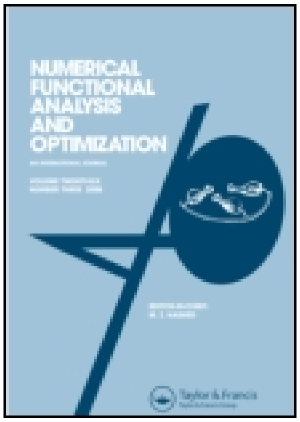

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# Spectral Analysis of Some Iterations in the Chandrasekhar's *H*-Functions

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## ABSTRACT

Two very general, fast and simple iterative methods were proposed by Bosma and de Rooij (Bosma, P. B., de Rooij, W. A. (1983). Efficient methods to calculate Chandrasekhar's $H$ functions. *Astron. Astrophys.* 126:283–292.) to determine Chandrasekhar's $H$-functions. The methods are based on the use of the equation $h = \tilde{F}(h)$, where $\tilde{F} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)^T$ is a nonlinear map from $R^n$ to $R^n$. Here $\tilde{f}_i = 1/(\sqrt{1-c} + \sum_{k=1}^{n} (c_k \mu_k h_k/\mu_i + \mu_k))$, $0 < c \leq 1$, $\quad i = 1, 2, \ldots, n$. One such method is essentially a nonlinear Gauss-Seidel iteration with respect to $\tilde{F}$. The other ingenious approach is to normalize each iterate after a nonlinear Gauss-Jacobi iteration with respect to $\tilde{F}$ is taken. The purpose of this article is two-fold. First, we prove that both methods converge locally. Moreover, the convergence rate of the second iterative method is shown to be strictly less than $(\sqrt{3}-1)/2$. Second, we show that both the Gauss-Jacobi method and Gauss-Seidel method with respect to some other known alternative forms of the Chandrasekhar's $H$-functions either do not converge or essentially stall for $c = 1$.

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## 1. INTRODUCTION

The Chandrasekhar's $H$-equation and its variants arise in the solution of exit distribution problems in neutron transport and radiative transfer. The simplest form of this equation is

$$H(\mu) = 1 + \frac{c}{2}\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu + \mu'}\,d\mu' := F(H)(\mu). \tag{1a}$$

In (1a), $c \in (0, 1]$, which denotes the fraction of scattering per collision, is a real parameter and $H \in C[0, 1]$ is the unknown. Since it is known (Mullikin, 1968) that the Fréchet derivative $F'(H_1^*)$, where $H_1^*$ is a unique positive solution of Eq. (1a) for $c = 1$, has spectral radius 1, the difficulty in solving Eq. (1a) arises dramatically as $c$ approaches 1 from the left. Much effort (Bosma and de Rooij, 1983; Chandrasekhar, 1960; Decker and Kelley, 1985; Kelly, 1988; Kelley and Suresh, 1983; Kelley and Xue, 1993, and the work cited therein) has been put in determining the problem of the $H$-function at near-conservative and conservative scattering (i.e., $c$ is near 1 and $c = 1$). Let $\{\mu_i\}_{i=1}^n$ be the quadrature set and $\{c_i\}_{i=1}^n$ be the corresponding weights. Then Eq. (1a) reduces to

$$h_i = 1 + \frac{c}{2}\mu_i h_i \sum_{k=1}^{n} \frac{c_k h_k}{\mu_i + \mu_k}, \quad i = 1, 2, \ldots, n, \tag{1b}$$

where $h_i := H(\mu_i)$. Let $h = [h_1, h_2, \ldots, h_n]^T$, we write Eq. (1b) as a vector equation of the form

$$h = F(h), \tag{1c}$$

where $F$ is a vector-valued function from $R^n$ to $R^n$. It is also known (see e.g., Kelley, 0000) that Eq. (1c) has a unique positive solution for $c = 1$, and two positive solutions otherwise. Moreover, if $\bar{h}_i$ and $\tilde{h}_i$ are two positive solutions of Eq. (1b) for $0 < c < 1$, then either $\bar{h}_i \geq \tilde{h}_i$ for all $i$ or $\bar{h}_i \leq \tilde{h}_i$ for all $i$. We shall denote the minimum positive solution of Eq. (1c) by $h^*$.

To set up the iterative procedures, some authors have used an equivalent form of Eq. (1a)

$$h_i = \frac{1}{1 - \frac{c}{2}\mu_i \sum_{k=1}^{n} (c_k h_k / \mu_i + \mu_k)}, \quad i = 1, 2, \ldots, n. \tag{2a}$$

Similarly, we write Eq. (2a) in vector form

$$h = \bar{F}(h). \tag{2b}$$

The iterative scheme $h^{(p+1)} = \bar{F}(h^{(p)})$ gives a good improvement over that of Eq. (1) at the near-conservative scattering. However, the spectral radius of $\bar{F}'(h^*)$ for $c = 1$ is still 1, which amounts to no improvement at conservative scattering. Note that the comparison of Eqs. (1) and (2) for the Matrix-valued analogs of the Chandrasekhar *H*-function in multigroup neutron transport was given in (Bowden et al., 1976; Kelley, 1980). From hereon, we shall term such one-step iteration a Gauss-Jacobi method with respect to $\tilde{F}$. Using the discrete version of a known identity (Chandrasekhar, 1960), we have

$$\frac{c}{2}\sum_{i=1}^{n} c_i h_i^* = 1 - \sqrt{1-c}, \tag{3}$$

we arrive at an alternative form Eq. (1a),

$$h_i = \frac{1}{\sqrt{1-c} + \frac{c}{2}\sum_{k=1}^{n}\left(c_k\mu_k h_k/\mu_i + \mu_k\right)}, \quad i = 1, 2, \ldots, n. \tag{4a}$$

In vector form, we write Eq. (4a) as

$$h = \tilde{F}(h). \tag{4b}$$

It was proved in (Bosma and de Rooij, 1983), that the continuous version of Eq. (4b) allows two solutions, one being entirely positive, the physically relevant solution $H_c^*(\mu)$, and the other one being entirely negative. Similar techniques applied to Eq. (4b) would yield the same assertions. Two very successful iterative procedures, which are based on the use of Eq. (4), were proposed by Bosma and de Rooij (1983). Their methods are general, fast, and simple and do not impose any fundamental restriction on the accuracy of the calculations for any $0 < c \leq 1$. One of the iterative schemes proposed in (Bosma and de Rooij, 1983) is essentially the Gauss-Seidel iterative technique applied to the nonlinear system (4). Specifically, Bosma and de Rooij considered the following iteration:

$$\begin{aligned} h_i^{(p+1)} &= \frac{1}{\sqrt{1-c} + \frac{c}{2}\sum_{k=1}^{i-1}\frac{c_k\mu_k h_k^{(p+1)}}{\mu_i+\mu_k} + \frac{c}{2}\sum_{k=i}^{n}\frac{c_k\mu_k h_k^{(p)}}{\mu_i+\mu_k}} \\ &:= \tilde{f}_i\left(h_1^{(p+1)}, \ldots, h_{i-1}^{(p+1)}, h_i^{(p)}, \ldots, h_n^{(p)}\right), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{5a}$$

$$h_i^{(0)} = 1, \quad i = 1, 2, \ldots, n. \tag{5b}$$

The second approach is to normalize each iterate after a Gauss-Jacobi iteration with respect to $\tilde{F}$ is taken. If we write the right hand side of Eq. (4a) as $h_i(\mu_i; h)$, then the second iteration of Bosma and de Rooij can be formulated as

$$h_i^{(p+1)} = \frac{h_i(\mu_i; h^{(p)})}{h_i(0; h^{(p)})} \tag{6a}$$

or, equivalently,

$$h_i^{(p+1)} = \frac{\sqrt{1-c} + \frac{c}{2}\sum_{k=1}^{n} c_k h_k^{(p)}}{\sqrt{1-c} + \frac{c}{2}\sum_{k=1}^{n}\frac{c_k\mu_k h_k^{(p)}}{\mu_i+\mu_k}}, \quad i = 1, 2, \ldots, n, \tag{6b}$$

$$h_i^{(0)} = 1, \quad i = 1, 2, \ldots, n. \tag{6c}$$

We shall call, respectively, the iterations of types (5) and (6) a Gauss-Seidel method with respect to $\bar{F}$ and a normalization method with respect to $\tilde{F}$. However, as noted in (Bosma and de Rooij, 1983), such schemes do not work for nonlinear systems (1) and (2).

The purpose of this article is to give a rigorous justification for why the Gauss-Seidel iterative procedure and the normalization approach work so well at the nonlinear system (4), and so poorly at the nonlinear systems (1) and (2). In particular, we prove that both methods with respect to Eq. (4) converge locally. Moreover, the convergence rate of the second iterative method is shown to be strictly less than $(\sqrt{3}-1)/2$. Finally, we show that both the Gauss-Jacobi method and Gauss-Seidel method with respect to some other known alternative forms of the Chandrasekhar's $H$-functions either do not converge or essentially stall for $c=1$.

## 2. THE GAUSS-JACOBI METHOD

Our prerequisites complies only some elementary facts from the theory of nonnegative matrices. If $A=(a_{ij})$ and $B=(b_{ij})$ are $n\times n$ matrices, we shall write $A\geq B$ if $a_{ij}\geq b_{ij}$ for $1\leq i,j\leq n$. And $A>B$ if $a_{ij}>b_{ij}$ for $1\leq i,j\leq n$. A matrix $A$ is called nonnegative or positive, if $A\geq 0$ or $A>0$, respectively. The notation $|A|$ means that $|A|=(|a_{ij}|)$. Let $A\in R^{n\times n}$, the spectrum of $A$ will be denote $\sigma(A)$ and the spectral of $A$ will be denoted $\rho(A)$, i.e., $\sigma(A)$ is the set of eigenvalues of $A$ and $\sigma(A)=\max\{|\lambda|:|\lambda|\in\sigma(A)\}$.

To study the convergence behavior of the iteration defined in Eq. (4), we first consider the spectral properties of the Jacobian matrices $\bar{F}'(h^*)$ and $\tilde{F}'(h^*)$. A direct calculation would yield

$$(\bar{F}'(h^*))_{ij}=\frac{cc_j\mu_i h_i^{*2}}{2(\mu_i+\mu_j)}\quad i,j=1,2,\ldots,n, \tag{7}$$

and

$$(\tilde{F}'(h^*))_{ij}=\frac{-cc_j\mu_j h_i^{*2}}{2(\mu_i+\mu_j)}\quad i,j=1,2,\ldots,n. \tag{8}$$

Let

$$\bar{D}_1=\frac{c}{2}\operatorname{diag}\left[\mu_1 h_1^{*2},\ldots,\mu_n h_n^{*2}\right]\quad\text{and}\quad \bar{D}_2=\operatorname{daig}[c_1,c_2,\ldots,c_n].$$

Similarly, we set

$$\tilde{D}_1=\frac{c}{2}\operatorname{diag}\left[h_1^{*2},h_2^{*2},\ldots,h_n^{*2}\right]\quad\text{and}\quad \tilde{D}_2=\operatorname{diag}[c_1\mu_1,\ldots,c_n\mu_n].$$

Then

$$\bar{F}'(h^*)=\bar{D}_1 H\bar{D}_2, \tag{9a}$$

where

$$H = \left(\frac{1}{\mu_i + \mu_j}\right)_{n\times n}, \tag{9b}$$

and

$$-\tilde{F}'(h^*) = \tilde{D}_1 H \tilde{D}_2. \tag{9c}$$

We first show the following lemma.

**Lemma 1.** *The matrix H is symmetric positive definite.*

**Proof.** Let $H_k = (1/(\mu_i + \mu_j))_{k\times k}$. Using $1/(\mu_k + \mu_k)$ as a pivot element to eliminate the elements to the left of $1/(\mu_k + \mu_k)$, we see that

$$\begin{aligned}\det A_k &= \left(\frac{1}{\mu_k + \mu_k}\right)\left(\prod_{i,j=1}^{k-1} \frac{(\mu_k - \mu_i)(\mu_k - \mu_j)}{(\mu_n + \mu_i)(\mu_n + \mu_j)}\right) \det A_{k-1} \\ &:= \alpha(k) \det A_{k-1}.\end{aligned}$$

Clearly, $\alpha(k) > 0$ for $k = 1, 2, \ldots, n$. Hence, an induction would yield that $\det A_k > 0$ for all $k = 1, 2, \ldots, n$. Therefore, $H$ is a symmetric positive definite matrix as asserted. ■

We are now ready to state our first result.

**Theorem 1.** *The following assertions hold*

(*i*) $\sigma(\bar{F}'(h^*)) = \sigma(-\tilde{F}'(h^*))$.
(*ii*) *The eigenvalues of* $\bar{F}'(h^*)$, *hence*, $-\tilde{F}'(h^*)$, *are real and positive.*
(*iii*) $\rho(\bar{F}'(h^*)) = \rho(-\tilde{F}'(h^*)) = (k_1/(\sqrt{1-c} + k_1))$, *for some* $0 < k_1 1 - \sqrt{1-c}$. *In particular, for* $c = 1$, $\rho(\bar{F}'(h^*)) = \rho(-\tilde{F}'(h^*)) = 1$.

**Proof.** To prove (i), we see that the characteristic polynomial of $\bar{F}'(h^*)$ is

$$\det(\bar{F}'(h^*) - \lambda I) = \det \bar{D}_1 \det(H - \lambda D) \det \bar{D}_2,$$

where $D = \bar{D}_1^{-1}\bar{D}_2^{-1}$.

Similarly,

$$\det(\tilde{F}'(h^*) - \lambda I) = \det \tilde{D}_1 \det(H - \lambda D) \det \tilde{D}_2.$$

We thus conclude that the first assertion of the theorem holds as asserted.

Using the facts that $D^{-1/2}HD^{-1/2}$ is symmetric positive definite and $D^{-1}H$ and $D^{-1/2}HD^{-1/2}$ are similar, we conclude that the eigenvalues of $\bar{F}'(h^*)$ are real and positive.

To see (iii), we use the celebrated Perron-Frobenius theorem, let $\bar{h} = [\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n]$, where $\bar{h}_i > 0$ for all $i = 1, 2, \ldots, n$, be the eigenvector corresponding to the eigenvalue $\rho(\bar{F}'(h^*)) := \gamma$. Writing $\bar{F}'(h^*)\bar{h} = \gamma\bar{h}$ in component form, we get

$$\frac{c}{2}\mu_i h_i^* \sum_{k=1}^{n} \frac{c_k \bar{h}_k}{\mu_i + \mu_k} = \frac{\gamma \bar{h}_i}{h_i^*} \tag{10}$$

Substituting Eq. (2a) into the right hand side of Eq. (10), we have

$$\frac{c}{2}\mu_i h_i^* \sum_{k=1}^{n} \frac{c_k \bar{h}_k}{\mu_i + \mu_k} + \frac{c}{2}\gamma\mu_i \bar{h}_i \sum_{k=1}^{n} \frac{c_k h_k^*}{\mu_i + \mu_k} = \gamma \bar{h}_i. \tag{11}$$

Multiplying $c_i$ on both sides of Eq. (11) and summing the resulting equation over the index $i$, we obtain that

$$\frac{c}{2}\sum_{i=1}^{n} c_i \bar{h}_i \sum_{k=1}^{n} \frac{c_k \mu_k h_k^*}{\mu_i + \mu_k} + \frac{c}{2}\gamma \sum_{i=1}^{n} c_i \bar{h}_i \sum_{k=1}^{n} \frac{c_k \mu_i \bar{h}_k}{\mu_i + \mu_k} = \gamma \sum_{i=1}^{n} c_i \bar{h}_i. \tag{12}$$

Set

$$k_1(i) = \frac{c}{2}\sum_{k=1}^{n} \frac{c_k \mu_k h_k^*}{\mu_i + \mu_k}. \tag{13}$$

Then $0 < k_1(i) < 1 - \sqrt{1-c}$. And Eq. (12) reduces to

$$\sum_{i=1}^{n} c_i \bar{h}_i k_1(i) + \gamma \sum_{i=1}^{n} c_i \bar{h}_i (1 - \sqrt{1-c} - k_1(i)) = \gamma \sum_{i=1}^{n} c_i \bar{h}_i. \tag{14}$$

Since $\bar{h}_i > 0$ for all $i$, there exists $k_1$, $0 < k_1 < 1 - \sqrt{1-c}$, such that

$$k_1 \sum_{i=1}^{n} c_i \bar{h}_i = \sum_{i=1}^{n} c_i \bar{h}_i k_1(i). \tag{15}$$

Using Eq. (15), we have, via Eq. (14), that

$$\gamma = \frac{k_1}{\sqrt{1-c} + k_1}.$$

We thus complete the proof of the theorem. ■

**Remark.**

1. It is known (Mullikin, 1968) that $\rho(F'(h^*)) = 1 - \sqrt{1-c}$. A direct calculation would yield that $\rho(F'(h^*)) \geq \rho(\bar{F}'(h^*))$. Moreover, the equality above holds only if $c = 1$.
2. For $c = 1$, the Gauss-Jacobi method with respect to either $F$, $\bar{F}$ or $\tilde{F}$ does not work well at all. However, for $0 < c < 1$ some improvement is expected if either one of the nonlinear systems (2) and (3) are chosen over (1).

3. We expect that similarly assertions in Theorem 1 can be applied as well to the infinite dimensional case by using the ideas in Anselone's book (Anselone, 1971) on collective compact operators.

## 3. THE GAUSS-SEIDEL METHOD

To see the effective of the application of the Gauss-Seidel technique on the nonlinear system (4), we need to reformulate (5) as

$$h^{(p+1)} = \tilde{G}(h^{(p)}), \tag{16a}$$

where $\tilde{G}$ is an appropriate defined nonlinear mapping from $R^n$ to $R^n$. Let $\tilde{G} = [\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n]$, where $\tilde{g}_i,\ i = 1, 2, \ldots, n$ are the functionals from $R^n$ to $R$. Then $\tilde{g}_i$ can be recursively defined as

$$\tilde{g}_1 = \tilde{f}_1, \tag{16b}$$

and

$$\tilde{g}_i(h) = \tilde{f}_i(\tilde{g}_1(h), \ldots, \tilde{g}_{i-1}(h), h_i, \ldots, h_n). \tag{16c}$$

Here $\tilde{f}_i i = 1, 2, \ldots, n,$ are defined in Eq. (5a). We are ready to state the following Theorem.

**Theorem 2.** *Let $\tilde{F}'(h^*) = \tilde{D} + \tilde{L} + \tilde{U}$, where $\tilde{D}$, $\tilde{L}$, and $\tilde{U}$ are, respectively, the diagonal part, strictly lower triangular part and strictly upper triangular part of $\tilde{F}'(h^*)$. Then $\tilde{G}'(h^*) = (I - \tilde{L})^{-1}(\tilde{D} + \tilde{U})$.*

**Proof.** Clearly, $h^*$ is also a fixed point of $\tilde{G}$. For $1 \le j < i$, we have

$$\frac{\partial \tilde{g}_i}{\partial h_j}(h^*) = \sum_{k=1}^{i-1} \left( \frac{\partial \tilde{f}_i}{\partial h_k}(h^*) \right) \left( \frac{\partial \tilde{g}_k}{\partial h_j}(h^*) \right). \tag{17a}$$

For $i \le j \le n$, we get

$$\frac{\partial \tilde{g}_i}{\partial h_j}(h^*) = \sum_{k=1}^{i-1} \left( \frac{\partial \tilde{f}_i}{\partial h_k}(h^*) \right) \left( \frac{\partial \tilde{g}_k}{\partial h_j}(h^*) \right) + \frac{\partial \tilde{f}_i}{\partial h_j}(h^*). \tag{17b}$$

Combing Eqs. (17a) and (17b), we have that

$$\tilde{G}(h^*) = \tilde{L}\tilde{G}(h^*) + \tilde{D} + \tilde{U},$$

or, equivalently,

$$\tilde{G}(h^*) = (I - \tilde{L})^{-1}(\tilde{D} + \tilde{U}).$$

■

**Remark.** Suppose one applies the Gauss-Seidel technique on the nonlinear system (2), and writes the corresponding one point iteration as $h^{(p+1)} = \bar{G}(h^{(p)})$. Moreover, let $\bar{F}'(h^*) = \bar{D} + \bar{L} + \bar{U}$,where $\bar{D}$, $\bar{L}$, and $\bar{U}$ are, respectively, the diagonal part, strictly lower triangular part and strictly upper triangular part of $\bar{F}'(h^*)$. Then $\bar{G}'(h^*) = (I - \bar{L})^{-1}(\bar{D} + \bar{U})$.

**Lemma 2.** *The spectral* $\rho(\bar{G}'(h^*))$ *of* $\bar{G}'(h^*)$ *is* $(k_1 - k_2 + k_2(\sqrt{1-c}))/(k_1 - k_2(1-\sqrt{1-c}) + \sqrt{1-c})$, *where* $k_1$ *and* $k_2$ *are constants such that* $0 < k_2 < k_1 < 1 - \sqrt{1-c}$.

**Proof.** Since $(I - \bar{L})^{-1} = I + \bar{L} + \bar{L}^2 + \cdots + \bar{L}^{n-1}$, we see that $\bar{G}(h^*) > 0$. Let $\bar{g} = [\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n]$, where $\bar{g}_i > 0$ for all $i = 1, 2, \ldots, n$, be the eigenvector corresponding to the eigenvalue $\rho(\bar{G}(h^*)) := \bar{\gamma}$. A similar procedure as done in obtaining Eq. (12) would yield

$$\frac{c}{2}\sum_{i=1}^{n} c_i\mu_i h_i^* \left(\lambda \sum_{k=1}^{i-1} \frac{c_k\bar{h}_k}{\mu_i + \mu_k} + \sum_{k=i}^{n} \frac{c_k\bar{h}_k}{\mu_i + \mu_k}\right) + \frac{c}{2}\gamma \sum_{i=1}^{n} c_i\bar{h}_i \sum_{k=1}^{n} \frac{c_k\mu_i h_k^*}{\mu_i + \mu_k} = \gamma \sum_{i=1}^{n} c_i\bar{h}_i. \tag{18}$$

Let $\bar{k}_2$ be the positive constant such that

$$\sum_{i=1}^{n} c_i\mu_i h_i^* \sum_{k=1}^{i-1} \frac{c_k\bar{h}_k}{\mu_i + \mu_k} = \bar{k}_2 \sum_{i=1}^{n} c_i\mu_i h_i^* \sum_{k=1}^{n} \frac{c_k\bar{h}_k}{\mu_i + \mu_k}. \tag{19}$$

Clearly $0 < \bar{k}_2 < 1$. Let $k_1$ be defined as in Theorem 1, and $k_2 = k_1\bar{k}_2$. It follows from Eqs. (18) and (19) that Eq. (18) would reduce to

$$k_2(\bar{\gamma} - 1)(1 - \sqrt{1-c}) + k_1 + \bar{\gamma}(1 - \sqrt{1-c} - k_1) = \bar{\gamma}_1. \tag{20}$$

A direct calculation would give

$$\bar{\gamma} = \frac{k_1 - k_2 + k_2(\sqrt{1-c})}{k_1 - k_2(1 - \sqrt{1-c}) + \sqrt{1-c}}$$

as asserted. ■

**Remark.**

1. For $c = 1$, $\bar{\gamma} = \gamma = 1$. Hence, both the Gauss-Seidel method and the Gauss-Jacobi method do not work on the nonlinear system (2). For $0 < c < 1$, we have that $\rho(\bar{G}'(h^*)) < \rho(\bar{F}'(h^*))$. This, in turn, suggests that some improvement is expected when the Gauss-Seidel method is applied to Eq. (2).
2. Similar assertions hold on the nonlinear system (1).

**Theorem 3.** $\rho(\tilde{G}'(h^*)) < \rho(\bar{G}'(h^*))$. *In particular, if* $c = 1$, $\rho(\tilde{G}'(h^*)) < 1$.

**Proof.** Let $\tilde{K} = -(I + \tilde{L})^{-1}(\tilde{D} + \tilde{U})$, where $\tilde{D}$, $\tilde{L}$, and $\tilde{U}$ are given in Theorem 2. A similar technique as given in the proof of Theorem 1(i) would yield that

$$\sigma(\tilde{K}) = \sigma(\tilde{G}'(h^*)).$$

Now,

$$\begin{aligned}\tilde{G}'(h^*) &= (I-\tilde{L})^{-1}(\tilde{D}+\tilde{U})\\ &= (I+\tilde{L}+\cdots+\tilde{L}^{n-1})(\tilde{D}+\tilde{U}).\end{aligned}$$

and

$$\tilde{K} = -\left(I-\tilde{L}+\tilde{L}^2-\cdots+(-1)^{n-1}\tilde{L}^{n-1}\right)(\tilde{D}+\tilde{U}).$$

Noting that $-\tilde{D}$, $-\tilde{U}$, and $-\tilde{L}$ are nonnegative, we see that

$$|\tilde{G}'(h^*)| \le \tilde{K} \quad \text{and} \quad |\tilde{G}'(h^*)| \ne \tilde{K}.$$

Clearly, $\tilde{G}'(h^*)$ and $\tilde{K}$ are irreducible. A well-known result (see e.g., Minc, 1988) would give

$$\rho(\tilde{G}'(h^*)) < \rho(\tilde{K}) = \rho(\bar{G}'(h^*)).$$

It follows from the above and Lemma 2 that the last assertion holds. ■

## 4. THE NORMALIZATION METHOD

In vector form, Eq. (6) can be written as

$$h^{(p+1)} = K(h^{(p)}). \tag{21}$$

A direct calculation yield that the Jacobian matrix $K'(h^*)$ is

$$K'(h^*) = \frac{c}{2}h^* w^T + \tilde{F}'(h^*), \tag{22}$$

where $w^T = [c_1, c_2, \ldots, c_n]$.

To see that the spectral radius of $K'(h^*)$ is less than one, we need to derive the following lemma:

**Lemma 3.** *For* $c = 1$, $\sum_{k=1}^{n} c_k \mu_k h_k^* = 2/\sqrt{3}$.

**Proof.** Multiplying $c_i\mu_i^2$ on both sides of Eq. (1b), and summing the resulting equation over the index $i$, we get

$$\sum_{i=1}^{n} c_i\mu_i^2 h_i^* = \sum_{i=1}^{n} c_i\mu_i^2 + \frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{(\mu_i^3+\mu_k^3)c_i h_i^* c_k h_k^*}{\mu_i+\mu_k}. \tag{23}$$

Simplifying the second term on the right hand side of Eq. (23) and using the fact that $\sum_{i=1}^{n} c_i\mu_i^2 = \int_0^1 \mu^2 d\mu = \frac{1}{3}$ for $n \ge 2$, we have

$$\left(\frac{1}{2}\sum_{i=1}^{n} c_i h_i^* \mu_i\right)^2 = \frac{1}{3},$$

and, hence, $\sum_{i=1}^{n} c_i h_i^* \mu_i = 2/\sqrt{3}$ as asserted.

**Remark.** A continuous version of equality established in the above lemma can be found in (Chandrasekhar, 1960).

**Theorem 4.** *Let* $w^T = [c_1, c_2, \ldots, c_n]$, *then* $\rho(K'(h^*)) < (c/2)\rho(h^* w^T)$, *and hence* $\rho(K'(h^*)) < 1 - \sqrt{1-c}$. *In particular, if* $c = 1$, $\rho(K'(h^*)) < 1$.

**Proof.** To see the first assertion of the theorem, it suffices to show that

$$\frac{c}{2} c_j h_i^* \geq |K'(h^*)|, \quad \text{for all} \quad i, j. \tag{24}$$

To this end, let $k_1(i)$ be given as in Eq. (13). For $c = 1$, using Eq. (1b), we see that

$$h_i^* = \frac{1}{k_1(i)}.$$

Hence,

$$\frac{\mu_j h_i^*}{\mu_i + \mu_j} = \frac{\mu_j}{k_1(i)(\mu_i + \mu_j)} \leq \frac{1}{k_1(i)(1 + \mu_i)} \leq \sqrt{3}. \tag{25}$$

The last inequality is justified by the inequality $k_1(i)(1 + \mu_i) \geq (1/2)\sum_{k=1}^{n} c_k \mu_k h_k^*$ and Lemma 3. Now, inequality (24) follows easily from Eq. (25). The first assertion of Theorem 4 now follows from the fact that $h_i^*$ are increasing in $c$ for all $i$. The last assertion of the theorem follows from Eq. (3). ■

In the following, we shall give a tighter upper bound for the spectral radius of $K'(h^*)$. To this and, we first recall a result in (Rothblum and Tan, 1985).

**Theorem 5. [Theorem 5.1 of Rothblum and Tan (1985)].** *Let* $P$ *be an* $n \times n$ *nonnegative irreducible matrix, and let* $r$ *be a positive right eigenvector of* $P$ *corresponding to the eigenvalue* $\rho = \rho(P)$. *Also, let* $a \in R^n$. *Then*

$$\sigma(P - ra^T) = [\sigma(P) - \{\rho\}] \cup \{\rho - a^T r\}.$$

**Lemma 4.** *For* $c - 1$, let $\lambda_2$ *be the second largest eigenvalue of* $-\tilde{F}'(h^*)$. *Then the spectral radius of* $K'(h^*)$ *equals to* $\max\{\lambda_2, 1 - \sqrt{1-c} - (k_1/\sqrt{1-c} + k_1)\}$, *where* $k_1$ *is given as in Theorem* 1.

**Proof.** We first note that $h^*$ is a positive right eigenvector of $-\tilde{F}'(h^*)$. The assertion of the lemma now follows from Theorems 1 and 5. ■

**Theorem 6.** *For* $c = 1$, $\rho(K'(h^*)) < ((\sqrt{3} - 1)/2)$.

**Proof.** For $c = 1$, the spectral radius of $K'(h^*)$ is clearly equal to $\lambda_2$. Now,

$$\lambda_2 < \operatorname{trace}\big(-\tilde{F}'(h^*)\big) - 1 = \frac{1}{4}\sum_{i=1}^{n} c_i h_i^{*2} - 1.$$

By Eq. (25) and Lemma 3, we see that

$$\frac{1}{4}\sum_{i=1}^{n} c_i h_i^{*2} \le \frac{\sqrt{3}}{4}\sum_{i=1}^{n}\frac{c_i(\mu_i+\mu_j)h_i^*}{\mu_j}$$
$$\le \frac{\sqrt{3}}{4}\sum_{i=1}^{n} c_i(1+\mu_i)h_i^* = \frac{\sqrt{3}+1}{2}.$$

Therefore, $\lambda_2 < (\sqrt{3}-1)/2$ as asserted. ■

**Concluding Remarks.**

1. Our numerical results suggest that $\rho(\tilde{G}'(h^*)) \approx 0.25$ and $\rho(K'(h^*)) \approx 0.06$. Moreover, the convergence behavior of the iterations (5) and (6) are independent of the number of quadrature points chosen. This, in turn, suggests that $\rho(\tilde{G}'(h^*))$ and $\rho(K'(h^*))$ are independent of the dimension of the Jacobian matrices $\tilde{G}'(h^*)$ and $K'(h^*)$. Moreover, our analysis does not give an indication as to how good these iterations really are. In light of above comments, it is worthwhile to pursue these matters further.
2. We have obtained some new estimates concerning the spectra of the Fréchet derivatives of certain operators with respect to various formulation of Eq. (1). It is of interest to see if all such estimates apply as well to the infinite dimensional case.
3. It is certainly worthwhile to see how Bosma and de Rooij's ideas can be extended to systems, such as the $H$-equations arising in polarized light, multigroup neutron transport or simple transport model with an angular shift (see e.g., Chandrasekhar, 1960; Coron, 1990; Ganapol, 1992; Juang, 1995; Kelley, 1980; Kelley and Xue, 1993).

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