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Vortex structure of quantum eigenstates and classical periodic orbits in two-dimensional harmonic oscillators

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Abstract

The connection between the wavefunctions and the classical periodic orbits in a 2D harmonic oscillator is analytically constructed by using the representation of $SU(2)$ coherent states. It is found that the constructed wavefunction generally corresponds to an ensemble of classical trajectories and its localization is extremely efficient. With the constructed wavefunction, we also analyse the property of the probability current density associated with the classical periodic orbit. The appearance of vortex structure in the quantum flow is clearly found to arise from the wave interference.

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1. Introduction

The classical–quantum interface has been a central issue in quantum theory and has remained an intriguing subject for nearly a century [1–5]. Although the elliptical stationary states in the Kepler problem have been developed [6, 7], it is a usually elusive question to build high-order wavefunctions that mimic the familiar classical periodic orbits for two-dimensional (2D) and three-dimensional (3D) quantum systems. Recently, the progress in modern semiconductor technology has made it possible to design nanostructure devices with quantum ballistic properties [8, 9]. One of the foremost results is that there are some striking phenomena in open quantum ballistic cavities associated with the wavefunctions in terms of classical periodic orbits [10–12]. Therefore, to construct the connection between the wavefunctions and classical periodic trajectories is of great importance for understanding quantum–classical correspondence as well as quantum transport in mesoscopic systems.

The quantum wavefunction intrinsically contains more information than mere probabilities. For example, the quantum probability flow in the hydrodynamic interpretation

[13] can be easily connected with some relevant macroscopic quantum phenomena, such as superconductivity [14] and Bose–Einstein condensation [15], through the presence of vortices. The theory of quantum vortices, first pointed out by Dirac [16], has been shown to play an important role in quantum mechanics. In recent years, there has been growing attention to the occurrence of quantum vortices in the current flow of mesoscopic structures especially in semiconductor open quantum dots [17–19]. However, the physical relevance of quantum vortices is not completely clear so far.

In this paper, we use the representation of the $SU(2)$ coherent state to analytically make a connection between the wavefunctions and the classical trajectories in a 2D harmonic oscillator with commensurate frequencies. The prominent feature of the wavefunction corresponding to classical periodic orbits is the appearance of vortex structures in the flow of probability current density. The formation of vortex structures is clearly demonstrated to be the result of quantum interference effects. Moreover, we modify the analytical wavefunction to obtain a partially coherent state. With the partially coherent state, we find that the localization of the wave pattern is very efficient; only a few degenerate eigenfunctions are already sufficient to localize wave patterns on high-order periodic orbits.

2. Wavefunctions associated with Lissajous orbits

The Hamiltonian for a 2D quantum harmonic oscillator is generally given by

$$H = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y} + \frac{m_x \omega_x^2 x^2}{2} + \frac{m_y \omega_y^2 y^2}{2}. \quad (1)$$

The eigenfunctions of equation (1) can be expressed as

$$\Phi_{m,n}(x, y; X, Y) = \frac{1}{\sqrt{2^{m+n-1} \pi m! n!}} \frac{1}{\sqrt{XY}} H_m \left(\frac{\sqrt{2}x}{X} \right) H_n \left(\frac{\sqrt{2}y}{Y} \right) \exp \left[- \left(\frac{x}{X} \right)^2 - \left(\frac{y}{Y} \right)^2 \right] \quad (2)$$

where $X = \sqrt{2\hbar/(m_x \omega_x)}$ and $Y = \sqrt{2\hbar/(m_y \omega_y)}$. The eigenvalues associated with the eigenfunctions $\Phi_{m,n}(x, y; X, Y)$ are given by

$$E_{m,n} = \left(m + \frac{1}{2} \right) \hbar \omega_x + \left(n + \frac{1}{2} \right) \hbar \omega_y. \quad (3)$$

As is well known, the classical trajectories for the 2D harmonic oscillators with commensurate frequencies are periodic orbits, called Lissajous figures [20]. According to Bohr's correspondence principle, the classical limit of a quantum system should be achieved when the quantum numbers go to infinity. However, the conventional eigenstates $\Phi_{m,n}(x, y; X, Y)$ do not manifest the characteristics of classical periodic orbits even in the correspondence limit of large quantum numbers. Recently, the wavefunctions associated with the classical elliptical trajectories in a 2D isotropic harmonic oscillator have been analytically constructed by using the representation of $SU(2)$ coherent states [21, 22]. Mathematically, the $SU(2)$ coherent states are a superposition of degenerate eigenstates. Here we derive the $SU(2)$ coherent states for the 2D harmonic oscillators with commensurate frequencies to construct the wavefunctions associated with the Lissajous figures.

Consider a 2D harmonic oscillator with frequencies in the ratio $\omega_x : \omega_y = q : p$, where p and q are integers, the eigenvalues can be rewritten in the form

$$E_{m,n} = \left[\left(m + \frac{1}{2} \right) q + \left(n + \frac{1}{2} \right) p \right] \hbar \omega \quad (4)$$

where ω is the common factor of the frequencies ω_x and ω_y . For $q : p$ quantum harmonic oscillators, it is explicit that a family of the eigenstates $\Phi_{pK,q(N-K)}(x, y; X, Y)$ with $K = 0, 1, 2, \dots, N$ are degenerate for a positive integer N and the eigenvalue of these eigenstates is given by $E_N = [pqN + (p + q)/2]\hbar\omega$. As in the Schwinger representation of the $SU(2)$ algebra, the coherent state for $q : p$ quantum harmonic oscillators is given by

$$\Psi_N^{p,q}(x, y; X, Y, \tau) = \frac{1}{(1 + |\tau|^2)^{N/2}} \sum_{K=0}^N \binom{N}{K}^{1/2} \tau^K \Phi_{pK,q(N-K)}(x, y; X, Y) \quad (5)$$

where the parameter τ is, in general, complex and $|\tau|^2$ is approximately the ratio of the mean energies in the x - and y -axes. With the $SU(2)$ coherent state in equation (5), the mean energies in the x - and y -axes are derived to be

$$E_{N,x} = \left[pqN \left(\frac{|\tau|^2}{1 + |\tau|^2} \right) + \frac{q}{2} \right] \hbar\omega \quad E_{N,y} = \left[pqN \left(\frac{1}{1 + |\tau|^2} \right) + \frac{p}{2} \right] \hbar\omega. \quad (6)$$

It can be found that $E_N = E_{N,x} + E_{N,y}$ and the ratio $E_{N,x}/E_{N,y}$ approaches $|\tau|^2$ for $N \gg 1$.

For making a connection with the classical periodic orbits, it is convenient to express the parameter τ as the polar representation, i.e. $\tau = A \exp(i\phi)$. Provided that p and q are relatively prime, the wave patterns of $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ are found to be localized on a single, nonrepeated orbit that is related to the Lissajous figure by

$$x(t) = \sqrt{2\langle x^2 \rangle} \cos \left(q\omega t - \frac{\phi}{p} \right) \quad y(t) = \sqrt{2\langle y^2 \rangle} \cos(p\omega t) \quad (7)$$

where

$$\langle x^2 \rangle = \left(\frac{A^2}{1 + A^2} pN + \frac{1}{2} \right) \frac{X^2}{2} \quad \langle y^2 \rangle = \left(\frac{1}{1 + A^2} qN + \frac{1}{2} \right) \frac{Y^2}{2}. \quad (8)$$

Note that $\langle x^2 \rangle$ and $\langle y^2 \rangle$ are, respectively, the expectation values of x^2 and y^2 for the coherent state $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})$. Figure 1 shows the calculated results of the $SU(2)$ coherent states for the frequency ratios of 1:1, 2:1, 3:2 and 4:3 with $A = 1$, $\phi = \pi/2$, $X = Y$ and $N = 20$. For comparison, the corresponding Lissajous figures are shown in figure 2. It can be seen that the distributions of $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ are in good agreement with the classical periodic orbits. Moreover, the behaviour of $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ illustrates geometrically Bohr's correspondence principle: the velocity of the classical particle is at a minimum at the apogees of the motion, and therefore the probability density has a peak at these points. Although classical and quantal probability densities become indistinguishable in the large quantum number limit, interference effects, which are characteristic of quantum mechanics, do not have any analogy in classical mechanics.

On the other hand, if p and q have a common factor M , the wave patterns of $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ are found to correspond to an ensemble of classical periodic orbits. The total number of the corresponding periodic orbits is M and their trajectories are given by

$$x_k(t) = \sqrt{2\langle x^2 \rangle} \cos \left(q\omega t - \frac{\phi_k}{p} \right) \quad y(t) = \sqrt{2\langle y^2 \rangle} \cos(p\omega t) \quad (9)$$

where $\phi_k = \phi + 2\pi k$ and $k = 0, 1, 2, \dots, M - 1$. For demonstration, figure 3 shows the wave pattern of the $SU(2)$ coherent state for $q : p = 3 : 3$ with $A = 1$, $\phi = \pi/2$, $X = Y$ and $N = 40$. It can be seen that the $SU(2)$ coherent state for $q : p = 3 : 3$ is associated with three classical periodic orbits (two elliptical orbits and one circular orbit). In other words, the $SU(2)$ coherent state generally corresponds to an ensemble of classical trajectories, not

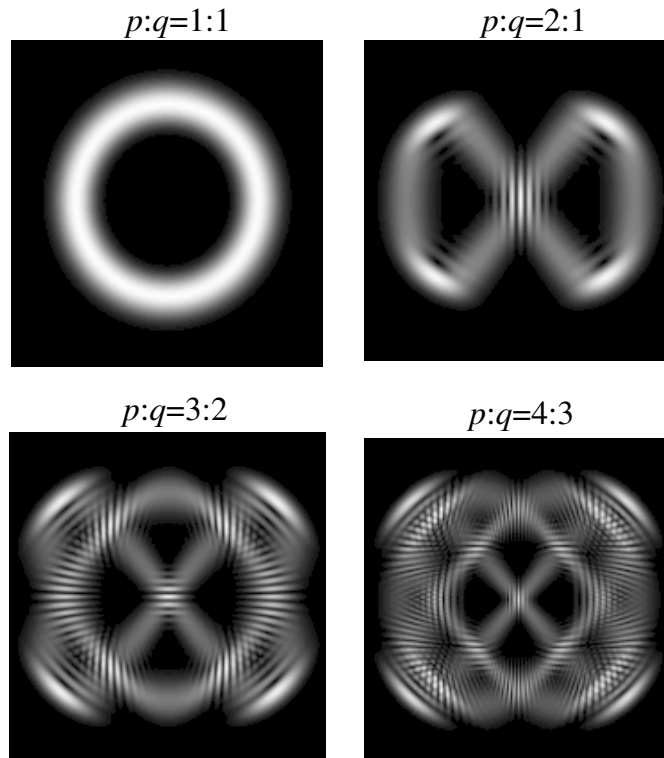


Figure 1. The calculated results of the $SU(2)$ coherent states $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ for the frequency ratios of 1:1, 2:1, 3:2 and 4:3 with $A = 1$, $\phi = \pi/2$, $X = Y$ and $N = 20$.

a single classical trajectory except that p and q are relatively prime. Even so, it must be emphasized that the corresponding classical orbits are not mutually independent but constitute an indivisible pattern through quantum interference effects. As described in the following section, quantum interference effects lead to the formation of vortex structures in the quantum flow.

Although the coherent state $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})$ comprises $N + 1$ degenerate eigenstates, the number of dominant eigenstates for wave localization is rather small for high-order states. To manifest the efficiency of wave localization, a partially coherent state is defined as

$$\Psi_{N,M}^{p,q}(x, y; X, Y, A e^{i\phi}) = \left[\sum_{K=J}^{N-J} \binom{N}{K} A^2 \right]^{-1/2} \sum_{K=J}^{N-J} \binom{N}{K}^{1/2} (A e^{i\phi})^K \Phi_{pK,q(N-k)}(x, y; X, Y) \quad (10)$$

where the index $M = N - 2J + 1$ represents the number of eigenstates used in the state $\Psi_{N,M}^{p,q}(x, y; X, Y, A e^{i\phi})$. Figure 4 displays the wave patterns of $|\Psi_{N,M}^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ with $M = 5$ and $N = 20$ corresponding to the classical periodic orbits shown in figure 2. It is clear that only five eigenstates are already sufficient to localize the wave patterns on the classical trajectories. This property is in good agreement with the recent finding that the wavefunctions seen in open quantum dots may be interpreted as arising from single eigenstates of closed billiards [23].

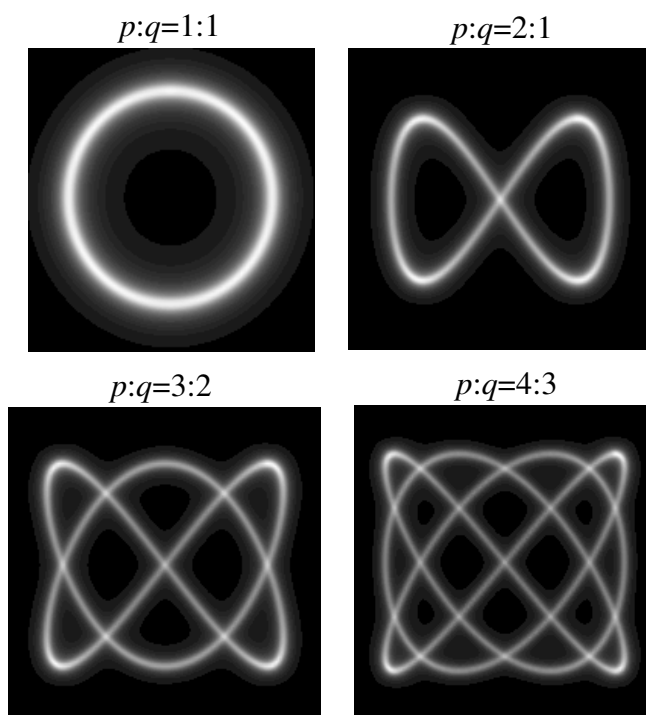


Figure 2. The classical Lissajous figures for the frequency ratios of 1:1, 2:1, 3:2 and 4:3 corresponding to the wave patterns displayed in figure 1.

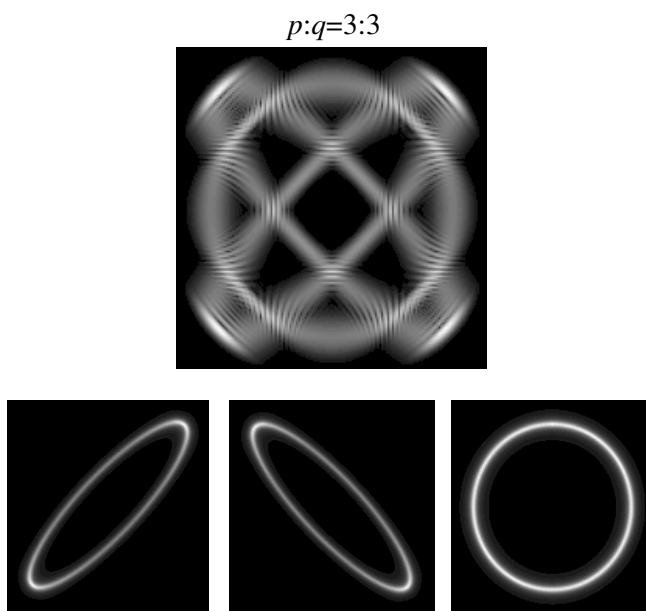


Figure 3. The calculated wave pattern for the $SU(2)$ coherent state $|\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ for $q : p = 3 : 3$ with $A = 1$, $\phi = \pi/2$, $X = Y$ and $N = 40$. For correspondence, the classical periodic orbits are shown below the wave pattern.

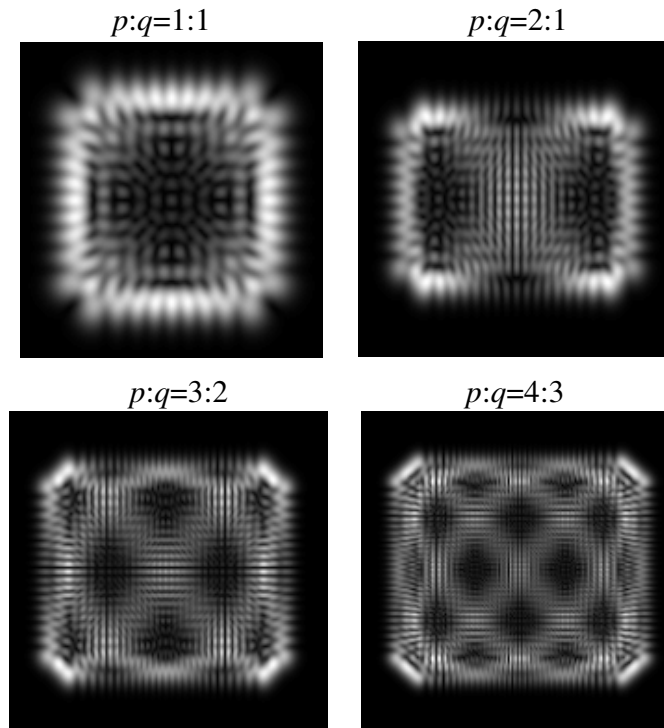


Figure 4. The wave patterns of $|\Psi_{N,M}^{p,q}(x, y; X, Y, A e^{i\phi})|^2$ with $M = 5$ and $N = 20$ corresponding to the classical periodic orbits displayed in figure 2 for showing the efficiency of wave localization.

3. Quantum vortices

Vortices are responsible for many observable phenomena known mainly to occur in macroscopic quantum systems, for example, superconductors or superfluids [24–26]. The order parameter equation in the study of these phenomena is the Ginzburg–Landau or Gross–Pitaevskii equation. However, the analysis of the wavefunction is greatly complicated because of the nonlinear character of the modelling equation. As pointed out already by Dirac [16], the vortices arising from the singular points of the quantum phase can manifest themselves in the linear Schrödinger equation. Recent works also show that the vortex problems play an important role in quantum mechanics [27–30]. Therefore, it is of great interest to analyse the vortex behaviour for the present wavefunction.

For analysing the property of phase singularities associated with the classical periodic orbits, it will be convenient to write $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})$ in polar form:

$$\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi}) = \sqrt{\rho(x, y)} \exp[i\chi(x, y)] \quad (11)$$

where

$$\rho(x, y) = |\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})|^2 \quad (12)$$

and

$$\chi(x, y) = \tan^{-1} \left\{ \frac{\text{Im} [\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})]}{\text{Re} [\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})]} \right\}. \quad (13)$$

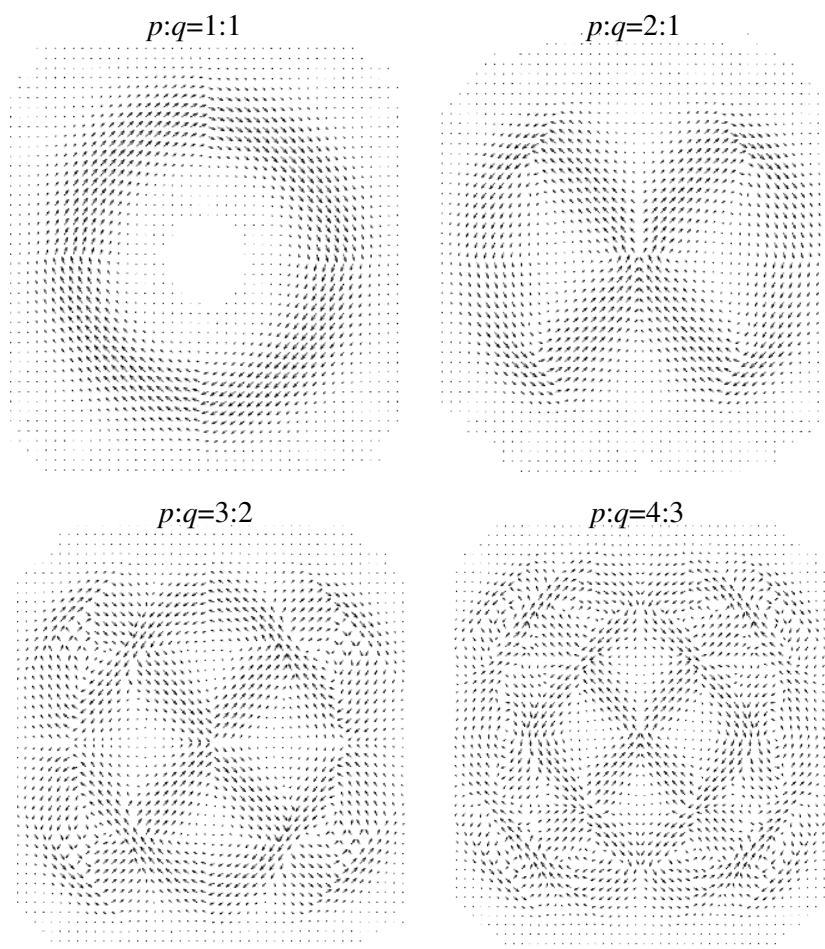


Figure 5. The calculated results for the probability current density corresponding to the wave patterns shown in figure 1.

In terms of the probability density $\rho(x, y)$ and the phase distribution $\chi(x, y)$, the probability current density is analytically given by [31]

$$\vec{J}(x, y) = \frac{\hbar}{m} \rho(x, y) \nabla \chi(x, y). \quad (14)$$

Note that when $\phi = \pm n\pi$ and n is an integer, the coherent state $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})$ is a standing wave and has no vortices. In other words, the vortices regularly appear in the coherent state $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi})$ with the exception of $\phi = \pm n\pi$. Hereafter we focus on the case of $\phi \neq \pm n\pi$, unless otherwise specified.

Using equations (11)–(14), the probability current densities have been calculated. Figure 5 shows the calculated results for the wavefunctions displayed in figure 1. It can be seen that quantum interference effects lead to the vortex structure in the quantum probability flow. Such vortices are topological singularities in the sense that on these points the phase of the wavefunction is not defined. In the case of $p = q = 1$ the quantum flow forms a

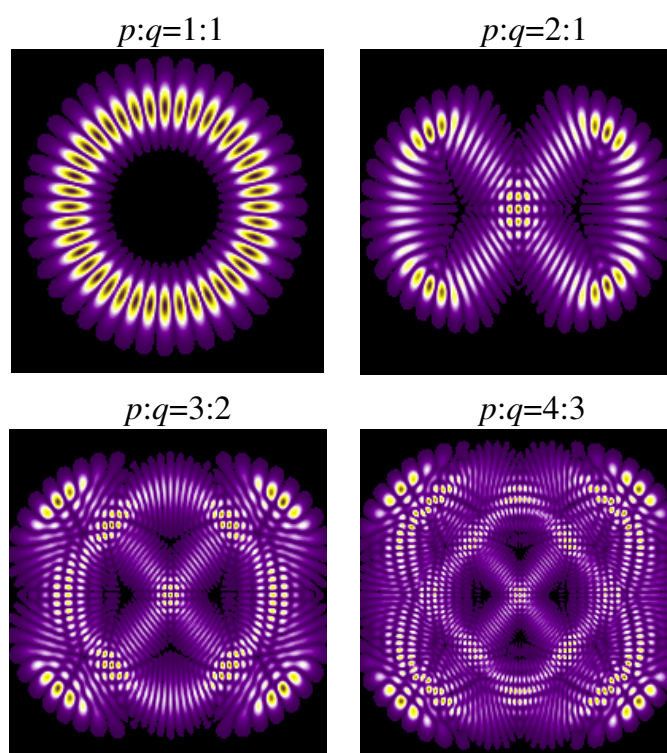


Figure 6. The standing wave patterns corresponding to the coherent states shown in figure 1.

single vortex, whereas increasing the indices of p and q can result in a complex structure with multiple vortices. Among the multiple vortices the tunnelling process is expected to exist in the quantum probability flow for establishing a complete periodic orbit. Note that the present vortices are similar in mathematical nature to those earlier discovered by Onsager and Feynman in superfluid helium [32, 33] and later found in many other quantum systems such as superconductors, plasma and chemical reactions [34–36]. Even so, an intriguing point is that the present vortices are not only related to classical periodic orbits but also entirely due to quantum interference. It is worthwhile mentioning that the standing wave pattern can be described by the superposition of two travelling waves with counterclockwise directions, i.e. $\Psi_N^{p,q}(x, y; X, Y, A e^{i\phi}) \pm \Psi_N^{p,q}(x, y; X, Y, A e^{-i\phi})$. Figure 6 displays the standing wave patterns corresponding to the coherent states shown in figure 1.

Finally, it should be remarked that the parameters in equation (7), which gives a connection between the wave pattern of a coherent state and classical Lissajous figures, are deduced from numerical calculations. It is difficult for us to give an analytical derivation for the expression of equation (7) at present. We have analysed a different physical system [37], namely a square billiard, and some of the results are quite similar to the present system. The relationship between the wavefunctions and the classical trajectories can be developed more manifestly in a square billiard. An understanding of [37] should be helpful in gaining some appreciation for the content of equation (7).

4. Conclusions

We have used the representation of the $SU(2)$ coherent state to make a connection between the wavefunctions and the classical trajectories in a 2D harmonic oscillator with commensurate frequencies. We modify the constructed coherent state to investigate the efficiency of wave localization. It is found that only a few degenerate eigenstates are already sufficient to localize wave patterns on classical periodic orbits. Also, the property of phase singularities in the quantum probability current has been analysed. The formation of vortex structures in the probability current density is clearly demonstrated to be the result of quantum interference effects.

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