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Bipanconnectivity and edge-fault-tolerant bipancyclicity of hypercubes ☆

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Abstract

A bipartite graph is *bipancyclic* if it contains a cycle of every even length from 4 to $|V(G)|$ inclusive. It has been shown that Q_n is bipancyclic if and only if $n \geqslant 2$. In this paper, we improve this result by showing that every edge of $Q_n - E'$ lies on a cycle of every even length from 4 to $|V(G)|$ inclusive where E' is a subset of $E(Q_n)$ with $|E'| \leqslant n-2$. The result is proved to be optimal. To get this result, we also prove that there exists a path of length l joining any two different vertices x and y of Q_n when $h(x,y) \leqslant l \leqslant |V(G)|-1$ and $l - h(x,y)$ is even where $h(x,y)$ is the Hamming distance between x and y.

Keywords: Hypercube; Hamiltonian; Bipancyclic; Bipanconnected; Interconnection networks

1. Introduction

Network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. There are a lot of mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network which is optimum from all aspects. Fault-tolerance is highly desirable in massive parallel systems that have a relative high probability of failure. A number of fault-tolerant considerations for specific multiprocessor architectures have been discussed.

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation we follow [3]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a,b) \mid (a,b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets V_0 and V_1 such that every edge joins V_1 with V_2. Two vertices a and b are *adjacent* if $(a,b) \in E$. Let E' be a subset of E. We use $G - E'$ to denote the graph with vertex set V and edge set $E - E'$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. We also write the path $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted

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by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *Hamiltonian cycle* is a cycle of length $V(G)$.

The ring embedding problem, which deals with all the possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks [4,5,7]. In general, a graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive [2]. The concept of pancyclicity has been extended to vertex-pancyclicity [6] and edge-pancyclicity [1]. Bipancyclicity is essentially a restriction of the concept of pancyclicity to bipartite graphs whose cycles are necessarily of even length. A bipartite graph is *vertex-bipancyclic* [10] if every vertex lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Similarly, a bipartite graph is *edge-bipancyclic* if every edge lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. Obviously, every edge-bipancyclic graph is vertex-bipancyclic. A bipartite graph G is *k-edge-fault-tolerant edge-bipancyclic* if $G - F$ remains edge-bipancyclic for any $F \subset E(G)$ with $|F| \leqslant k$.

Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ be an n-bit binary strings. For $0 \leqslant k < n$, we use u^k to denote the binary string $v_{n-1}v_{n-2}\ldots v_1v_0$ such that $v_k = 1 - u_k$ and $u_i = v_i$ for all $i \neq k$. The *Hamming weight* of u, denoted by $w(u)$, is the number of is such that $u_i = 1$. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be two n-bit binary strings. The *Hamming distance* $h(u, v)$ between two vertices u and v is the number of different bits in the corresponding strings of both vertices. The *n-dimensional hypercube*, denoted by Q_n, consists of all n-bit binary strings as its vertices and two vertices u and v are adjacent if and only if $h(u, v) = 1$. Thus, Q_n is a bipartite graph with bipartition $\{u \mid w(u)$ is odd$\}$ and $\{u \mid w(u)$ is even$\}$. An edge (u, v) in $E(Q_n)$ is of *dimension i* if $u = v^i$. It is known that $d_{Q_n}(u, v) = h(u, v)$.

Hypercube, Q_n, is one of the most popular interconnection network topologies [9]. In [11], it is proved that Q_n is bipancyclic if and only if $n \geqslant 2$. In [8], it is proved that $Q_n - F$ is Hamiltonian if $F \subset E(Q_n)$ with $|F| \leqslant n - 2$ and $n \geqslant 2$. In this paper, we improve these two results by showing that $Q_n - F$ is edge-bipancyclic where $n \geqslant 2$ and $F \subset E(Q_n)$ with $|F| \leqslant n - 2$. In other words, we prove that Q_n is $(n - 2)$-edge-fault-tolerant edge-bipancyclic if $n \geqslant 2$. In particular, let F be any subset of $E(Q_n)$ with $|F| \leqslant n-2$ and e be any edge of $E(Q_n) - F$. Then, e is in a Hamiltonian cycle of $Q_n - F$.

Let u be any vertex of Q_n and F be $\{(u, u^i) \mid 1 \leqslant i < n\}$. Obviously, $|F| = n - 1$ and $\deg_{Q_n - F}(u) = 1$. Thus, (u, u^0) does not lie on any cycle of $Q_n - F$. Hence, our result is optimal.

To prove Q_n is $(n - 2)$-edge-fault-tolerant edge-bipancyclic if $n \geqslant 2$, we also prove that Q_n is bipanconnected. The concept of bipanconnectivity is derived from the concept of panconnectivity [12]. A graph G is *panconnected* if there exists a path of length l joining any two different vertices x and y with $d(x, y) \leqslant l \leqslant |V(G)| - 1$. It is easy to see that any bipartite graph with at least three vertices is not panconnected. For this reason, we say a bipartite graph is *bipanconnected* if there exists a path of length l joining any two different vertices x and y with $d(x, y) \leqslant l \leqslant |V(G)| - 1$ such that $2|(l - d(x, y))$.

In the following section, we prove that Q_n is bipanconnected. In the final section, we prove that Q_n is $(n - 2)$-edge-fault-tolerant edge-bipancyclic if $n \geqslant 2$.

2. Bipanconnectivity

For convenience, we use Q^0_{n-1} to denote the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 0\}$ and Q^1_{n-1} to denote the subgraph of Q_n induced by $\{x \in V(Q_n) \mid x_0 = 1\}$. Thus, Q^0_{n-1} and Q^1_{n-1} are isomorphic to Q_{n-1}.

Theorem 1. *Q_n is bipanconnected if $n \geqslant 2$.*

Proof. We prove this theorem by induction on n. Obviously, the theorem holds for $n = 2$. Assume that the theorem is true for every integer $2 \leqslant k < n$. Let $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and $v = v_{n-1}v_{n-2}\ldots v_1v_0$ be any two vertices in Q_n.

Case 1: $h(u, v) < n$. Without loss of generality, we may assume that $u_0 = v_0 = 0$. Then $u_0, v_0 \in V(Q^0_{n-1})$. By induction hypothesis, there exists a path of length l joining u and v for any $h(u, v) \leqslant l \leqslant 2^{n-1} - 1$ such that $2|(l - h(u, v))$.

Suppose that $2^{n-1} \leqslant l \leqslant 2^n - 1$ with $2|(l - h(u,v))$. Let P_0 be one of the longest paths of Q^0_{n-1} joining u and v. With the above discussion, $l(P_0) = 2^{n-1} - 1$ if $h(u,v)$ is odd, or $l(P_0) = 2^{n-1} - 2$ if $h(u,v)$ is even. Obviously, $2|(l(P_0) - h(u,v))$. Let $l_1 = l - l(P_0) - 1$. Then l_1 is odd and $1 \leqslant l_1 < 2^{n-1}$. Let (x,y) be any edge on P_0. We can write P_0 as $\langle u, P_1, x, y, P_2, v\rangle$. By definition, x^0 and y^0 are vertices in Q^1_{n-1}. Since $h(x,y) = 1$, $h(x^0, y^0) = 1$. By induction hypothesis, there exists a path P_3 of length l_1 in Q^1_{n-1} joining x^0 and y^0. Thus, $\langle u, P_1, x, x^0, P_3, y^0, y, P_2, v\rangle$ is a path of length l in Q_n joining u and v.

Case 2: $h(u,v) = n$. Without loss of generality, we may assume that $u_0 = 0$ and $v_0 = 1$, i.e., $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Let y be a neighbor of v in Q^1_{n-1} and x be a neighbor of y in Q^0_{n-1}, i.e., $x_0 = 0$, $y_0 = 1$, and $h(x,y) = h(y,v) = 1$. Thus, $h(x,u) = n - 2$.

Suppose that $n \leqslant l \leqslant 2^{n-1} + 1$ with $2|(l - n)$. By induction hypothesis, there exists a path P of length $l - 2$ in Q^0_{n-1} joining u and x. Then $\langle u, P, x, y, v\rangle$ is a path of length l in Q_n joining u and v.

Suppose that $2^{n-1} + 2 \leqslant l \leqslant 2^n - 1$ with $2|(l - n)$. Let P_0 be one of the longest paths of Q^0_{n-1} joining u and x, i.e., $l(P_0) = 2^{n-1} - 1$ or $2^{n-1} - 2$. Obviously, $l(P_0) - h(x,u)$ is even. Let $l_1 = l - l(P_0) - 1$. Then l_1 is odd and $1 \leqslant l_1 \leqslant 2^{n-1}$. By induction hypothesis, there exists a path P of length l_1 in Q^1_{n-1} joining y and v. Thus, $\langle u, P_0, x, y, P, v\rangle$ is a path of length l in Q_n joining u and v.

The theorem is proved. □

3. Edge-fault-tolerant bipancyclic

Lemma 1. *Q_3 is 1-edge-fault-tolerant edge-bipancyclic.*

Proof. It is known that the possible cycle lengths of Q_3 are 4, 6, and 8. Let f be any faulty edge of Q_3. Since Q_3 is edge symmetric, we may assume that $f = (000, 001)$. Let

$$A = \big\{\langle 000 \to 010 \to 110 \to 100 \to 000\rangle, \langle 001 \to 011 \to 111 \to 101 \to 001\rangle, \langle 010 \to 110 \to 111 \to 011 \to 010\rangle, \langle 100 \to 110 \to 111 \to 101 \to 100\rangle\big\};$$

$$B = \big\{\langle 000 \to 010 \to 011 \to 111 \to 110 \to 100 \to 000\rangle, \langle 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 001\rangle, \langle 011 \to 010 \to 110 \to 100 \to 101 \to 111 \to 011\rangle\big\};$$

$$C = \big\{\langle 000 \to 010 \to 011 \to 001 \to 101 \to 111 \to 110 \to 100 \to 000\rangle, \langle 000 \to 010 \to 110 \to 111 \to 011 \to 001 \to 101 \to 100 \to 000\rangle\big\}.$$

Sets A, B, and C contain a set of 4-cycles, 6-cycles, and 8-cycles, respectively, of Q_3. Note that f is not in any cycle in $A \cup B \cup C$. Let e be any edge of Q_3 such that $e \neq f$. We can observe that e lies on one of the 4-cycles, 6-cycles, and 8-cycles in sets A, B, and C, respectively. Thus, the lemma is proved. □

Theorem 2. *Q_n is $(n-2)$-edge-fault-tolerant edge-bipancyclic.*

Proof. We prove this theorem by induction. Obviously, the theorem is true for $n = 2$. By Lemma 1, the theorem is true for $n = 3$. Assume that the theorem is true for all $3 \leqslant k < n$. Let F be any subset of $E(Q_n)$ with $|F| \leqslant n - 2$. For $0 \leqslant i < n$, let F_i denote the set of i-dimensional edges in F. Thus, $\sum_{i=0}^{n-1} |F_i| = |F|$. Without loss of generality, we assume that $|F_0| \geqslant |F_1| \geqslant \cdots \geqslant |F_{n-1}|$. Moreover, we use F^0 to denote the set $E(Q^0_{n-1}) \cap F$ and F^1 to denote the set $E(Q^1_{n-1}) \cap F$. Thus, $F = F^0 \cup F_0 \cup F^1$ and $|F^0| + |F^1| \leqslant n - 3$.

Let e be any edge of $E(Q_n) - F$ and l be any even integer with $4 \leqslant l \leqslant 2^n$. To prove this theorem, we need to construct a cycle of length l containing e.

Case 1: e is not of dimension 0. Without loss of generality, we may assume that $e \in E(Q^0_{n-1})$.

Suppose that $4 \leqslant l \leqslant 2^{n-1}$. Since $|F^0| \leqslant n - 3$, by induction hypothesis there exists a cycle of length l in $Q^0_{n-1} - F$ containing e. In particular, we use C_0 to denote such a cycle of length 2^{n-1}.

Suppose that $2^{n-1} + 2 \leqslant l \leqslant 2^n$. Let $l_1 = l - 2^{n-1}$. Then $2 \leqslant l_1 \leqslant 2^{n-1}$. Since $|E(C_0) - \{e\}| = 2^{n-1} - 1 > 2(n-2) = 2|F|$ for $n \geqslant 3$, there exists an edge (u,v) on C_0 such that $(u,v) \neq e$ and

$\{(u,u^0),(v,v^0),(u^0,v^0)\}\cap F=\emptyset$. We may write C_0 as $\langle u,P_0,v,u\rangle$. Obviously, e lies on P_0, $h(u^0,v^0)=1$, and $\{u^0,v^0\}\subseteq V(Q^1_{n-1})$. Suppose that $l_1=2$. Then $\langle u,P_0,v,v^0,u^0,u\rangle$ is a cycle of length l in Q_n-F. Suppose that $l_1\geqslant 4$. Since $|F^1|\leqslant n-3$, by induction hypothesis there exists a cycle C_1 of length l_1 in $Q^1_{n-1}-F$ containing (u^0,v^0). We can write C_1 as $\langle u^0,v^0,P_1,u^0\rangle$. Then $\langle u,P_0,v,v^0,P_1,u^0,u\rangle$ is a cycle of length l in Q_n-F containing e.

Case 2: e is of dimension 0.

Subcase 2.1: $|F_0|<n-2$. Let $\overline{Q}^0_{n-1}$ denote the subgraph of Q_n induced by $\{x\in V(Q_n)\mid x_1=0\}$ and $\overline{Q}^1_{n-1}$ denote the subgraph of Q_n induced by $\{x\in V(Q_n)\mid x_1=1\}$. Thus, $\overline{Q}^0_{n-1}$ and $\overline{Q}^1_{n-1}$ are isomorphic to Q_{n-1}. Without loss of generality, we may assume that $e\in E(\overline{Q}^0_{n-1})$. We claim that $|E(\overline{Q}^0_{n-1})\cap F| + |E(\overline{Q}^1_{n-1})\cap F|\leqslant n-3$.

Suppose that $|F|\leqslant n-3$. Obviously, $|E(\overline{Q}^0_{n-1})\cap F|+|E(\overline{Q}^1_{n-1})\cap F|\leqslant n-3$. Suppose that $|F|=n-2$. Then, $|F_1|\geqslant 1$. Again, $|E(\overline{Q}^0_{n-1})\cap F|+|E(\overline{Q}^1_{n-1})\cap F|\leqslant n-3$. Accordingly, $|E(\overline{Q}^0_{n-1})\cap F| + |E(\overline{Q}^1_{n-1})\cap F|\leqslant n-3$.

Suppose that $4\leqslant l\leqslant 2^{n-1}$. Since $|E(\overline{Q}^0_{n-1})\cap F|\leqslant n-3$, by induction hypothesis there exists a cycle of length l in $\overline{Q}^0_{n-1}-F$ containing e. In particular, we use C_0 to denote such a cycle of length 2^{n-1}.

Suppose that $2^{n-1}+2\leqslant l\leqslant 2^n$. Let $l_1=l-2^{n-1}$. Then $2\leqslant l_1\leqslant 2^{n-1}$. Since $|E(C_0)-\{e\}|=2^{n-1}-1>2(n-2)=2|F|$ for $n\geqslant 3$, there exists an edge (u,v) on C_0 such that $(u,v)\neq e$ and $\{(u,u^1),(v,v^1),(u^1,v^1)\}\cap F=\emptyset$. We may write C_0 as $\langle u,P_0,v,u\rangle$. Obviously, e is on P_0, $h(u^1,v^1)=1$, and $\{u^1,v^1\}\subseteq V(\overline{Q}^1_{n-1})$. Suppose that $l_1=2$. Then $\langle u,P_0,v,v^1,u^1,u\rangle$ is a cycle of length l in Q_n-F. Suppose that $l_1\geqslant 4$. Since $|E(\overline{Q}^1_{n-1})\cap F|\leqslant n-3$, by induction hypothesis there exists a cycle C_1 of length l_1 in $\overline{Q}^1_{n-1}-F$ containing (u^1,v^1). Write C_1 as $\langle u^1,v^1,P_1,u^1\rangle$. Then $\langle u,P_0,v,v^1,P_1,u^1,u\rangle$ is a cycle of length l in Q_n-F containing e.

Subcase 2.2: $|F_0|=n-2$. Then $E(Q^0_{n-1})\cap F=\emptyset$ and $E(Q^1_{n-1})\cap F=\emptyset$. Assume that $e=(u,v)$ with $u\in V(Q^0_{n-1})$ and $v\in V(Q^1_{n-1})$.

Suppose that $l=4l'$ for $1\leqslant l'\leqslant 2^{n-2}$. Since there are $(n-1)$ neighbors of u in Q^0_{n-1}, there exists a neighbor x of u in Q^0_{n-1} such that $(x,x^0)\notin F$. Obviously, $h(u,x)=h(u^0,v^0)=1$. By Theorem 1, there exists a path P_0 of length $2l'-1$ in Q^0_{n-1} joining u and x and there exists a path P_1 in Q^1_{n-1} of length $2l'-1$ joining x^0 and u^0. Then $\langle u,P_0,x,x^0,P_1,u^0,u\rangle$ is a cycle in Q_n-F containing e of length l.

Suppose that $l=4l'+2$ for $1\leqslant l'\leqslant 2^{n-2}-1$. Let $A=\{w\mid w\in V(Q^0_{n-1})$ and $h(u,w)=2\}$. Obviously, $|A|=C\binom{n-1}{2}\geqslant n-2$. There exists an element x in A such that $(x,x^0)\notin F$. Obviously, $h(u,x)=h(u^0,x^0)=2$. By Theorem 1, there exists a path P_0 in Q^0_{n-1} of length $2l'$ joining u and x and there exists a path P_1 in Q^1_{n-1} of length $2l'$ joining x^0 and u^0. Then $\langle u,P_0,x,x^0,P_1,u^0,u\rangle$ is a cycle in Q_n-F containing e of length l.

The theorem is proved. □

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