Available online at www.sciencedirect.com

SCIENCE DIRECT

Discrete Applied Mathematics 129 (2003) 421–431

DISCRETE
APPLIED
MATHEMATICS

www.elsevier.com/locate/dam

Using transforming matrices to generate DNA clone grids

Hua-Min Huang[a], Frank K. Hwang[b,1], Jian-Feng Ma[b,1]

[a] *Department of Mathematics, National Central University, Chung-Li, Taiwan*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan*

Received 9 June 2000; received in revised form 27 March 2002; accepted 9 October 2002

Abstract

Ina yeast artificial chromosome library, DNA clones may be stored in n-dimensional grids. Barillot, Lacroix and Cohen proposed to use the grid lines as pools in a pooling design (used in physical mapping). To screen the clones in a given grid, they noted that it is important to take several copies of the grid, but rearrange the clones such that two clones are in the same grid line at most once. For ease of implementation, biologists proposed using transforming matrices to transform one copy into another. We give a construction of a set of transforming matrices which produce a maximum number of such copies whenever q is a prime power.

1. Introduction

Consider DNA clones stored in a yeast-artificial-chromosome (YAC) library (see [4]). The clones of a YAC library are typically grown on nylon filters in rectangular arrays or “grids” (see Fig. 1). A pool is a set of clones to be tested together. Since each pool needs much time to be prepared for testing, usually non-adaptive algorithms are used, namely, all pools are tested simultaneously, to screen the clone library for clones containing a specified DNA segment. One convenient way to collect the pools is to take the rows and columns of the grids as pools. We assume the screening is confined to one grid, say G, at a time, which is of size $q \times q$, and among the q^2 clones at most d are positive (specific clones which we need to identify). In absence

E-mail address: huang@math.ncu.edu.tw (H.-M. Huang).

[1] Supported in part by the National Science Council of Republic of China under grant NSF 88-2115-21-9-016.

doi:10.1016/S0166-218X(02)00595-4

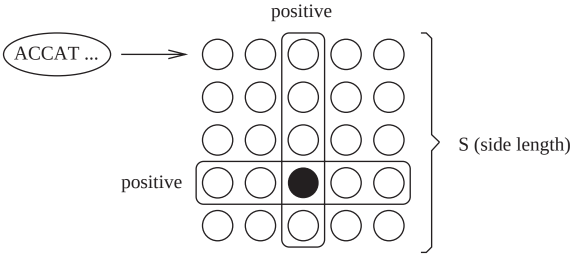

Fig. 1. A DNA subsequence is stored in a clone. A positive clone renders its row and its column positive.

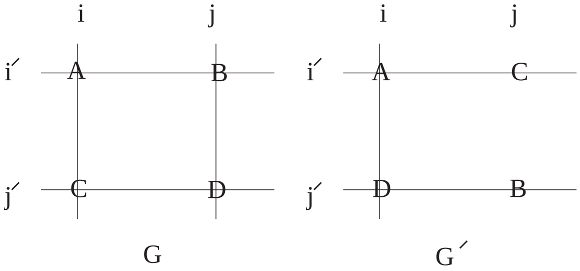

Fig. 2. If two positive clones A and D are not in the same row and column, then we need other grids to find the positive clones.

of experimental error, the testing outcome is negative (or 0) if all clones in the pool are negative. If at least one clone is positive in the pool, the test outcome is positive (or 1).

If there is a positive clone in G, then we get a positive outcome for the row and column containing the positive clone. For this reason, all positive clones are located at the intersections of positive rows and positive columns. Unfortunately, we cannot determine which intersections are the locations of positive clones when we have more than one positive clone.

Suppose for example (see Fig. 2) that A at (i',i) and D at (j',j) are positive clones which are not in the same row and column. Then the test produces four positive lines (i,i',j,j') with four intersections. In this case, we cannot be certain which two intersections are the locations of positive clones. Therefore when d is more than one, other grids, which rearrange the clones, are needed to differentiate positive clones from other clones at these intersections.

Two clones are called *collinear* in a grid if they lie either in the same column or in the same row (See Fig. 3). Barillot et al. [1] suggested using a second grid G_1 in which the rows and the columns are again partitions of the q^2 clones, but two clones collinear in G_0 are no longer collinear in G_1 (see Fig. 3).

We now consider a generalization of the 2-dimensional grid to n-dimensions.

0	1	2		0	5	7
3	4	5		4	6	2
6	7	8		8	1	3
	G_0				G_1	

Fig. 3. Any two clones are collinear just once in the two grids.

Let Z_q be the set of all integers modulo q. The general abstract model of an *n-dimensional grid G* on q^n objects is a bijective (one to one) mapping

$$G\colon \{0,1,2,\ldots,q^n-1\} \to (Z_q)^n = \{[x_1,\ldots,x_n]^{\mathrm{t}} \mid x_i \in Z_q\}$$

onto column n-tuples over Z_q. Two clones x, y are *collinear* if their images $G(x), G(y)$ differ in exactly one component. We also call the image $G(x)$ the coordinate of the clone x.

The standard grid G_0 is defined to be the q-nary representation of $\{0,1,2,\ldots,q^n-1\}$, where

$G_0(0) = [0,0,\ldots,0]^{\mathrm{t}}$,
$G_0(1) = [1,0,\ldots,0]^{\mathrm{t}}$,
$\vdots$
$G_0(q^n-1) = [q-1,q-1,\ldots,q-1]^{\mathrm{t}}$.

A set of grids is *compatible* if any two clones x, y collinear in a given grid are not collinear in the other grid. Since each clone is collinear with $n(q-1)$ other clones in a single grid, while the total numbers of clones other than the given clone is q^n-1, at most $\lfloor (q^n-1)/(n(q-1)) \rfloor$ compatible grids can exists. If a set of compatible grids attains this upper bound, we refer to this set as a *full* set of compatible grids. For $n=2$, Hwang [5] showed there exists a factorization of the complete graph of order q^2 into 2-dimensional grids with side length q if and only if there exists a set of $q-1$ mutually orthogonal latin squares of order q.

This implies that if q is a prime power, then there is a full set of compatible grids. On the other hand, if $q=6$ there is no other grid compatible with the standard grid. In [3], it is shown that the complete graph of order q^n can be packed with a full set of n-dimensional grids of side length q for every prime power q. However, no explicit construction of such a packing is given.

For easier implementation, De Jong et al. [2] and Barillot et al. [1], proposed using transforming matrices to construct compatible grids. Let G be an arbitrary n-dimensional grid, and A_0 the $n \times n$ identity matrix. Then an $n \times n$ matrix A, called a *transforming matrix*, is *efficient* if $A_0 \cdot G$ and $A \cdot G$ are compatible. A set of transforming matrices $\{A_0, A_1, \ldots, A_{k-1}\}$ is an efficient set if $\{A_0 \cdot G, A_1 \cdot G, \ldots, A_{k-1} \cdot G\}$ are compatible for any grid G. An efficient set is *full* if the induced compatible set of grids is full. In this paper, we give a necessary and sufficient condition for an efficient set. We also give a simple construction of a full set of transforming matrices when q is a prime power.

2. Coordinate transformation and efficient matrix

Let G_0 be the standard grid, $E_0 = [1,0,0,\dots,0]^{\mathrm{t}}, E_1 = [0,1,0,\dots,0]^{\mathrm{t}},\dots,E_{n-1} = [0,0,\dots,0,1]^{\mathrm{t}}$ be column vectors of the identity matrix of order n. Let $A=[V_0,V_1,\dots,V_{n-1}]$ be an invertible $n\times n$ matrix over Z_q, where $V_0,V_1,\dots,V_{n-1}$ are column vectors of A. We can use the matrix A to define a coordinate transformation on $(Z_q)^n$ by assigning

$$\begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

to a new coordinate

$$\begin{bmatrix} \widetilde{x_0} \\ \vdots \\ \widetilde{x_{n-1}} \end{bmatrix}$$

if

$$\begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} = A\cdot \begin{bmatrix} \widetilde{x_0} \\ \vdots \\ \widetilde{x_{n-1}} \end{bmatrix} = \widetilde{x_0}\cdot V_0 + \cdots + \widetilde{x_{n-1}}\cdot V_{n-1}.$$

Let G be the new grid induced by this new coordinate system. Two distinct clones $x=\widetilde{x_0}\cdot V_0+\cdots+\widetilde{x_{n-1}}\cdot V_{n-1}$, $y=\widetilde{y_0}\cdot V_0+\cdots+\widetilde{y_{n-1}}\cdot V_{n-1}$ are collinear in the new grid if and only if there is an $0\leqslant i\leqslant n-1$ such that $\widetilde{x_j}=\widetilde{y_j}$ for all $j\neq i$, i.e.

$$x-y=\alpha\cdot V_i \quad \text{for some } 0\neq\alpha\in Z_q, \quad 0\leqq i\leqq n-1.$$

Two clones x,y are collinear in the standard grid G_0 if and only if

$$x-y=\alpha\cdot E_i \quad \text{for some } 0\neq\alpha\in Z_q, \quad 0\leqq i\leqq n-1.$$

Since it is well known that the column vectors of an invertible matrix are not parallel to each other, so we have the following lemma:

Lemma 1. *Let A_0 be the identity matrix, G_0 be the standard grid. A set of matrices $\{A_0,A_1,\dots,A_{k-1}\}$ is an efficient set of transforming matrices and generates a set of compatible grids $\{G_0,G_1,\dots,G_{k-1}\}$ if and only if no two column vectors among the matrices $A_0,A_1,\dots,A_{k-1}$ are parallel to each other.*

Remark. By setting k=2, we obtain a necessary and sufficient condition for an efficient matrix (as versus Barillot's sufficient condition).

Example 1.

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

is a full set of transforming matrices and generates the following compatible grids over Z_5:

$$G_0 = \begin{bmatrix} (0,0) & (0,1) & (0,2) & (0,3) & (0,4) \\ (1,0) & (1,1) & (1,2) & (1,3) & (1.4) \\ (2,0) & (2,1) & (2,2) & (2,3) & (2,4) \\ (3,0) & (3,1) & (3,2) & (3,3) & (3,4) \\ (4,0) & (4,1) & (4,2) & (4,3) & (4,4) \end{bmatrix},$$

$$G_1 = \begin{bmatrix} (0,0) & (1,2) & (2,4) & (3,1) & (4,3) \\ (1,1) & (2,3) & (3,0) & (4,2) & (0.4) \\ (2,2) & (3,4) & (4,1) & (0,3) & (1,0) \\ (3,3) & (4,0) & (0,2) & (1,4) & (2,1) \\ (4,4) & (0,1) & (1,3) & (2,0) & (3,2) \end{bmatrix},$$

$$G_2 = \begin{bmatrix} (0,0) & (1,4) & (2,3) & (3,2) & (4,1) \\ (1,3) & (2,2) & (3,1) & (4,0) & (0.4) \\ (2,1) & (3,0) & (4,4) & (0,3) & (1,2) \\ (3,4) & (4,3) & (0,2) & (1,1) & (2,0) \\ (4,2) & (0,1) & (1,0) & (2,4) & (3,3) \end{bmatrix}.$$

Note that the non-existence of an efficient set over Z_q does not imply the non-existence of a compatible grid sets. For example, there is no 2×2 efficient matrix over Z_4. However, Since there are 3 mutually orthogonal Latin squares of order 4, we can find a new grid compatible with the standard grid.

Barillot et al. [1] proposed using an efficient set of transforming matrices to construct compatible grids and claimed the following to be a necessary condition:

“An efficient transforming matrix must have a determinant and all the subdeterminants non-null and prime with q”.

It turns out that this condition is sufficient but not necessary for $n > 2$.

Example 2. For $q = 3$, the transforming matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix},$$

sends the standard 3-dimensional grid G_0 (in 3 layers)

$$\begin{bmatrix} (000) & (001) & (002) \\ (010) & (011) & (012) \\ (020) & (021) & (022) \end{bmatrix} \quad \begin{bmatrix} (100) & (101) & (102) \\ (110) & (111) & (112) \\ (120) & (121) & (122) \end{bmatrix}$$

$$\begin{bmatrix} (200) & (201) & (202) \\ (210) & (211) & (212) \\ (220) & (221) & (222) \end{bmatrix}$$

to the grid G_1

$$\begin{bmatrix} (000) & (102) & (201) \\ (121) & (220) & (022) \\ (212) & (011) & (110) \end{bmatrix} \quad \begin{bmatrix} (210) & (012) & (111) \\ (001) & (100) & (202) \\ (122) & (221) & (020) \end{bmatrix}$$

$$\begin{bmatrix} (120) & (222) & (021) \\ (211) & (010) & (112) \\ (002) & (101) & (200) \end{bmatrix}.$$

It is easily checked that although the matrix A contains submatrices with zero determinants. The matrix A is an efficient matrix.

Let q be a prime power. If we change the coefficients from Z_q to a finite field of order q, then it is possible to construct a full set of transforming matrices with algebraic machinery. In the next section, we review some basic facts of finite fields. Details can be found in [6].

3. Finite fields

A finite non-empty set F with binary operation $+, \cdot$ is called the *finite field*, if for all x, y, z in F, the following conditions are satisfied:

(1) $x + y = y + x, x \cdot y = y \cdot x$,
(2) $x + (y + z) = (x + y) + z$,
(3) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
(4) $x \cdot (y + z) = x \cdot y + x \cdot z$, furthermore there is a zero elements 0, identity element e, $-x$ for all x, y^{-1} for all $y \neq 0$ such that,
(5) $x + 0 = x, x + (-x) = 0$,
(6) $y \cdot e = y, y \cdot y^{-1} = e$.

If F is a finite field of size q, then q must be a prime power. On the other hand, we can make a finite set of size q a finite field with the following constructions:

(1) If q is a prime number, then Z_q, the set of all integers modulo q is a finite field.
(2) If $q = p^r$ is a prime power, then the set of all polynomials, with coefficients in Z_p modulo an irreducible polynomial $p(x)$ of degree r, is a finite field.

The structure of a finite field of size q are essentially unique and denoted by $GF(q)$ referred to the *Galois field of order q*.

For any positive integer n, there exists an irreducible polynomial $p(x)$ of degree n over $GF(q)$, such that the set $\{x, x^2, \ldots, x^{q^n-2}, x^{q^n-1} = 1\}$ modulo $p(x)$ is the set of all non-zero polynomials of degree less than n. In other words, the set of all non-zero polynomials of degree less than n forms a cyclic group with respect to polynomial multiplication modulo $p(x)$. The set of all scalars (polynomials of zero degree) is a cyclic subgroup of order $q-1$, of the form $x^m, x^{2m}, \ldots, x^{(q-1)m}$, where $m = q^n - 1/q - 1$. We call such $p(x)$ a *primitive polynomial*. The set of all polynomials over Z_q of degree less than n modulo a primitive polynomial is also a finite field of order q^n.

Let $F = GF(q)$, $p(x) = p_0 + p_1 \cdot x + \cdots + p_{n-1} \cdot x^{n-1} + x^n$, $p_0 \neq 0$ be a primitive polynomial of degree n, and let

$$S = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -p_0 \\ 1 & 0 & 0 & \cdot & 0 & -p_1 \\ 0 & 1 & 0 & 0 & \cdot & -p_2 \\ \cdot & 0 & 1 & 0 & 0 & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & 0 & 1 & -p_{n-1} \end{bmatrix}$$

be the *companion matrix* of $p(x)$. It is easy to verify that S is an invertible matrix. Let

$$E_0 = [1, 0, 0, \ldots, 0]^{\mathrm{t}}, \quad E_1 = [0, 1, 0, \ldots, 0]^{\mathrm{t}}, \cdots E_{n-1} = [0, 0, \ldots, 0, 1]^{\mathrm{t}}.$$

Since $x \cdot (a_0 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1}) (\text{mod } p(x)) = (a_0 \cdot x + a_1 \cdot x^2 + \cdots + a_{n-2} \cdot x^{n-1}) - a_{n-1} \cdot (p_0 + p_1 \cdot x + \cdots + p_{n-1} \cdot x^{n-1})$.

The mapping:

$$\phi\colon a_0 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1} \rightarrow a_0 \cdot E_0 + a_1 \cdot E_1 + \cdots + a_{n-1} \cdot E_{n-1}$$

is an isomorphism (preserving addition and scalar multiplication) sending polynomials of degree less than n to the column n-vectors over $GF(p)$ and satisfies:

$\phi(x^i) = S^i \cdot E_0$,

$\phi(x^i \cdot (a_0 + a_1 \cdot x + \cdots + a_{n-1} \cdot x^{n-1})) = S^i \cdot (a_0 \cdot E_0 + a_1 \cdot E_1 + \cdots + a_{n-1} \cdot E_{n-1})$, for all i.

Example 3. $p(x)=x^2+x+2$ is a primitive polynomial over $GF(3)$ with companion matrix

$$S=\begin{bmatrix}0 & -2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix},$$

$$\{x, x^2=2x+1, x^3=2x+2, x^4=2, x^5=2x, x^6=x+2, x^7=x+1, x^8=1\}$$

is the set of all non-zero elements of $GF(9)$.

The corresponding 2-vectors are:

$$\phi(x)=S\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix},\quad \phi(x^2)=S^2\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix},$$

$$\phi(x^3)=S^3\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix},\quad \phi(x^4)=S^4\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix},$$

$$\phi(x^5)=S^5\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix},\quad \phi(x^6)=S^6\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix},$$

$$\phi(x^7)=S^7\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\quad \phi(x^8)=S^8\cdot\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

Following the same reasoning as in Section 2, we have the following lemma:

Lemma 2. *A set of matrices $\{A_0, A_1, \ldots, A_{k-1}\}$ over the Galois field $GF(q)$, where A_0 is an identity matrix, is an efficient set of transforming matrices and generates a set of compatible $\{G_0, G_1, \ldots, G_{k-1}\}$ if and only if no two column vectors among the matrices $A_0, A_1, \ldots, A_{k-1}$ are parallel to each other.*

4. Construction of a full set of transforming matrices over $GF(q)$

Our discussion in this section is based on the framework established in Section 3.

Let $p(x)$ be a primitive polynomial of degree n over $GF(q)$, S be the companion matrix of $p(x)$, and ϕ be the isomorphism defined in the previous section. Let $m=(q^n-1)/(q-1), k=\lfloor m/n\rfloor$, A_0 be the identity matrix, and let $A_i=(S^n)^i$ for $1\leqslant i\leqslant k-1$. In this section, we prove that the set of matrices $\{A_0, A_1, \ldots, A_{k-1}\}$ is a full set of transforming matrices.

Lemma 3. *The column vectors of A_i are $S^{n\cdot i}\cdot E_0, S^{n\cdot i+1}\cdot E_0, \ldots, S^{n\cdot i+n-1}\cdot E_0$.*

Proof. Straightforward verification.

Lemma 4. *Any two column vectors among the set of transforming matrices $\{A_0, A_1, \ldots, A_{k-1}\}$ are not parallel to each other.*

Proof. The $n \cdot k$ column vectors are $E_0, S \cdot E_0, S^2 \cdot E_0, \ldots, S^{n \cdot k-1} \cdot E_0$. Let $V_i = S^i \cdot E_0, V_j = S^j \cdot E_0$, where $0 \leqslant i < j < m$, be 2 distinct column vectors. If V_i is parallel to V_j then there is a scalar α in $GF(q)$ such that $\alpha \cdot V_i = V_j$. Since ϕ is an isomorphism and $\phi(x^i) = S^i \cdot E_0 = V_i, \phi(x^j) = S^j \cdot E_0 = V_j$, this implies that $(\alpha \cdot x^i) = (x^j)$. But any scalar in $GF(q)$ is of the form $x^{r \cdot m}$ for some $1 \leqslant r \leqslant q-1$ and $0 \leqslant i < j < m$, leading to a contradiction. □

Lemma 5. *The matrices $A_i, i = 0, 1, 2, \ldots,$ are invertible.*

Proof. Since A_0 is the identity matrix and the companion matrix S is invertible, the matrices $A_i = (S^n)^i$ must be invertible for all i. □

Theorem 6. *$\{A_0, A_1, \ldots, A_{k-1}\}$ is a full set of transforming matrices.*

Proof. By Lemmas 4 and 5, A_i is invertible for all i and no column vectors among $\{A_0, A_1, \ldots, A_{k-1}\}$ are parallel to each other. Theorem 6 now follows from Lemma 2. □

Example 4. Let $F = \{0, 1, 2, 3, 4, \}$ be the Galois field of order 5, $p(x) = 2 + x^2 + x^3$ be a primitive polynomial over $GF(5)$.

$$S = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad S^3 = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix}, \quad S^6 = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix}, \ldots,$$

where S is the companion matrix of $p(x)$. The following matrices form a full set of transforming matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 3 \\ 0 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 4 \\ 3 & 1 & 0 \\ 0 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 1 \\ 4 & 4 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 4 \\ 1 & 2 & 4 \\ 3 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 2 \\ 4 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 \\ 2 & 4 & 1 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}.$$

Example 5. Let $F = \{0, 1, z, z+1\}$ be the Galois field of order 4, where $z^2 = z + 1$. Let $p(x) = x^3 + x^2 + x + z$ be a primitive polynomial over F. The companion matrix of $p(x)$ is

$$\begin{bmatrix} 0 & 0 & z \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The polynomials x^i modulo x^3+x^2+x+z, $i=0,1,2,\ldots,21$ are:

$$1,\quad x,\quad x^2,\quad x^3=z+x+x^2,\quad x^4=z+(1+z)x,\quad x^5=zx+(1+z)x^2,$$

$$x^6=1+(1+z)x+x^2,\quad x^7=z+zx^2,\quad x^8=(1+z)+zx^2,\quad x^9=(1+z)+x+zx^2,$$

$$x^{10}=(1+z)+x+(1+z)x^2,\quad x^{11}=1+zx^2,\quad x^{12}=(1+z)+(1+z)x+zx^2,$$

$$x^{13}=(1+z)+x+x^2,\quad x^{14}=z+zx,\quad x^{15}=zx+zx^2,\quad x^{16}=(1+z)+zx,$$

$$x^{17}=(1+z)x+zx^2,\quad x^{18}=(1+z)+zx+x^2,\quad x^{19}=z+zx+(1+z)x^2,$$

$$x^{20}=1+x+x^2,\quad x^{21}=z.$$

The corresponding non-zero 2-vectors are

$$\begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix},\ \begin{bmatrix}z\\1\\1\end{bmatrix},\ \begin{bmatrix}z\\1+z\\0\end{bmatrix},\ \begin{bmatrix}0\\z\\1+z\end{bmatrix},\ \begin{bmatrix}1\\1+z\\1\end{bmatrix},\ \begin{bmatrix}z\\0\\z\end{bmatrix},$$

$$\begin{bmatrix}1+z\\0\\z\end{bmatrix},\ \begin{bmatrix}1+z\\1\\z\end{bmatrix},\ \begin{bmatrix}1+z\\1\\1+z\end{bmatrix},\ \begin{bmatrix}1\\0\\z\end{bmatrix},\ \begin{bmatrix}1+z\\1+z\\z\end{bmatrix},\ \begin{bmatrix}1+z\\1\\1\end{bmatrix},\ \begin{bmatrix}z\\z\\0\end{bmatrix},$$

$$\begin{bmatrix}0\\z\\z\end{bmatrix},\ \begin{bmatrix}1+z\\z\\0\end{bmatrix},\ \begin{bmatrix}0\\1+z\\z\end{bmatrix},\ \begin{bmatrix}1+z\\z\\1\end{bmatrix},\ \begin{bmatrix}z\\z\\1+z\end{bmatrix},\ \begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}z\\0\\0\end{bmatrix}.$$

The full set of 3 by 3 matrices over $GF(4)$ consists of

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}z&z&0\\1&1+z&z\\1&0&1+z\end{bmatrix},\quad \begin{bmatrix}1&z&1+z\\1+z&0&0\\1&z&z\end{bmatrix},$$

$$\begin{bmatrix} 1+z & 1+z & 1 \\ 1 & 1 & 0 \\ z & 1+z & z \end{bmatrix}, \quad \begin{bmatrix} 1+z & 1+z & z \\ 1+z & 1 & z \\ z & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1+z & 0 \\ z & z & 1+z \\ z & 0 & z \end{bmatrix}, \quad \begin{bmatrix} 1+z & z & 1 \\ z & z & 1 \\ 1 & 1+z & 1 \end{bmatrix}.$$

Acknowledgements

We like to thank two referees for many helpful comments and suggestions.

References

[1] E. Barillot, B. Lacroix, D. Cohen, Theoretical analysis of library screening using a N-dimensional pooling strategy, Nucleic Acids Res. 19 (1991) 6241–6247.

[2] P.J. De Jong, C. Aslanidis, J. Alleman, C. Chen, Genome mapping and sequencing, Proceedings of the Cold Spring Harbor Workshop, New York, 1990, p. 48.

[3] S. El-Zanati, M. Plantholt, C. Vanden Eynden, Graph decompositions into generalized cubes, Ars Combin. 49 (1998) 237–247.

[4] E.D. Green, M.V. Olson, Systematic screening of yeast artificial chromosome libraries by use of the polymerase chain reaction, Proc. Nat. Acad. Sci. USA 87 (1990) 1213–1217.

[5] F.K. Hwang, An isomorphic factorization of the complete graph, J. Graph Theory 19 (1995) 333–337.

[6] R.J. McEliece, Finite fields for computer scientists and engineers, Kluwer Academic Publishers, Dordrecht, 1987.