(\phi_{hyp}\right)_z \left(\left(\phi_{hyp}\right)_z \dot{f}_{\bar{z}} + \dot{f}_{z\bar{z}}\right)\right) dz \wedge d\bar{z} \\ &= \vartheta\mu\, dz \wedge d\bar{z} - d\xi, \end{aligned}$$

where

$$\xi = 2\left(\phi_{hyp}\right)_z \dot{f}_{\bar{z}} d\bar{z} - \phi_{hyp}\, d\dot{f}_z. \tag{4.6}$$

Since $\vartheta\mu$ is a (1, 1)-tensor for Γ, $\delta(\vartheta\mu\, dz\wedge d\bar{z})=0$, so that $\delta L_\mu\omega=-\delta d\xi$. We have

$$\langle d\xi, F_1-F_2\rangle=\langle\xi,\partial'(F_1-F_2)\rangle=\langle\xi,\partial''(L_1-L_2)\rangle=\langle\delta\xi, L_1-L_2\rangle.$$

Set $\chi=\delta\xi+L_\mu\check{\theta}$. The 1-form χ on Ω is closed,

$$d\chi=\delta(d\xi)+L_\mu d\check{\theta}=\delta(-L_\mu\omega)+L_\mu\delta\omega=0,$$

and satisfies

$$\delta\chi=\delta(L_\mu\check{\theta}+\delta\xi)=L_\mu\delta\check{\theta}=L_\mu\check{u}.$$

Using (3.4), (3.7), (3.8) and part (ii) of Lemma 3.2, we get

$$\begin{aligned}L_\mu\check{\theta}_{\gamma^{-1}}=&-\dot{f}_z\frac{\gamma''}{\gamma'}\,dz+\phi_{hyp}\left(\left(\dot{f}_{zz}\circ\gamma\,\gamma'-\dot{f}_{zz}\right)dz+\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\,d\bar{z}\right)+\dot{f}_z\frac{\overline{\gamma''}}{\overline{\gamma'}}\,d\bar{z}\\&+\frac{1}{2}\left(-\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\frac{\gamma''}{\gamma'}\,dz-\log|\gamma'|^2\left(\dot{f}_{zz}\circ\gamma\,\gamma'-\dot{f}_{zz}\right)dz\right.\\&\left.-\log|\gamma'|^2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\,d\bar{z}+\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\frac{\overline{\gamma''}}{\overline{\gamma'}}\,d\bar{z}\right)\\&-\left(\log|c(\gamma)|^2+2\log 2\right)d\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)-\frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz-\frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right)\\=&-\dot{f}_z\frac{\gamma''}{\gamma'}dz+\dot{f}_z\circ\gamma\frac{\overline{\gamma''}}{\overline{\gamma'}}\,d\bar{z}-d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\right)\\&+\phi_{hyp}\,d\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)-\left(\log|c(\gamma)|^2+2\log 2\right)d\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\\&-\frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz-\frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right).\end{aligned}$$

Using

$$\delta\xi_{\gamma^{-1}}=-2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\,d\bar{z}-\phi_{hyp}\,d\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)+\log|\gamma'|^2 d\left(\dot{f}_z\circ\gamma\right),$$

we get

$$\begin{aligned}\chi_{\gamma^{-1}}=&d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma+\dot{f}_z\right)-\left(\log|c(\gamma)|^2+2\log 2\right)\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\right)\\&-\left(\dot{f}_z\circ\gamma+\dot{f}_z\right)\frac{\gamma''}{\gamma'}dz-2\frac{\gamma''}{\gamma'}\dot{f}_{\bar{z}}\,d\bar{z}-\frac{\dot{c}(\gamma)}{c(\gamma)}\left(\frac{\gamma''}{\gamma'}dz-\frac{\overline{\gamma''}}{\overline{\gamma'}}d\bar{z}\right).\end{aligned}$$

Using parts (ii) and (iii) of Lemma 3.2 and

$$-\frac{2\dot{c}}{cz+d}=\frac{\dot{c}}{c}\frac{\gamma''}{\gamma'}(z),$$

we finally obtain

$$\chi_{\gamma^{-1}} = d\left(\frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma+\dot{f}_z+2\frac{\dot{c}(\gamma)}{c(\gamma)}\right)\right.$$
$$\left.-\left(\log|c(\gamma)|^2+2+2\log 2\right)\left(\dot{f}_z\circ\gamma-\dot{f}_z\right)\right)$$
$$= dl_{\gamma^{-1}}.$$

We have

$$\begin{aligned}\langle d\xi, F_1-F_2\rangle+\langle L_\mu\check{\theta}, L_1-L_2\rangle &= \langle\chi, L_1-L_2\rangle = \langle dl, L_1-L_2\rangle\\ &= \langle l, \partial'(L_1-L_2)\rangle = \langle l, \partial''(V_1-V_2)\rangle\\ &= \langle\delta l, V_1-V_2\rangle.\end{aligned}$$

Using $L_\mu\check{u} = d\delta l$ we get

$$\langle L_\mu\check{u}, W_1-W_2\rangle = \langle\delta l, \partial'(W_1-W_2)\rangle = \langle\delta l, V_1-V_2\rangle$$

so that

$$L_\mu S = \frac{i}{2}\langle\vartheta\mu\, dz\wedge d\bar{z}, F_1-F_2\rangle,$$

as asserted.

Finally, we justify the differentiation under the integral sign. Set

$$l_\gamma = l_\gamma^{(0)}+l_\gamma^{(1)},$$

where

$$\begin{aligned}l_{\gamma^{-1}}^{(0)} &= \frac{\dot{c}(\gamma)}{c(\gamma)}\log|\gamma'|^2-\left(\log|c(\gamma)|^2+2+2\log 2\right)(\dot{f}_z\circ\gamma-\dot{f}_z),\\ l_{\gamma^{-1}}^{(1)} &= \frac{1}{2}\log|\gamma'|^2\left(\dot{f}_z\circ\gamma+\dot{f}_z\right).\end{aligned}$$

Next, we use part (i) of Lemma 3.2. According to it, the function $l_\gamma^{(0)}$ is continuous on $\mathcal{C}\setminus\{\gamma(\infty)\}$. Since

$$(\delta l^{(1)})_{\gamma_1^{-1},\gamma_2^{-1}} = \frac{1}{2}\left(\log|\gamma_2'\circ\gamma_1|^2(\dot{f}_z\circ\gamma_1-\dot{f}_z)-\log|\gamma_1'|^2(\dot{f}_z\circ\gamma_2\gamma_1-\dot{f}_z\circ\gamma_1)\right),$$

we also conclude that $(\delta l^{(1)})_{\gamma_1,\gamma_2}$, and hence the function $(\delta l)_{\gamma_1,\gamma_2}$, are continuous on $\mathcal{C}\setminus\{\gamma_1(\infty),\ (\gamma_1\gamma_2)(\infty)\}$. Now let $W_1^{(n)}\Subset W_1$ and $W_2^{(n)}\Subset W_2$ be a sequence of 1-chains in Ω_1 and Ω_2 obtained from W_1 and W_2 by "cutting" Γ-contracting at 0 paths at points $p_n'\in\Omega_1$ and $p_n''\in\Omega_2$, where $p_n',\ p_n''\to 0$ as $n\to\infty$. Clearly,

$$S = \lim_{n\to\infty} S_n,$$

where

$$S_n = \frac{i}{2}\left(\langle\omega, F_1-F_2\rangle-\langle\check{\theta}, L_1-L_2\rangle+\langle\check{u}, W_1^{(n)}-W_2^{(n)}\rangle\right).$$

Our previous arguments show that

$$L_\mu S_n = \frac{i}{2}\langle \vartheta\mu\, dz \wedge d\bar{z}, F_1 - F_2\rangle - \langle (\delta l)(p_n'), U_1\rangle + \langle (\delta l)(p_n''), U_2\rangle.$$

Since the function δl is continuous at $p = 0$ and $U_1 = U_2$, we get

$$\lim_{n\to\infty} L_\mu S_n = \frac{i}{2}\,\langle \vartheta\mu, F_1 - F_2\rangle.$$

Moreover, the convergence is uniform in some neighborhood of Γ in $\mathfrak{D}(\Gamma)$, since $f^{\varepsilon\mu}$ is holomorphic at $\varepsilon = 0$. Thus

$$L_\mu S = \lim_{n\to\infty} L_\mu S_n,$$

which completes the proof. □

For fixed Riemann surface Y denote by P_F and P_{QF} sections of $\mathfrak{P}(X) \to \mathfrak{T}(X)$ corresponding to the Fuchsian uniformization of $X' \in \mathfrak{T}(X)$ and to the simultaneous uniformization of $X' \in \mathfrak{T}(X)$ and Y respectively.

Corollary 4.1. *On the Teichmüller space* $\mathfrak{T}(X)$*,*

$$P_F - P_{QF} = \frac{1}{2}\,\partial S_Y.$$

Remark 4.1. Conversely, Theorem 4.1 follows from the Corollary 4.1 and the symmetry property (4.3).

Remark 4.2. In the Fuchsian case the maps J_1 and J_2 are identities and a similar computation shows that $\boldsymbol{\vartheta} = 0$, in accordance with $S = 8\pi(2g-2)$ being a constant function on $\mathfrak{T}(X) \times \mathfrak{T}(\bar{X})$.

4.3. Second variation. Here we compute $d\boldsymbol{\vartheta} = \bar{\partial}\boldsymbol{\vartheta}$. First, we have the following statement.

Lemma 4.2. *The quasi-Fuchsian projective connection* P_{QF} *is a holomorphic section of the affine bundle* $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{\;f^{\varepsilon\mu}\;} & \Omega^{\varepsilon\mu} \\ \big\downarrow{\scriptstyle \pi_{QF}} & & \big\downarrow{\scriptstyle \pi_{QF}^{\varepsilon\mu}} \\ X \sqcup Y & \xrightarrow{\;F^{\varepsilon\mu}\;} & X^{\varepsilon\mu} \sqcup Y^{\varepsilon\mu} \end{array}$$

where $\mu \in \Omega^{-1,1}(\Gamma)$. We have

$$\mathcal{S}\left(\left(\pi_{QF}^{\varepsilon\mu}\right)^{-1}\right) \circ F^{\varepsilon\mu}\left(F_z^{\varepsilon\mu}\right)^2 + \mathcal{S}\left(F^{\varepsilon\mu}\right) = \mathcal{S}\left(f^{\varepsilon\mu}\right) \circ \pi_{QF}^{-1}\left(\pi_{QF}^{-1}\right)_z^2 + \mathcal{S}\left(\pi_{QF}^{-1}\right).$$

Since $f^{\varepsilon\mu}$ and, obviously $F^{\varepsilon\mu}$, are holomorphic at $\varepsilon = 0$, we get

$$\frac{\partial}{\partial\bar{\varepsilon}}\bigg|_{\varepsilon=0} \mathcal{S}\left(\left(\pi_{QF}^{\varepsilon\mu}\right)^{-1}\right) = 0.$$

□

Using Corollary 4.1, Lemma 4.3 and the result [ZT87b]

$$\bar{\partial} P_F = -i\,\omega_{WP},$$

which follows from (1.2) since P_S is a holomorphic section of $\mathfrak{P}_g \to \mathfrak{S}_g$, we immediately get

Corollary 4.2. *For fixed Y*

$$\partial\bar{\partial} S_Y = -2\bar{\partial}(P_F - P_{QF}) = -2d(P_F - P_{QF}) = 2\,i\,\omega_{WP},$$

so that $-S_Y$ is a Kähler potential for the Weil-Petersson metric on $\mathfrak{T}(X)$.

Remark 4.3. The equation $d(P_F - P_{QF}) = -i\,\omega_{WP}$ was first proved in [McM00] and was used for the proof that moduli spaces are Kähler hyperbolic (note that the symplectic form ω_{WP} used there is twice the one we are using here, and there is a missing factor $1/2$ in the computation in [McM00]). Specifically, the Kraus-Nehari inequality asserts that $P_F - P_{QF}$ is a bounded antiderivative of $-\,i\,\omega_{WP}$ with respect to Teichmüller and Weil-Petersson metrics [McM00]. In this regard, it is interesting to estimate the Kähler potential S_Y on $\mathfrak{T}(X)$. From the basic inequality of the distortion theorem (see, e.g., [Dur83])

$$\left|\frac{h''(z)}{h'(z)} - \frac{2\bar{z}}{(1-|z|^2)}\right| \leq \frac{4}{(1-|z|^2)},$$

where h is a univalent function in the unit disk, we immediately get

$$|(\phi_{hyp})_z|^2 \leq 4e^{\phi_{hyp}},$$

so that the bulk term in S_Y is bounded on $\mathfrak{T}(X)$ by $20\pi(2g-2)$. It can also be shown that other terms in S_Y have at most "linear growth" on $\mathfrak{T}(X)$, in accordance with the boundness of ∂S_Y.

The following result follows from Corollary 4.2 and the symmetry property (4.3). For completeness, we give its proof in the form that is generalized verbatim to Kleinian groups.

Theorem 4.2. *The following formula holds on $\mathfrak{D}(\Gamma)$,*

$$d\boldsymbol{\vartheta} = \bar{\partial}\partial S = -2i\,\omega_{WP},$$

so that $-S$ is a Kähler potential of the Weil-Petersson metric on $\mathfrak{D}(\Gamma)$.

Proof. Let $\mu, \nu \in \Omega^{-1,1}(\Gamma)$. First, using the Cartan formula, we get

$$\begin{aligned} d\boldsymbol{\vartheta}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu}\right) =& L_\mu(\boldsymbol{\vartheta}(\nu)) - L_\nu(\boldsymbol{\vartheta}(\mu)) - \boldsymbol{\vartheta}\left(\left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\varepsilon_\nu}\right]\right) \\ =& L_\mu(L_\nu S) - L_\nu(L_\mu S) = 0, \end{aligned}$$

which just manifests that $\partial^2 = 0$. On the other hand,

$$d\boldsymbol{\vartheta}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right) = L_\mu(\boldsymbol{\vartheta}(\bar{\nu})) - L_{\bar{\nu}}(\boldsymbol{\vartheta}(\mu)) - \boldsymbol{\vartheta}\left(\left[\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right]\right)$$
$$= - L_{\bar{\nu}} \iint\limits_{\Gamma\backslash\Omega} \vartheta\mu$$
$$= - \iint\limits_{\Gamma\backslash\Omega} (L_{\bar{\nu}}\boldsymbol{\vartheta})\mu,$$

since $\boldsymbol{\vartheta}$ is a $(1,0)$-form.

The computation of $L_{\bar{\nu}}\boldsymbol{\vartheta}$ repeats verbatim the one given in [ZT87b]. Namely, consider the commutative diagram (3.2) with $i = 1, 2$, and, for brevity, omit the index i. Since $(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu} = F^{\varepsilon\hat{\nu}} \circ J^{-1}$, the property **SD1** of the Schwarzian derivative (applicable when at least one of the functions is holomorphic) yields

$$\mathcal{S}(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu}\ (f_z^{\varepsilon\nu})^2 + \mathcal{S}(f^{\varepsilon\mu}) = \mathcal{S}(F^{\varepsilon\hat{\nu}}) \circ J^{-1}(J_z^{-1})^2 + \mathcal{S}(J^{-1}). \tag{4.7}$$

We obtain

$$\left.\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right|_{\varepsilon=0} \mathcal{S}(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu}\ (f_z^{\varepsilon\nu})^2 = \left.\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right|_{\varepsilon=0} \mathcal{S}(F^{\varepsilon\hat{\nu}}) \circ J^{-1}(J_z^{-1})^2$$
$$= \left.\frac{\partial}{\partial\bar{\varepsilon}_\nu}\right|_{\varepsilon=0} F_{zzz}^{\varepsilon\hat{\nu}} \circ J^{-1}(J_z^{-1})^2$$
$$= -\frac{1}{2}\rho\overline{\nu(z)},$$

where in the last line we have used Ahlfors formula (3.11). Finally,

$$d\boldsymbol{\vartheta}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right) = \iint\limits_{\Gamma\backslash\Omega} \mu\bar{\nu}\rho = -2i\,\omega_{WP}\left(\frac{\partial}{\partial\varepsilon_\mu}, \frac{\partial}{\partial\bar{\varepsilon}_\nu}\right).$$

□

4.4. Quasi-Fuchsian reciprocity. The existence of the function S on the deformation space $\mathfrak{D}(\Gamma)$ satisfying the statement of Theorem 4.1 is a global form of quasi-Fuchsian reciprocity. Quasi–Fuchsian reciprocity of McMullen [McM00] follows from it as a immediate corollary.

Let $\mu, \nu \in \Omega^{-1,1}(\Gamma)$ be such that μ vanishes outside Ω_1 and ν – outside Ω_2, so that Lie derivatives L_μ and L_ν stand for the variation of X for fixed Y and variation of Y for fixed X respectively.

Theorem 4.3. *(McMullen's quasi-Fuchsian reciprocity).*

$$\iint_X (L_\nu\mathcal{S}(J_1^{-1}))\ \mu = \iint_Y (L_\mu\mathcal{S}(J_2^{-1}))\ \nu.$$

Proof. Immediately follows from Theorem 4.1, since

$$L_\nu L_\mu S = 2 \iint_X (L_\nu \mathcal{S}(J_1^{-1}))\ \mu,$$
$$L_\mu L_\nu S = 2 \iint_Y (L_\mu \mathcal{S}(J_2^{-1}))\ \nu,$$

and $[L_\mu, L_\nu] = 0$. □

In [McM00], quasi-Fuchsian reciprocity was used to prove that $d(P_F - P_{QF}) = -i\,\omega_{WP}$. For completeness, we give here another proof of this result using earlier approach in [ZT87a], which admits generalization to other deformation spaces.

Proposition 4.1. *On the deformation space* $\mathfrak{D}(\Gamma)$*,*

$$\partial \boldsymbol{\vartheta} = 0.$$

Proof. Using the same identity (4.7) which follows from the commutative diagram (3.2), we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} \mathcal{S}(J^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu}\ (f_z^{\varepsilon\nu})^2 &= \frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} \mathcal{S}(F^{\varepsilon\hat{\nu}}) \circ J^{-1}(J_z^{-1})^2 - \frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} \mathcal{S}(f^{\varepsilon\nu})\\ &= \frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} F_{zzz}^{\varepsilon\hat{\nu}} \circ J^{-1}(J_z^{-1})^2 - \frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} f_{zzz}^{\varepsilon\nu},\end{aligned}$$

where we replaced μ by ν and omit index $i = 1, 2$. Differentiating (3.3) three times with respect to z we get

$$\frac{\partial}{\partial\varepsilon_\nu}\bigg|_{\varepsilon=0} f_{zzz}^{\varepsilon\nu}(z) = -\frac{6}{\pi}\iint\limits_{\mathbb{C}} \frac{\nu(w)}{(z-w)^4}\ d^2w = -\frac{6}{\pi}\iint\limits_{\Gamma\backslash\Omega} K(z,w)\nu(w)d^2w, \tag{4.8}$$

where

$$K(z,w) = \sum_{\gamma\in\Gamma} \frac{\gamma'(w)^2}{(z-\gamma w)^4}.$$

It is well-known that for harmonic ν the integral in (4.8) is understood in the principal value sense (as $\lim_{\delta\to 0}$ of integral over $\mathbb{C} \setminus \{|w-z| \le \delta\}$). Therefore, using Ahlfors formula (3.10) we obtain

$$(L_\nu\boldsymbol{\vartheta})(z) = \frac{12}{\pi}\iint\limits_{\Gamma\backslash\Omega} K(z,w)\nu(w)d^2w,$$

and

$$\begin{aligned}\partial\boldsymbol{\vartheta}(\mu,\nu) &= L_\mu\boldsymbol{\vartheta}(\nu) - L_\nu\boldsymbol{\vartheta}(\mu)\\ &= \iint\limits_{\Gamma\backslash\Omega}(L_\mu\boldsymbol{\vartheta})(z)\,\nu(z)d^2z - \iint\limits_{\Gamma\backslash\Omega}(L_\nu\boldsymbol{\vartheta})(w)\,\mu(w)d^2w = 0,\end{aligned}$$

since kernel $K(z,w)$ is obviously symmetric in z and w, $K(z,w) = K(w,z)$. □

5. Holography

Let Γ be a marked, normalized, purely loxodromic quasi-Fuchsian group of genus $g > 1$. The group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ acts on the closure $\overline{\mathbb{U}}^3 = \mathbb{U}^3 \cup \hat{\mathbb{C}}$ of the hyperbolic 3-space $\mathbb{U}^3 = \{Z = (x, y, t) \in \mathbb{R}^3 \,|\, t > 0\}$. The action is discontinuous on $\mathbb{U}^3 \cup \Omega$ and $M = \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is a hyperbolic 3-manifold, compact in the relative topology of $\overline{\mathbb{U}}^3$, with the boundary $X \sqcup Y \simeq \Gamma\backslash\Omega$. According to the holography principle, the on-shell gravity theory on M, given by the Einstein-Hilbert action functional with the cosmological term, is equivalent to the "off-shell" gravity theory on its boundary $X \sqcup Y$, given by the Liouville action functional. Here we give a precise mathematical formulation of this principle.

5.1. Homology and cohomology set-up. We start by generalizing homological algebra methods in Sect. 2 to the three-dimensional case.

5.1.1. Homology computation. Denote by $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(\mathbb{U}^3 \cup \Omega)$ the standard singular chain complex of $\mathbb{U}^3 \cup \Omega$, and let R be a fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ such that $R \cap \Omega$ is the fundamental domain $F = F_1 - F_2$ for the group Γ in Ω (see Sect. 2). To have a better picture, consider first the case when Γ is a Fuchsian group. Then R is a region in $\overline{\mathbb{U}}^3$ bounded by the hemispheres which intersect $\hat{\mathbb{C}}$ along the circles that are orthogonal to $\mathbb{R}$ and bound the fundamental domain F (see Sect. 2.2.1). The fundamental region R is a three-dimensional CW-complex with a single 3-cell given by the interior of R. The 2-cells – the faces D_k, D_k', E_k and E_k', $k = 1, \dots, g$, are given by the parts of the boundary of R bounded by the intersections of the hemispheres and the arcs $a_k - \bar{a}_k$, $a_k' - \bar{a}_k'$, $b_k - \bar{b}_k$ and $b_k' - \bar{b}_k'$ respectively (see Fig. 1). The 1-cells – the edges, are given by the 1-cells of $F_1 - F_2$ and by e_k^0, e_k^1, f_k^0, f_k^1 and d_k, $k = 1, \dots, g$, defined as follows. The edges e_k^0 are intersections of the faces E_{k-1} and D_k joining the vertices $\bar{a}_k(0)$ to $a_k(0)$, the edges e_k^1 are intersections of the faces D_k and E_k' joining the vertices $\bar{a}_k(1)$ to $a_k(1)$; $f_k^0 = e_{k+1}^0$ are intersections of E_k and D_{k+1} joining $\bar{b}_k(0)$ to $b_k(0)$, f_k^1 are intersections of D_k' and E_k joining $\bar{b}_k(1)$ to $b_k(1)$, and d_k are intersections of E_k' and D_k' joining $\bar{a}_k'(1)$ to $a_k'(1)$. Finally, the 0-cells – the vertices, are given by the vertices of F. This property means that the edges of R do not intersect in $\mathbb{U}^3$. When Γ is a quasi-Fuchsian group, the fundamental region R is a topological polyhedron homeomorphic to the geodesic polyhedron for the corresponding Fuchsian group $\tilde{\Gamma}$.

As in the two-dimensional case, we construct the 3-chain representing M in the total complex $\mathrm{Tot}\,\mathsf{K}$ of the double homology complex $\mathsf{K}_{\bullet,\bullet} = \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ as follows. First, identify R with $R \otimes [\,] \in \mathsf{K}_{3,0}$. We have $\partial'' R = 0$ and

$$\begin{aligned}\partial' R &= -F + \sum_{k=1}^{g} \left(D_k - D_k' - E_k + E_k'\right)\\ &= -F + \partial'' S,\end{aligned}$$

where $S \in \mathsf{K}_{2,1}$ is given by

$$S = \sum_{k=1}^{g} \left(E_k \otimes [\beta_k] - D_k \otimes [\alpha_k]\right).$$

Secondly,

$$
\begin{aligned}
\partial' S = &\sum_{k=1}^{g} \left((b_k - \bar{b}_k) \otimes [\beta_k] - (a_k - \bar{a}_k) \otimes [\alpha_k]\right) \\
&- \sum_{k=1}^{g} \left(\left(f_k^1 - f_k^0\right) \otimes [\beta_k] - \left(e_k^1 - e_k^0\right) \otimes [\alpha_k]\right) \\
= &\, L - \partial'' E,
\end{aligned}
$$

where $L = L_1 - L_2$ and $E \in \mathsf{K}_{1,2}$ is given by

$$
\begin{aligned}
E = &\sum_{k=1}^{g} \left(e_k^0 \otimes [\alpha_k|\beta_k] - f_k^0 \otimes [\beta_k|\alpha_k] + f_k^0 \otimes [\gamma_k^{-1}|\alpha_k\beta_k]\right) \\
&- \sum_{k=1}^{g-1} f_g^0 \otimes [\gamma_g^{-1} \dots \gamma_{k+1}^{-1}|\gamma_k^{-1}].
\end{aligned}
$$

Therefore $\partial' E = V = V_1 - V_2$ and the 3-chain $R - S + E \in (\operatorname{Tot} \mathsf{K})_3$ satisfies

$$
\partial(R - S + E) = -F - L + V = -\Sigma, \tag{5.1}
$$

as asserted.

5.1.2. Cohomology computation. The $\mathrm{PSL}(2, \mathbb{C})$-action on $\mathbb{U}^3$ is the following. Represent $Z = (z, t) \in \mathbb{U}^3$ by a quaternion

$$
\boldsymbol{Z} = x \cdot \mathbf{1} + y \cdot \boldsymbol{i} + t \cdot \boldsymbol{j} = \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix},
$$

and for every $c \in \mathbb{C}$ set

$$
\boldsymbol{c} = \operatorname{Re} c \cdot \mathbf{1} + \operatorname{Im} c \cdot \boldsymbol{i} = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix}.
$$

Then for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{C})$ the action $Z \mapsto \gamma Z$ is given by

$$
\boldsymbol{Z} \mapsto (\boldsymbol{a}\boldsymbol{Z} + \boldsymbol{b})(\boldsymbol{c}\boldsymbol{Z} + \boldsymbol{d})^{-1}.
$$

Explicitly, for $Z = (z, t) \in \mathbb{U}^3$ setting $z(Z) = z$ and $t(Z) = t$ gives

$$
z(\gamma Z) = \left((az + b)(\overline{cz + d}) + a\bar{c}\, t^2\right) J_\gamma(Z), \tag{5.2}
$$

$$
t(\gamma Z) = t\, J_\gamma(Z), \tag{5.3}
$$

where

$$
J_\gamma(Z) = \frac{1}{|cz + d|^2 + |ct|^2}.
$$

Note that $J_\gamma^{3/2}(Z)$ is the Jacobian of the map $Z \mapsto \gamma Z$, hence it satisfies the transformation property

$$J_{\gamma_1\circ\gamma_2}(Z) = J_{\gamma_1}(\gamma_2 Z) J_{\gamma_2}(Z). \tag{5.4}$$

From (5.2) and (5.3) we get the following formulas for the derivatives:

$$\frac{\partial z(\gamma Z)}{\partial z} = (\overline{cz+d})^2 J_\gamma^2(Z), \tag{5.5}$$

$$\frac{\partial z(\gamma Z)}{\partial \bar z} = -(\bar c t)^2 J_\gamma^2(Z), \tag{5.6}$$

$$\frac{\partial z(\gamma Z)}{\partial t} = 2t\bar c(\overline{cz+d}) J_\gamma^2(Z). \tag{5.7}$$

In particular,

$$\frac{\partial z(Z)}{\partial z} = \gamma'(z) + O(t^2), \quad \frac{\partial z(\gamma Z)}{\partial \bar z} = O(t^2), \quad \frac{\partial z(Z)}{\partial t} = O(t), \tag{5.8}$$

as $t \to 0$ and $z \in \hat{\mathbb{C}} \backslash \{\gamma^{-1}(\infty)\}$, where for $z \in \mathbb{C}$ we continue to use the two-dimensional notations

$$\gamma(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \gamma'(z) = \frac{1}{(cz+d)^2}, \quad \frac{\gamma''}{\gamma'}(z) = \frac{-2c}{cz+d}.$$

The hyperbolic metric on $\mathbb{U}^3$ is given by

$$ds^2 = \frac{|dz|^2 + dt^2}{t^2},$$

and is $\mathrm{PSL}(2,\mathbb{C})$-invariant. Denote by

$$w_3 = \frac{1}{t^3} dx \wedge dy \wedge dt = \frac{i}{2t^3} dz \wedge d\bar z \wedge dt$$

the corresponding volume form on $\mathbb{U}^3$. The form w_3 is exact on $\mathbb{U}^3$,

$$w_3 = dw_2, \quad \text{where} \quad w_2 = -\frac{i}{4t^2} dz \wedge d\bar z. \tag{5.9}$$

The 2-form $w_2 \in \mathsf{C}^{2,0}$ is no longer $\mathrm{PSL}(2,\mathbb{C})$-invariant. A straightforward computation using (5.5)–(5.7) gives for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{C})$,

$$\begin{aligned}(\delta w_2)_{\gamma^{-1}} =& \gamma^* w_2 - w_2 \\ =& \frac{i}{2} J_\gamma(Z) \left(|c|^2 dz \wedge d\bar z - \frac{c(\overline{cz+d})}{t} dz \wedge dt + \frac{\bar c(cz+d)}{t} d\bar z \wedge dt \right).\end{aligned}$$

Since $d\delta w_2 = \delta d w_2 = \delta w_3 = 0$ and $\mathbb{U}^3$ is simply connected, this implies that there exists $w_1 \in \mathsf{C}^{1,1}$ such that $dw_1 = \delta w_2$. Explicitly,

$$(w_1)_{\gamma^{-1}} = -\frac{i}{8} \log\left(|ct|^2 J_\gamma(Z)\right) \left(\frac{\gamma''}{\gamma'} dz - \overline{\frac{\gamma''}{\gamma'}} d\bar z \right). \tag{5.10}$$

Using (5.4) and (5.8) we get for $\delta w_1 \in C^{1,2}$,

$$
\begin{aligned}
(\delta w_1)_{\gamma_1^{-1},\gamma_2^{-1}} = &-\frac{i}{8}\left(\log J_{\gamma_1}(Z) + \log\frac{|c(\gamma_2)|^2}{|c(\gamma_2\gamma_1)|^2}\right)\left(\frac{\gamma_2''}{\gamma_2'}\circ\gamma_1\gamma_1'\,dz - \overline{\frac{\gamma_2''}{\gamma_2'}}\circ\gamma_1\overline{\gamma_1'}\,d\bar z\right)\\
&-\frac{i}{8}\left(\log J_{\gamma_2}(\gamma_1 Z) + \log\frac{|c(\gamma_2\gamma_1)|^2}{|c(\gamma_1)|^2}\right)\left(\overline{\frac{\gamma_1''}{\gamma_1'}}\,d\bar z - \frac{\gamma_1''}{\gamma_1'}\,dz\right)\\
&+ B_{\gamma_1^{-1},\gamma_2^{-1}}(Z). \qquad (5.11)
\end{aligned}
$$

Here $B_{\gamma_1^{-1},\gamma_2^{-1}}(Z) = O(t\log t)$ as $t \to 0$, uniformly on compact subsets of $\mathbb{C} \setminus \{\gamma_1^{-1}(\infty), (\gamma_2\gamma_1)^{-1}(\infty)\}$.

Clearly the 1-form δw_1 is closed,

$$d(\delta w_1) = \delta(dw_1) = \delta(\delta w_2) = 0.$$

Since $\mathbb{U}^3$ is simply connected, there exists $w_0 \in \mathsf{C}^{0,2}$ such that $w_1 = dw_0$. Moreover, using $H^3(\Gamma, \mathbb{C}) = 0$ we can always choose the antiderivative w_0 such that $\delta w_0 = 0$. Finally, set $\Phi = w_2 - w_1 - w_0 \in (\operatorname{Tot}\mathsf{C})^2$, so that

$$D\Phi = w_3. \qquad (5.12)$$

5.2. Regularized Einstein-Hilbert action. In two dimensions, the critical value of the Liouville action for a Riemann surface $X \simeq \Gamma\backslash\mathbb{U}$ is proportional to the hyperbolic area of the surface (see Sect. 2). It is expected that in three dimensions the critical value of the Einstein-Hilbert action functional with cosmological term is proportional to the hyperbolic volume of the 3-manifold $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ (plus a term proportional to the induced area of the boundary). However, the hyperbolic metric diverges at the boundary of $\overline{\mathbb{U}}^3$ and for quasi-Fuchsian group Γ (as well as for general Kleinian group [1]) the hyperbolic volume of $\Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ is infinite. In [Wit98], Witten proposed a regularization of the action functional by truncating the 3-manifold M by surface $f = \varepsilon$, where the cut-off function $f \in C^\infty(\mathbb{U}^3, \mathbb{R}_{>0})$ vanishes to the first order on the boundary of $\overline{\mathbb{U}}^3$. Every choice of the function f defines a metric on $\mathbb{U}^3$

$$ds^2 = \frac{f^2}{t^2}(|dz|^2 + dt^2),$$

belonging to the conformal class of the hyperbolic metric. On the boundary of $\overline{\mathbb{U}}^3$ it induces the metric

$$\lim_{t\to 0}\frac{f^2(z,t)}{t^2}|dz|^2.$$

Clearly for the case of quasi-Fuchsian group Γ (or for the general Kleinian case considered in the next section), the cut-off function f should be Γ-automorphic. Existence of such a function is guaranteed by the following result, which we formulate for the general Kleinian case.

[1] Note that we are using the definition of the Kleinian groups as in [Mas88]. In the theory of hyperbolic 3-manifolds these groups are called Kleinian groups of the second kind.

Lemma 5.1. *Let Γ be a non-elementary purely loxodromic, geometrically finite Kleinian group with region of discontinuity Ω, normalized so that $\infty \notin \Omega$. For every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ there exists the Γ-automorphic function $f \in C^\infty(\mathbb{U}^3 \cup \Omega)$ which is positive on $\mathbb{U}^3$ and satisfies*

$$f(Z) = te^{\phi(z)/2} + O(t^3), \quad \text{as } t \to 0,$$

uniformly on compact subsets of Ω.

Proof. Note that $\Gamma\backslash\Omega$ is isomorphic to a finite disjoint union of compact Riemann surfaces. Let R be a fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ which is compact in the relative topology of $\overline{\mathbb{U}}^3$. I. Kra has proved in [Kra72a] (the construction in [Kra72a] suggested by M. Kuga generalizes verbatim to our case) that there exist a bounded open set V in $\overline{\mathbb{U}}^3$ such that $R \subset V$ and a function $\eta \in C^\infty(\mathbb{U}^3 \cup \Omega)$ – partition of unity for Γ on $\mathbb{U}^3 \cup \Delta$, satisfying the following properties.

(i) $0 \leq \eta \leq 1$ and $\operatorname{supp} \eta \subset \overline{V}$.
(ii) For each $Z \in \mathbb{U}^3 \cup \Omega$ there is a neighborhood U of Z and a finite subset J of Γ such that $\eta|_{\gamma(U)} = 0$ for each $\gamma \in \Gamma \setminus J$.
(iii) $\sum_{\gamma\in\Gamma} \eta(\gamma Z) = 1$ for all $Z \in \mathbb{U}^3 \cup \Omega$.

Let $B = \overline{V} \cap \{(z,t) \mid z \in \Lambda\}$. Since $R \cap \Omega$ is compact, then (shrinking V if necessary) there exists a $t_0 > 0$ such that B does not intersect the region $\{(z,t) \in \overline{V} \mid t \leq t_0\}$. Define the function $\hat{f} : \overline{V} \to \mathbb{R}$ by

$$\hat{f}(z,t) = \begin{cases} te^{\phi(z)/2} & \text{if } (z,t) \in \overline{V} \text{ and } t \leq t_0/2, \\ 1 & \text{if } (z,t) \in \overline{V} \text{ and } t \geq t_0, \end{cases}$$

and extend it to a smooth function $\hat{f}$ on $\overline{V}$, positive on $\overline{V} \cap \mathbb{U}^3$. Set

$$f(Z) = \sum_{\gamma\in\Gamma} \eta(\gamma Z)\hat{f}(\gamma Z).$$

By the property (ii), for every $Z \in \mathbb{U}^3 \cup \Omega$ this sum contains only finitely many non-zero terms, so that the function f is well-defined. By properties (i) and (iii) it is positive on $\mathbb{U}^3$. To prove the asymptotic behavior, we use elementary formulas

$$z(\gamma Z) = \frac{az+b}{cz+d} + O(t^2) = \gamma(z) + O(t^2),$$
$$t(\gamma Z) = \frac{t}{|cz+d|^2} + O(t^3) \quad \text{as } t \to 0,$$

where $z \neq \gamma^{-1}(\infty)$. Since ϕ is smooth on Ω and

$$e^{\phi(\gamma z)/2} = e^{\phi(z)/2}|cz+d|^2,$$

we get for $z \in \Omega$ such that $\gamma Z \in \overline{V}$ and t is small enough,

$$\begin{aligned} \hat{f}(\gamma Z) &= \left(\frac{t}{|cz+d|^2} + O(t^3)\right)\left(e^{\phi(\gamma z)/2} + O(t^2)\right) \\ &= te^{\phi(z)/2} + O(t^3), \end{aligned}$$

where the O-term depends on γ. Using properties (ii) and (iii) we finally obtain

$$f(Z) = \sum_{\gamma\in\Gamma} \eta(\gamma Z)\left(te^{\phi(z)/2} + O(t^3)\right)$$
$$= te^{\phi(z)/2} + O(t^3),$$

uniformly on compact subsets of Ω. □

Returning to the case when Γ is a normalized purely loxodromic quasi-Fuchsian group, for every $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ let f be a function given by the lemma. For $\varepsilon > 0$ let $R_\varepsilon = R \cap \{f \geq \varepsilon\}$ be the truncated fundamental region. For every chain c in $\mathbb{U}^3$ let $c_\varepsilon = c \cap \{f \geq \varepsilon\}$ be the corresponding truncated chain. Also let $F_\varepsilon = \partial' R_\epsilon \cap \{f = \varepsilon\}$ be the boundary of R_ε on the surface $f = \varepsilon$ and define chains L_ε and V_ε on $f = \varepsilon$ by the same equations $\partial' F_\varepsilon = \partial'' L_\varepsilon$ and $\partial' L_\varepsilon = \partial'' V_\varepsilon$ as chains L and V (see Sects. 2.2.1 and 2.3.1). Since the truncation is Γ-invariant, for every chain $c \in \mathsf{S}_\bullet(\mathbb{U}^3)$ and $\gamma \in \Gamma$ we have

$$(\gamma c)_\varepsilon = \gamma c_\varepsilon.$$

In particular, relations between the chains, derived in Sect. 5.1, hold for truncated chains as well.

Let M_ε be the truncated 3-manifold with the boundary $\partial' M_\varepsilon$. For ε sufficiently small $\partial' M_\varepsilon = X_\varepsilon \sqcup Y_\varepsilon$ is diffeomorphic to $X \sqcup Y$. Denote by $V_\varepsilon[\phi]$ the hyperbolic volume of M_ε. The hyperbolic metric induces a metric on $\partial' M_\varepsilon$, and $A_\varepsilon[\phi]$ denotes the area of $\partial' M_\varepsilon$ in the induced metric.

Definition 5.1. *The regularized on-shell Einstein-Hilbert action functional is defined by*

$$\mathcal{E}_\Gamma[\phi] = -4 \lim_{\varepsilon\to 0}\left(V_\varepsilon[\phi] - \frac{1}{2}A_\varepsilon[\phi] - 2\pi\chi(X)\log\varepsilon\right),$$

where $\chi(X) = \chi(Y) = 2 - 2g$ *is the Euler characteristic of* X.

The main result of this section is the following.

Theorem 5.1. *(Quasi-Fuchsian holography) For every* $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ *the regularized Einstein-Hilbert action is well-defined and*

$$\mathcal{E}_\Gamma[\phi] = \check{S}_\Gamma[\phi],$$

where $\check{S}_\Gamma[\phi]$ *is the modified Liouville action functional without the area term,*

$$\check{S}_\Gamma[\phi] = S_\Gamma[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^{\phi} d^2z - 8\pi\,(2g-2)\log 2.$$

Proof. It is sufficient to verify the formula,

$$V_\varepsilon[\phi] - \frac{1}{2}A_\varepsilon[\phi] = 2\pi\chi(X)\log\varepsilon - \frac{1}{4}\check{S}_\Gamma[\phi] + o(1) \quad \text{as } \varepsilon \to 0, \tag{5.13}$$

which is a counter-part of the formula (1.13) for quasi-Fuchsian groups.

The area form induced by the hyperbolic metric on the surface $f(Z)=\varepsilon$ is given by

$$\sqrt{1+\left(\frac{f_x}{f_t}\right)^2+\left(\frac{f_y}{f_t}\right)^2}\,\frac{dx\wedge dy}{t^2}.$$

Using

$$\frac{f_x}{f_t}(Z)=\frac{t}{2}\,\phi_x(z)+O(t^3)\quad\text{and}\quad\frac{f_y}{f_t}(Z)=\frac{t}{2}\,\phi_y(z)+O(t^3),$$

we have as $\varepsilon\to 0$,

$$\begin{aligned}A_\varepsilon[\phi]&=\iint\limits_{F_\varepsilon}\sqrt{1+\frac{t^2}{4}(\phi_x^2+\phi_y^2)(z)+O(t^4)}\,\frac{dx\wedge dy}{t^2}\\&=\iint\limits_{F_\varepsilon}\frac{dx\wedge dy}{t^2}+\frac{1}{2}\iint\limits_{F}\phi_z\phi_{\bar z}\,dx\wedge dy+o(1)\\&=\iint\limits_{F_\varepsilon}\frac{dx\wedge dy}{t^2}+\frac{i}{4}\langle\check\omega[\phi],F\rangle+o(1).\end{aligned}$$

Here we have introduced

$$\check\omega[\phi]=\omega[\phi]-e^{\phi}dz\wedge d\bar z=|\phi_z|^2dz\wedge d\bar z,\tag{5.14}$$

and have used that for $Z\in F_\varepsilon$,

$$t=\varepsilon e^{-\phi(z)/2}+O(\varepsilon^3),\tag{5.15}$$

uniformly for $Z=(z,t)$ where $z\in F$.

Next, using (5.1) and (5.12) we have,

$$\begin{aligned}V_\varepsilon[\phi]=&\ \langle w_3,R_\varepsilon\rangle\\=&\ \langle w_3,R_\varepsilon-S_\varepsilon+E_\varepsilon\rangle\\=&\ \langle D(w_2-w_1-w_0),R_\varepsilon-S_\varepsilon+E_\varepsilon\rangle\\=&\ \langle w_2-w_1-w_0,\partial\,(R_\varepsilon-S_\varepsilon+E_\varepsilon)\rangle\\=&-\langle w_2,F_\varepsilon\rangle+\langle w_1,L_\varepsilon\rangle-\langle w_0,V_\varepsilon\rangle.\end{aligned}$$

The terms in this formula simplify as $\varepsilon\to 0$. First of all, it follows from (5.9) that

$$-\langle w_2,F_\varepsilon\rangle=\frac{1}{2}\iint\limits_{F_\varepsilon}\frac{dx\wedge dy}{t^2}.$$

Secondly, using (5.15) and $J_\gamma(Z)=|\gamma'(z)|+O(t^2)$ as $t\to 0$, we have on L_ε,

$$\begin{aligned}(w_1)_{\gamma^{-1}}&=-\frac{i}{8}\log\left(|c\varepsilon|^2\,e^{-\phi}\,|\gamma'(z)|\right)\left(\frac{\gamma''}{\gamma'}dz-\overline{\frac{\gamma''}{\gamma'}}d\bar z\right)+o(1)\\&=-\frac{i}{8}\left(2\log\varepsilon-\phi+\frac{1}{2}\log|\gamma'|^2+\log|c(\gamma)|^2\right)\left(\frac{\gamma''}{\gamma'}dz-\overline{\frac{\gamma''}{\gamma'}}d\bar z\right)+o(1).\end{aligned}$$

Therefore, as $\varepsilon \to 0$,

$$\langle w_1, L_\varepsilon\rangle = -\frac{i}{4}\langle \varkappa, L\rangle(\log\varepsilon - \log 2) + \frac{i}{8}\langle \check{\theta}[\phi], L\rangle + o(1),$$

where 1-forms $\varkappa_\gamma$ and $\check{\theta}_\gamma[\phi]$ were introduced in Corollary 2.3 and formula (2.16) respectively. Finally,

$$\langle w_0, V_\varepsilon\rangle = \langle w_0, \partial' E_\varepsilon\rangle = \langle dw_0, E_\varepsilon\rangle = \langle \delta w_1, E_\varepsilon\rangle = \langle \delta w_1, E\rangle + o(1),$$

where we used that the 1-form δw_1 is smooth on $\mathbb{U}^3$ and continuous on $\mathbb{C}\setminus\Gamma(\infty)$. Since it is closed, we can replace the 1-chain E by the 1-chain $W = W_1 - W_2$ consisting of Γ-contracting paths at 0 (see Sect. 2.3). It follows from (5.11) that $\delta w_1 = \frac{i}{8}\check{u} + o(1)$ as $t \to 0$, where the 1-form $\check{u}_{\gamma_1,\gamma_2}$ was introduced in (2.17), so that

$$-\langle w_0, V_\varepsilon\rangle = -\frac{i}{8}\langle \check{u}, W\rangle + o(1).$$

Putting everything together, we have as $\varepsilon \to 0$,

$$V_\varepsilon[\phi] - \frac{1}{2}A_\varepsilon[\phi] = -\frac{i}{4}\langle \varkappa, L\rangle(\log\varepsilon - \log 2) - \frac{i}{8}\Big(\langle \check{\omega}[\phi], F\rangle - \langle \check{\theta}[\phi], L\rangle + \langle \check{u}, W\rangle\Big) + o(1).$$

Using Corollary 2.3, trivially modified for the quasi-Fuchsian case, and (2.27) conclude the proof. □

A fundamental domain F for Γ in Ω is called admissible, if it is the boundary in $\mathbb{C}$ of a fundamental region R for Γ in $\mathbb{U}^3 \cup \Omega$. As an immediate consequence of the theorem we get the following.

Corollary 5.1. *The Liouville action functional* $S_\Gamma[\phi]$ *is independent of the choice of admissible fundamental domain.*

Proof. Since $V_\varepsilon[\phi]$, $A_\varepsilon[\phi]$ are intrinsically associated with the quotient manifolds $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and $X \sqcup Y \simeq \Gamma\backslash\Omega$, the statement follows from the definition of the Einstein-Hilbert action and the theorem. □

Although we proved the same result in Sect. 2 using methods of homological algebra, the above argument easily generalizes to other Kleinian groups.

Remark 5.1. The truncation of the 3-manifold M by the function f does depend on the choice of the realization of the fundamental group of M as a normalized discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$. Different realizations of $\pi_1(M)$ result in different choices of the function f, since f has to satisfy the asymptotic behavior in Lemma 5.1, where the leading term $te^{\phi(z)/2}$ is not a well-defined function on M.

Remark 5.2. The cochain $w_0 \in \mathsf{C}^{0,2}$ was defined as a solution of the equation $dw_0 = w_1$ satisfying $\delta w_0 = 0$. However, in the computation in Theorem 5.1 this condition is not needed – any choice of an antiderivative for w_1 will suffice. This is due to the fact that the chain in $(\mathrm{Tot}\,\mathsf{K})_3$ that starts with $R \in \mathsf{K}_{3,0}$ does not contain a term in $\mathsf{K}_{0,3}$, hence $\partial' E = V$. Thus we can trivially add the term $\langle \delta w_0, R_\varepsilon - S_\varepsilon + E_\varepsilon\rangle = 0$ to $V_\varepsilon[\phi]$, which through the equation $D\Phi = w_3 - \delta w_0$ still gives $\langle w_0, V\rangle = \langle dw_0, E\rangle$. Thus the absence of $\mathsf{K}_{0,3}$-components in the chain in $(\mathrm{Tot}\,\mathsf{K})_3$ implies that each term in E produces two boundary terms in V which cancel out the integration constants in the definition of w_0. As a result, $S_\Gamma[\phi]$ does not depend on the choice of w_0. In the next section we generalize the Liouville action functional to Kleinian groups having the same property.

5.3. Epstein map. Construction of the regularized Einstein-Hilbert action in the previous section works for a larger class of cut-off surfaces than those given by equation $f = \varepsilon$. Namely, it follows from the proof of Theorem 5.1, that the statement holds for any family S_ε of cut-off surfaces such that for $Z = (z, t) \in S_\varepsilon$,

$$t = \varepsilon e^{-\phi(z)/2} + O(\varepsilon^3) \quad \text{as } \varepsilon \to 0, \tag{5.16}$$

uniformly for $z \in F$.

Given a conformal metric $ds^2 = e^{\phi(z)}|dz|^2$ on $\Omega \subset \hat{\mathbb{C}}$ there is a natural surface in $\mathbb{U}^3$ associated to it through the inverse of the hyperbolic Gauss map. Corresponding construction is due to C. Epstein [Eps84, Eps86] (see also [And98]) and is the following. For every $Z \in \mathbb{U}^3$ and $z \in \hat{\mathbb{C}}$ there is a unique horosphere H based at point z and passing through the point Z: H is a Euclidean sphere in $\mathbb{U}^3$ tangent to $z \in \mathbb{C}$ and passing through Z, or is a Euclidean plane parallel to the complex plane for $z = \infty$. Denote by $[Z, z]$ an affine parameter of the horosphere H – the hyperbolic distance between the point $(0, 1) \in \mathbb{U}^3$ and the horosphere H considered as positive if the point $(0, 1)$ is outside H and negative otherwise. Denote the corresponding horosphere by $H(z, [Z, z])$. The Epstein map $\mathcal{G} : \Omega \to \mathbb{U}^3$ is defined by

$$e^{\phi(z)/2}|dz| = e^{[\mathcal{G}(z), z]} \frac{2|dz|}{1 + |z|^2}$$

and it is Γ-invariant

$$\mathcal{G} \circ \gamma = \gamma \circ \mathcal{G} \text{ for all } \gamma \in \Gamma$$

if $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$.

Remark 5.3. Note that our definition of the Epstein map corresponds to the case $f = id$ in Definition 3.9 in [And98].

Geometrically, the image of the Epstein map is the Epstein surface $\mathcal{H} = \mathcal{G}(\Omega)$, which is the envelope of the family of horospheres $H(z, \varrho(z))$ with

$$\varrho(z) = \log\left(\frac{1}{2}(1 + |z|^2)\right) + \frac{\phi(z)}{2},$$

parametrized by $z \in \Omega$, where $\mathcal{G}(z)$ is the point of tangency of the horosphere $H(z, \varrho(z))$ with the surface $\mathcal{H}$. Explicit computation gives

$$\mathcal{G}(w) = \left(w + \frac{2\phi_{\bar{w}}(w)}{e^{\phi(w)} + |\phi_w(w)|^2}, \ \frac{2e^{\phi(w)/2}}{e^{\phi(w)} + |\phi_w(w)|^2}\right), \quad w \in \Omega. \tag{5.17}$$

Remark 5.4. The square of Euclidean distance between points w and $\mathcal{G}(w)$ in $\overline{\mathbb{U}}^3$ is $4/(e^{\phi(w)} + |\phi_w(w)|^2)$. This gives a geometric interpretation of the density $|\phi_z|^2 + e^{\phi}$ of the (1,1)-form ω introduced in (2.4).

Now to a given $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ we associate the family $\phi_\varepsilon = \phi + 2\log 2 - 2\log\varepsilon \in \mathcal{CM}(\Gamma\backslash\Omega)$ with $\varepsilon > 0$, which corresponds to the family of conformal metrics

$ds_\varepsilon^2 = 4\varepsilon^{-2}e^{\phi(w)}|dw|^2$, and consider the corresponding Γ-invariant family $\mathcal{H}_\varepsilon$ of Epstein surfaces. It follows from the parametric representation

$$z = w + \frac{2\varepsilon^2\phi_{\bar{w}}(w)}{4e^{\phi(w)} + \varepsilon^2|\phi_w(w)|^2}, \tag{5.18}$$

$$t = \frac{4\varepsilon e^{\phi(w)/2}}{4e^{\phi(w)} + \varepsilon^2|\phi_w(w)|^2}, \tag{5.19}$$

that for ε small the surfaces $\mathcal{H}_\varepsilon$ embed smoothly in $\mathbb{U}^3$ and as $\varepsilon \to 0$,

$$z = w + O(\varepsilon^2),$$
$$t = \varepsilon e^{-\phi(w)/2} + O(\varepsilon^3),$$

uniformly for w in compact subsets of Ω. These formulas immediately give the desired asymptotic behavior (5.16). The choice of Epstein surfaces $\mathcal{H}_\varepsilon$ as cut-off surfaces for definition of the regularized Einstein-Hilbert action seems to be the most natural. It is quite remarkable that independently Eqs. (5.18), (5.19) appear in [Kra01] in relation with a general solution of "asymptotically AdS three-dimensional gravity".

6. Generalization to Kleinian Groups

6.1. Kleinian groups of Class A. Let Γ be a finitely generated Kleinian group with the region of discontinuity Ω, a maximal set of non-equivalent components $\Omega_1, \dots, \Omega_n$ of Ω, and the limit set $\Lambda = \hat{\mathbb{C}} \setminus \Omega$. As in the quasi-Fuchsian case, a path P is called Γ-contracting in Ω, if $P = P_1 \cup P_2$, where $p \in \Lambda \setminus \{\infty\}$ is a fixed point for Γ, paths $P_1 \setminus \{p\}$ and $P_2 \setminus \{p\}$ lie entirely in distinct components of Ω and are Γ-contracting at p in the sense of Definition 2.3. It follows from arguments in Sect. 2.3.1 that Γ-contracting paths in Ω are rectifiable.

Definition 6.1. *A Kleinian group Γ is of Class A if it satisfies the following conditions.*

A1 *Γ is non-elementary and purely loxodromic.*
A2 *Γ is geometrically finite.*
A3 *Γ has a fundamental region R in $\mathbb{U}^3 \cup \Omega$ which is a finite three-dimensional CW-complex with no 0-dimensional cells in $\mathbb{U}^3$ and such that $R \cap \Omega \subset \Omega_1 \cup \dots \cup \Omega_n$.*

In particular, Property **A1** implies that Γ is torsion-free and does not contain parabolic elements, and Property **A2** asserts that Γ has a fundamental region R in $\mathbb{U}^3 \cup \Omega$ which is a finite topological polyhedron. Property **A3** means that the region R can be chosen such that the vertices of R – endpoints of edges of R, lie on $\Omega \in \hat{\mathbb{C}}$ and the boundary of R in $\hat{\mathbb{C}}$, which is a fundamental domain for Γ in Ω, is not too "exotic".

The class A is rather large: it clearly contains all purely loxodromic Schottky groups (for which Property **A3** is vacuous), Fuchsian groups, quasi-Fuchsian groups, and free combinations of these groups.

As in the previous section, we say that the Kleinian group Γ is normalized if $\infty \in \Lambda$.

6.2. Einstein-Hilbert and Liouville functionals. For a finitely generated Kleinian group Γ let $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ be a corresponding hyperbolic 3-manifold, and let $\Gamma_1, \dots, \Gamma_n$ be the stabilizer groups of the maximal set $\Omega_1, \dots, \Omega_n$ of non-equivalent components of Ω. We have

$$\Gamma\backslash\Omega = \Gamma_1\backslash\Omega_1 \sqcup \cdots \sqcup \Gamma_n\backslash\Omega_n \simeq X_1 \sqcup \cdots \sqcup X_n,$$

so that Riemann surfaces $X_1, \dots, X_n$ are simultaneously uniformized by Γ. Manifold M is compact in the relative topology of $\overline{\mathbb{U}}^3$ with the disjoint union $X_1 \sqcup \cdots \sqcup X_n$ as the boundary.

6.2.1. Homology and cohomology set-up. Let $\mathsf{S}_\bullet \equiv \mathsf{S}_\bullet(\mathbb{U}^3 \cup \Omega)$, $\mathsf{B}_\bullet \equiv \mathsf{B}_\bullet(\mathbb{Z}\Gamma)$ be standard singular chain and bar-resolution homology complexes and $\mathsf{K}_{\bullet,\bullet} \equiv \mathsf{S}_\bullet \otimes_{\mathbb{Z}\Gamma} \mathsf{B}_\bullet$ – the corresponding double complex. When Γ is a Kleinian group of Class A, we can generalize homology construction from the previous section and define corresponding chains R, S, E, F, L, V in total complex $\operatorname{Tot}\mathsf{K}$ as follows. Let R be a fundamental region for Γ in $\mathbb{U}^3 \sqcup \Omega$ – a closed topological polyhedron in $\overline{\mathbb{U}}^3$ satisfying Property **A3**. The group Γ is generated by side pairing transformations of $R \cap \mathbb{U}^3$ and we define the chain $S \in \mathsf{K}_{2,1}$ as the sum of terms $-s \otimes \gamma^{-1}$ for each pair of sides s, s' of $R \cap \mathbb{U}^3$ identified by a transformation γ, i.e., $s' = -\gamma s$. The sides are oriented as components of the boundary and the negative sign stands for the opposite orientation. We have

$$\partial' R = -F + \partial'' S, \tag{6.1}$$

where $F = \partial' R \cap \Omega \in \mathsf{K}_{2,0}$. Note that it is immaterial whether we choose $-s \otimes \gamma^{-1}$ or $-s' \otimes \gamma$ in the definition of S, since these terms differ by a ∂''-coboundary. Next, relations between generators of Γ determine the Γ-action on the edges of R, which, in turn, determines the chain $E \in \mathsf{K}_{1,2}$ through the equation

$$\partial' S = L - \partial'' E. \tag{6.2}$$

Here $L = \partial' S \cap \Omega \in \mathsf{K}_{1,1}$. Finally, Property **A3** implies that

$$\partial' E = V, \tag{6.3}$$

where the chain $V \in \mathsf{K}_{0,2}$ lies in Ω.

Next, let the 1-chain $W \in \mathsf{K}_{1,2}$ be a "proper projection" of the 1-chain E onto Ω, i.e., W is defined by connecting every two vertices belonging to the same edge of R either by a smooth path lying entirely in one component of Ω, or by a Γ-contracting path, so that $\partial' W = V$. The existence of such 1-chain W is guaranteed by Property **A3** and the following lemma, which is of independent interest.

Lemma 6.1. *Let Γ be a normalized, geometrically finite, purely loxodromic Kleinian group, and let R be the fundamental region of Γ in $\mathbb{U}^3 \cup \Omega$ such that $R \cap \Omega \subset \Omega_1 \cup \cdots \cup \Omega_n$ – a union of the maximal set of non-equivalent components of Ω. If an edge e of $R \cap \mathbb{U}^3$ has endpoints v_0 and v_1 belonging to two distinct components Ω_i and Ω_j, then there exists a Γ-contracting path in Ω joining vertices v_0 and v_1. In particular, Ω_i and Ω_j has at least one common boundary point, which is a fixed point for Γ.*

Proof. There exist sides s_1 and s_2 of R such that $e \subset s_1 \cap s_2$. For each of these sides there exists a group element identifying it with another side of R. Let $\gamma \in \Gamma$ be such an element for s_1. Since Γ is torsion-free and $v_0, v_1 \in \Omega$, the element γ identifies the edge e with the edge e' of R with endpoints $\gamma(v_0) \neq v_0$ and $\gamma(v_1) \neq v_1$. Since, by assumption, $R \cap \Omega \subset \Omega_1 \cup \cdots \cup \Omega_n$, we have that $\gamma(v_0) \in \Omega_i$ and γ fixes Ω_i. Similarly, $\gamma(v_1) \in \Omega_j$ and γ fixes Ω_j. Now assume that attracting fixed point p of γ is not ∞ (otherwise we replace γ by γ^{-1}). Join v_0 and $\gamma(v_0)$ by a smooth path P_1^0 inside Ω_i, and let $P_1^n = \gamma^n(P_1^0)$ be its n^{th} γ-iterate. Since γ fixes Ω_i, the path P_1^n lies entirely inside Ω_i. Since $\lim_{n\to\infty} \gamma^n(v_0) = p$, the path $P_1 = \cup_{n=0}^{\infty} P_1^n$ joins v_0 and p, and except for the endpoint p lies entirely in Ω_i. Clearly the path P_1^0 can be chosen so that the path P_1 is smooth everywhere except at p. The path P_2 joining points v_1 and p inside Ω_j is defined similarly, and the path $P = P_1 \cup P_2$ is Γ-contracting in Ω. □

Setting $\Sigma = F + L - V$ we get from (6.1)–(6.3) that

$$\partial(R - S + E) = -\Sigma.$$

Remark 6.1. Since $\mathbb{U}^3$ is acyclic, it follows from general arguments in [AT97] that for any geometrically finite purely loxodromic Kleinian group Γ with fundamental region R given by a closed topological polyhedron, there exist chains $S \in \mathsf{K}_{2,1}$, $E \in \mathsf{K}_{1,2}$, $T \in \mathsf{K}_{0,3}$ and chains $F \in \mathsf{K}_{2,0}$, $L \in \mathsf{K}_{1,1}$, $V \in \mathsf{K}_{0,2}$ on Ω, satisfying

$$\begin{aligned}\partial' R &= -F + \partial'' S,\\ \partial' S &= L - \partial'' E,\\ \partial' E &= V + \partial'' T.\end{aligned}$$

Property **A3** asserts that $T = 0$, and we get Eqs. (6.1)–(6.3).

Correspondingly, let $\mathsf{A}^\bullet \equiv \mathsf{A}^\bullet_{\mathbb{C}}(\mathbb{U}^3 \cup \Omega)$ and $\mathsf{C}^{\bullet,\bullet} \equiv \mathrm{Hom}(\mathsf{B}_\bullet, \mathsf{A}^\bullet)$ be the de Rham complex on $\mathbb{U}^3 \cup \Omega$ and the bar-de Rham complex respectively. The cochains $w_3, w_2, w_1, \delta w_1, w_0$ are defined by the same formulas as in Sect. 5.1. For $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ define the cochains $\omega[\phi], \theta[\phi], u$ by the same formulas (2.4), (2.5), (2.6), with the group elements belonging to Γ. Finally, define the cochains $\check{\theta}[\phi], \check{u}$ by (2.16) and (2.17).

6.2.2. Action functionals. Let Γ be a normalized Class A Kleinian group. For each $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ let f be the function constructed in Lemma 5.1. As in Sect. 5.2, we truncate the manifold M by the cut-off function f and define $V_\varepsilon[\phi]$, $A_\varepsilon[\phi]$.

Definition 6.2. *The regularized on-shell Einstein-Hilbert action functional for a normalized Class A Kleinian group* Γ *is defined by*

$$\mathcal{E}_\Gamma[\phi] = -4 \lim_{\varepsilon\to 0}\left(V_\epsilon[\phi] - \frac{1}{2}A_\epsilon[\phi] - \pi(\chi(X_1) + \cdots + \chi(X_n))\log\varepsilon\right).$$

As in the quasi-Fuchsian case, a fundamental domain F for a Kleinian group Γ in Ω is called admissible, if it is the boundary in $\mathbb{C}$ of a fundamental region R for Γ in $\mathbb{U}^3$ satisfying Property **A3**.

Definition 6.3. *The Liouville action functional* $S_\Gamma : \mathcal{CM}(\Gamma\backslash\Omega) \to \mathbb{R}$ *for a normalized Class A Kleinian group* Γ *is defined by*

$$S_\Gamma[\phi] = \frac{i}{2}\Big(\langle\omega[\phi], F\rangle - \langle\check{\theta}[\phi], L\rangle + \langle\check{u}, W\rangle\Big), \tag{6.4}$$

where F *is an admissible fundamental domain for* Γ *in* Ω.

Remark 6.2. When Γ is a purely loxodromic Schottky group (not necessarily classical Schottky group), the Liouville action functional defined above is, up to the constant term $4\pi(2g-2)\log 2$, the functional (1.8), introduced by P. Zograf and the first author [ZT87b].

Using these definitions and repeating verbatim arguments in Sect. 5 we have the following result.

Theorem 6.1. *(Kleinian holography) For every* $\phi \in \mathcal{CM}(\Gamma\backslash\Omega)$ *the regularized Einstein-Hilbert action is well-defined and*

$$\mathcal{E}_\Gamma[\phi] = \check{S}_\Gamma[\phi] = S_\Gamma[\phi] - \iint\limits_{\Gamma\backslash\Omega} e^{\phi} d^2z + 4\pi\,(\chi(X_1) + \cdots + \chi(X_n))\log 2.$$

Corollary 6.1. *The definition of a Liouville action functional does not depend on the choice of the admissible fundamental domain* F *for* Γ.

As in the Fuchsian and quasi-Fuchsian cases, the Euler-Lagrange equation for the functional S_Γ is the Liouville equation, and its single critical point given by the hyperbolic metric $e^{\phi_{hyp}}|dz|^2$ on $\Gamma\backslash\Omega$ is non-degenerate. For every component Ω_i denote by $J_i : \mathbb{U} \to \Omega_i$ the corresponding covering map (unique up to a $\mathrm{PSL}(2,\mathbb{R})$-action on $\mathbb{U}$). Then the density $e^{\phi_{hyp}}$ of the hyperbolic metric is given by

$$e^{\phi_{hyp}(z)} = \frac{|(J_i^{-1})'(z)|^2}{(\mathrm{Im}\ J_i^{-1}(z))^2} \quad \text{if } z \in \Omega_i\,, \quad i = 1, \dots, n. \tag{6.5}$$

Remark 6.3. As in Remark 2.3, let $\Delta[\phi] = -e^{-\phi}\partial_z\partial_{\bar{z}}$ be the Laplace operator of the metric $ds^2 = e^{\phi}|dz|^2$ acting on functions on $X_1 \sqcup \cdots \sqcup X_n$, let $\det\Delta[\phi]$ be its zeta-function regularized determinant, and let

$$\mathcal{I}[\phi] = \log\frac{\det\Delta[\phi]}{A[\phi]}.$$

Polyakov's "conformal anomaly" formula and Theorem 6.1 give the following relation between Einstein-Hilbert action $\mathcal{E}[\phi]$ for $M \simeq \Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and "analytic torsion" $\mathcal{I}[\phi]$ on its boundary $X_1 \sqcup \cdots \sqcup X_n \simeq \Gamma\backslash\Omega$,

$$\mathcal{I}[\phi+\sigma] + \frac{1}{12\pi}\mathcal{E}[\phi+\sigma] = \mathcal{I}[\phi] + \frac{1}{12\pi}\mathcal{E}[\phi], \quad \sigma \in C^\infty(X_1 \sqcup \cdots \sqcup X_n, \mathbb{R}).$$

6.3. Variation of the classical action. Here we generalize the theorems in Sect. 4 for quasi-Fuchsian groups to Kleinian groups.

6.3.1. Classical action. Let Γ be a normalized Class A Kleinian group and let $\mathfrak{D}(\Gamma)$ be its deformation space. For every Beltrami coefficient $\mu \in \mathcal{B}^{-1,1}(\Gamma)$ the normalized quasiconformal map $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ descends to an orientation preserving homeomorphism of the quotient Riemann surfaces $\Gamma\backslash\Omega$ and $\Gamma^\mu\backslash\Omega^\mu$. This homeomorphism extends to a homeomorphism of the corresponding 3-manifolds $\Gamma\backslash(\mathbb{U}^3 \cup \Omega)$ and $\Gamma^\mu\backslash(\mathbb{U}^3 \cup \Omega^\mu)$, which can be lifted to an orientation preserving homeomorphism of $\mathbb{U}^3$. In particular, a fundamental region of Γ is mapped to a fundamental region of Γ^μ. Hence Property **A3** is stable and every group in $\mathfrak{D}(\Gamma)$ is of Class A. Moreover, since ∞ is a fixed point of f^μ, every group in $\mathfrak{D}(\Gamma)$ is normalized.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let $S_{\Gamma'} = S_{\Gamma'}[\phi'_{hyp}]$ be the classical Liouville action for Γ'. Since the property of the fundamental domain F being admissible is stable, Corollary 6.1 asserts that the classical action gives rise to a well-defined real-analytic function $S : \mathfrak{D}(\Gamma) \to \mathbb{R}$.

As in Sect. 4, let $\vartheta \in \Omega^{2,0}(\Gamma)$ be the holomorphic quadratic differential on $\Gamma\backslash\Omega$, defined by

$$\vartheta = 2(\phi_{hyp})_{zz} - (\phi_{hyp})_z^2.$$

It follows from (6.5) that

$$\vartheta(z) = 2\mathcal{S}(J_i^{-1})(z) \quad \text{if } z \in \Omega_i, \ \ i = 1, \dots, n.$$

Define a $(1,0)$-form $\boldsymbol{\vartheta}$ on $\mathfrak{D}(\Gamma)$ by assigning to every $\Gamma' \in \mathfrak{D}(\Gamma)$ a corresponding $\vartheta \in \Omega^{2,0}(\Gamma')$.

For every $\Gamma' \in \mathfrak{D}(\Gamma)$ let P_F and P_K be Fuchsian and Kleinian projective connections on $X'_1 \sqcup \cdots \sqcup X'_n \simeq \Gamma'\backslash\Omega'$, defined by the Fuchsian uniformizations of Riemann surfaces $X'_1, \dots, X'_n$ and by their simultaneous uniformization by the Kleinian group Γ'. We will continue to denote corresponding sections of the affine bundle $\mathfrak{P}(\Gamma) \to \mathfrak{D}(\Gamma)$ by P_F and P_K respectively. The difference $P_F - P_K$ is a $(1,0)$-form on $\mathfrak{D}(\Gamma)$. As in Sect. 4.1,

$$\boldsymbol{\vartheta} = 2(P_F - P_K).$$

Correspondingly, the isomorphism

$$\mathfrak{D}(\Gamma) \simeq \mathfrak{D}(\Gamma_1, \Omega_1) \times \cdots \times \mathfrak{D}(\Gamma_n, \Omega_n)$$

defines embeddings

$$\mathfrak{D}(\Gamma_i, \Omega_i) \hookrightarrow \mathfrak{D}(\Gamma)$$

and pull-backs S_i and $(P_F - P_K)_i$ of the function S and the $(1,0)$-form $P_F - P_K$. The deformation space $\mathfrak{D}(\Gamma_i, \Omega_i)$ describes simultaneous Kleinian uniformization of Riemann surfaces $X_1, \cdots, X_n$ by varying the complex structure on X_i and keeping the complex structures on other Riemann surfaces fixed, and the $(1,0)$-form $(P_F - P_K)_i$ is the difference of corresponding projective connections.

6.3.2. First variation. Here we compute the (1, 0)-form ∂S on $\mathfrak{D}(\Gamma)$.

Theorem 6.2. *On the deformation space* $\mathfrak{D}(\Gamma)$*,*

$$\partial S = 2(P_F - P_K).$$

Proof. Since $F^{\varepsilon\mu} = f^{\varepsilon\mu}(F)$ is an admissible fundamental domain for $\Gamma^{\varepsilon\mu}$, and, according to Lemma 2.4, the 1-chain $W^{\varepsilon\mu} = f^{\varepsilon\mu}(W)$ consists of $\Gamma^{\varepsilon\mu}$-contracting paths in $\Omega^{\varepsilon\mu}$, the proof repeats verbatim the proof of Theorem 4.1. Namely, after the change of variables we get

$$L_\mu S = \frac{i}{2}\left(\langle L_\mu\omega, F\rangle - \langle L_\mu\check{\theta}, L\rangle + \langle L_\mu, \check{u}, W\rangle\right),$$

where

$$L_\mu\omega = \vartheta\mu dz \wedge d\bar{z} - d\xi$$

and 1-form ξ is given by (4.6). As in the proof of Theorem 4.1, setting $\chi = \delta\xi + L_\mu\check{\theta}$ we get that the 1-form χ on Ω is closed,

$$d\chi = \delta(d\xi) + L_\mu d\check{\theta} = \delta(-L_\mu\omega) + L_\mu\delta\omega = 0,$$

and satisfies

$$\delta\chi = \delta(L_\mu\check{\theta} + \delta\xi) = L_\mu\delta\check{\theta} = L_\mu\check{u} = d\delta l.$$

Since the 1-chain W consists either of smooth paths or of Γ-contracting paths in Ω, and function δl is continuous on W, the same arguments as in the proof of Theorem 4.1 allow to conclude that

$$L_\mu S = \frac{i}{2}\langle\vartheta\mu dz \wedge d\bar{z}, F\rangle.$$

□

Corollary 6.2. *Let* $X_1, \dots, X_n$ *be Riemann surfaces simultaneously uniformized by a Kleinian group* Γ *of Class A. Then on* $\mathfrak{D}(\Omega_i, \Gamma_i)$*,*

$$(P_F - P_K)_i = \frac{1}{2}\,\partial S_i.$$

6.3.3. Second variation.

Theorem 6.3. *On the deformation space* $\mathfrak{D}(\Gamma)$*,*

$$d\boldsymbol{\vartheta} = \bar{\partial}\partial S = -2i\,\omega_{WP},$$

so that $-S$ *is a Kähler potential of the Weil-Petersson metric on* $\mathfrak{D}(\Gamma)$*.*

The proof is the same as the proof of Theorem 4.2.

6.4. Kleinian Reciprocity. Let $\mu \in \Omega^{-1,1}(\Gamma)$ be a harmonic Beltrami differential, $f^{\varepsilon\mu}$ be a corresponding normalized solution of the Beltrami equation, and let $\upsilon = \dot{f}$ be the corresponding vector field on $\hat{\mathbb{C}}$,

$$\upsilon(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu(w)z(z-1)}{(w-z)w(w-1)} d^2w$$

(see Sect. 3.2). Then

$$\varphi_\mu(z) = \frac{\partial^3}{\partial z^3}\upsilon(z) = -\frac{6}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu(w)}{(w-z)^4} d^2w$$

is a quadratic differential on $\Gamma\backslash\Omega$, holomorphic outside the support of μ.

In [McM00] McMullen proposed the following generalization of quasi-Fuchsian reciprocity.

Theorem 6.4. *(McMullen's Kleinian Reciprocity) Let* Γ *be a finitely generated Kleinian group. Then for every* $\mu, \nu \in \Omega^{-1,1}(\Gamma)$,

$$\iint\limits_{\Gamma\backslash\Omega} \varphi_\mu \nu = \iint\limits_{\Gamma\backslash\Omega} \varphi_\nu \mu.$$

The proof in [McM00] is based on the symmetry of the kernel $K(z, w)$, defined in Sect. 4.2. Here we note that Theorem 6.2 provides a global form of Kleinian reciprocity for Class A groups from which Theorem 6.4 follows immediately.

Indeed, when Γ is a normalized Class A Kleinian group, Kleinian reciprocity is the statement

$$L_\mu L_\nu S = L_\nu L_\mu S,$$

since, according to (4.8),

$$-\frac{1}{2}L_\mu \boldsymbol{\vartheta}(z) = -\frac{6}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu(w)}{(w-z)^4} d^2w = \varphi_\mu(z)$$

and

$$\iint\limits_{\Gamma\backslash\Omega} \varphi_\mu \nu = -\frac{1}{2} \iint\limits_{\Gamma\backslash\Omega} L_\mu L_\nu S.$$

Acknowledgements. We greatly appreciate stimulating discussions with E. Aldrovandi on homological methods, and with I. Kra, M. Lyubich and B. Maskit on various aspects of the theory of Kleinian groups. We are grateful to K. Krasnov for valuable comments and to C. McMullen for the insightful suggestion to use Epstein surfaces for regularizing the Eistein-Hilbert action. The work of the first author was partially supported by NSF grants DMS-9802574 and DMS-0204628.

References

[Ahl61] Ahlfors, L.V.: Some remarks on Teichmüller's space of Riemann surfaces. Ann. Math. **74**(2), 171–191 (1961)

[Ahl87] Ahlfors, L.V.: *Lectures on quasiconformal mappings*. Monterey CA: Wadsworth & Brooks/Cole Advanced Books & Software, 1987, With the assistance of Clifford J. Earle Jr. Reprint of the 1966 original

[And98] Anderson, G.C.: Projective structures on Riemann surfaces and developing maps to $\mathbb{H}^3$ and $\mathbb{C}P^n$. Ph.D. thesis, Berkeley, 1998
[AT97] Aldrovandi, E., Takhtajan, L.A.: Generating functional in CFT and effective action for two-dimensional quantum gravity on higher genus Riemann surfaces. Commun. Math. Phys. **188**(1), 29–67 (1997)
[Ber60] Bers, L.: Simultaneous uniformization. Bull. Am. Math. Soc. **66**, 94–97 (1960)
[Ber70] Bers, L.: *Spaces of Kleinian groups*. In: Several Complex Variables, I (Proc. Conf. Univ. of Maryland, College Park, Md. 1970) Berlin: Springer, 1970, pp. 9–34
[Ber71] Bers, L.: *Extremal quasiconformal mappings*. In: Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y. 1969) Princeton NJ: Princeton, Univ. Press, 1971,pp. 27–52
[Dur83] Duren, P.L.: *Univalent functions*. New York: Springer-Verlag, 1983
[Eps84] Epstein, C.L.: Envelopes of horospheres and Weingarten surfaces in hyperbolic 3-space. Preprint, 1984
[Eps86] Epstein, C.L.: The hyperbolic Gauss map and quasiconformal reflections. J. Reine Angew. Math. **372**, 96–135 (1986)
[Kra72a] Kra, I.: *Automorphic forms and Kleinian groups*. Reading, MA: W. A. Benjamin Inc., 1972, Mathematics Lecture Note Series
[Kra72b] Kra, I.: On spaces of Kleinian groups. Comment. Math. Helv. **47**, 53–69 (1972)
[Kra00] Krasnov, K.: Holography and Riemann surfaces. Adv. Theor. Math. Phys. **4**(4), 929–979 (2000)
[Kra01] Krasnov, K.: On holomorphic factorization in asymptotically Ads 3d gravity. e-preprint hep-th/0109198, 2001
[Mas88] Maskit, B.: *Kleinian groups*. Berlin: Springer-Verlag, 1988
[McM00] McMullen, C.T.: The moduli space of Riemann surfaces is Kähler hyperbolic. Ann. Math. (2) **151**(1), 327–357 (2000)
[MM02] Manin, Y.I., Marcolli, M.: Holography principle and arithmetic of algebraic curves. Adv. Theor. Math. Phys. **5**, 617–650 (2002)
[OPS88] Osgood, B., Phillips, R., Sarnak, P.: Extremals of determinants of Laplacians. J. Funct. Anal. **80**(1), 148–211 (1988)
[Pol81] Polyakov, A.M.: Quantum geometry of bosonic strings. Phys. Lett. B **103**(3), 207–210 (1981)
[Tak92] Takhtajan, L.: Semi-classical Liouville theory, complex geometry of moduli spaces, and uniformization of Riemann surfaces. In: *New Symmetry Principles in Quantum Field Theory* (Cargèse 1991), New York: Plenum, 1992, pp. 383–406
[Wit98] Witten, E.: Anti de Sitter space and holography. Adv. Theor. Math. Phys. **2**(2), 253–291 (1998)
[Wol86] Wolpert, S.A.: Chern forms and the Riemann tensor for the moduli space of curves. Invent. Math. **85**(1), 119–145 (1986)
[ZT85] Zograf P.G., Takhtadzhyan L.A.: Action of the Liouville equation as generating function for accessory parameters and the potential of the Weil-Petersson metric on Teichmüller space. Funkt. Anal. i Prilozhen. **19**(3), 67–68 (1985) (Russian); English. transl. in: Funct. Anal. Appl. **19**(3), 219–220 (1985)
[ZT87a] Zograf P.G., Takhtadzhyan L.A.: On the Liouville equation, accessory parameters and the geometry of Teichmüller space for Riemann surfaces of genus 0. Mat. Sb. (N.S.) **132(174)**(2), 147–166 (1987) (Russian); English transl. in: Math. USSR Sb. **60**(1), 143–161 (1988)
[ZT87b] Zograf P.G., Takhtadzhyan L.A.: On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces. Mat. Sb. (N.S.) **132(174)**(3), 297–313 (1987) (Russian); English transl. in: Math. USSR SB. **60**(2), 297–313 (1988)

Communicated by P. Sarnak