Finite Blaschke products of contractions

Hwa-Long Gau [a,1], Pei Yuan Wu [b,*,2]

[a]*Department of Mathematics, National Central University, Chung-Li 32054, Taiwan*
[b]*Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan*

Abstract

Let A be a contraction on Hilbert space H and ϕ a finite Blaschke product. In this paper, we consider the problem when the norm of $\phi(A)$ is equal to 1. We show that (1) $\|\phi(A)\| = 1$ if and only if $\|A^k\| = 1$, where k is the number of zeros of ϕ counting multiplicity, and (2) if H is finite-dimensional and A has no eigenvalue of modulus 1, then the largest integer l for which $\|A^l\| = 1$ is at least $m/(n-m)$, where $n = \dim H$ and $m = \dim \ker(I - A^*A)$, and, moreover, $l = n - 1$ if and only if $m = n - 1$.
© 2003 Elsevier Science Inc. All rights reserved.

AMS classification: 47A30; 15A60; 47B35

Keywords: Contraction; Blaschke product; Compression of the shift; Toeplitz operator; Hankel operator

Let A be a contraction ($\|A\| \leqslant 1$) on a Hilbert space and ϕ a Blaschke product with k zeros $\lambda_1, \ldots, \lambda_k$ ($|\lambda_j| < 1$) counting multiplicity:

$$\phi(z) = \prod_{j=1}^{k} \frac{z - \lambda_j}{1 - \bar{\lambda}_j z}.$$

Then $\phi(A) = \prod_{j=1}^{k}(A - \lambda_j I)(I - \bar{\lambda}_j A)^{-1}$ is also a contraction. In this paper, we consider the problem when the norm of $\phi(A)$ is equal to 1. It is not difficult to

---

verify that $\|A\| = 1$ if and only if $\|(A - \lambda I)(I - \bar{\lambda}A)^{-1}\| = 1$ for some λ, $|\lambda| < 1$. In Section 1, we show that, more generally, $\phi(A)$ has norm 1 if and only if A^k does. Hence our problem can be simplified by assuming that all zeros of ϕ are 0. Then in Section 2, we define, for any contraction A, its *norm-one index* k_A as the supremum of nonnegative integers k for which $\|A^k\| = 1$. We have $\|A^k\| = 1$ for all $k \leqslant k_A$ while $\|A^k\| < 1$ for $k > k_A$. Note that, in general, $0 \leqslant k_A \leqslant \infty$. To obtain some upper and lower bounds for k_A, we exclude the trivial cases by assuming that A acts on a finite-dimensional space H and has no eigenvalue of modulus 1. In Theorems 2.2 and 2.4, we show that if $n = \dim H$ and $m = \dim \ker(I - A^*A)$, then $m/(n - m) \leqslant k_A \leqslant m$. Finally, in Section 3, we restrict ourselves to the class $\mathscr{S}_n$ of contractions A on an n-dimensional space which have no unit eigenvalue and satisfy $\mathrm{rank}(I - A^*A) = 1$. We show that such contractions can be characterized via their norm-one index: A is in $\mathscr{S}_n$ if and only if $k_A = n - 1$ (Theorem 3.1). (This was also obtained independently by Y.-C. Li and M.-H. Shih.) Contractions in $\mathscr{S}_n$ and their infinite-dimensional analogues have been investigated since the 1960s. They serve as the building blocks in the Jordan form for C_0 contractions as developed by Sz.-Nagy and Foiaş [1,13]. The study of the norms of functions of them via their functional model is connected naturally with that of certain properties of Toeplitz and Hankel operators. This we explore in Theorem 3.2 at the end of this paper.

The study of the norms of analytic functions of a contraction dates back to the von Neumann inequality of 1951 that a certain analytic function of a contraction is again a contraction (cf. [8, Problem 229]), which was further refined by Fan to an analogous assertion for strict contractions (cf. [5, Theorem 1]). These results somehow motivate our present undertaking.

1. Finite Blaschke products of contractions

The main theorem of this section reduces our study of the norms of finite Blaschke products of a contraction A to that of powers of A.

Theorem 1.1. *Let A be a contraction on Hilbert space H and let ϕ be a Blaschke product with k zeros counting multiplicity. Then*

(a) $\dim \ker(I - \phi(A)^*\phi(A)) = \dim \ker(I - A^{k*}A^k)$, *and*
(b) $\|\phi(A)\| = 1$ *if and only if* $\|A^k\| = 1$.

Note that if A is a contraction on a finite-dimensional space, then a vector x is in $\ker(I - A^*A)$ if and only if $\|Ax\| = \|x\|$. Thus, in the preceding theorem, (b) follows from (a) for finite-dimensional H. This is not the case if H is infinite-dimensional.

We start with the following lemma for the proof of Theorem 1.1(a).

Lemma 1.2. *Let A be a contraction and let $\phi_j(z) = (z - \lambda_j)/(1 - \bar{\lambda}_j z)$, where $|\lambda_j| < 1$, for $j = 1, \ldots, k$. If x is a vector satisfying $\|A\phi_2(A)\cdots\phi_k(A)x\| = \|x\|$, then it also satisfies $\|\phi_1(A)\phi_2(A)\cdots\phi_k(A)(I - \bar{\lambda}_1 A)x\| = \|(I - \bar{\lambda}_1 A)x\|$.*

Proof. Let $B_1 = A\phi_2(A)\cdots\phi_k(A)$, $B_j = \phi_j(A)\cdots\phi_k(A)$ for $j = 2, \ldots, k$, and $B_{k+1} = I$. From

$$\|x\| = \|B_1 x\| \leqslant \|B_j x\| \leqslant \|B_{j+1} x\| \leqslant \|x\|, \quad j = 2, \ldots, k,$$

we deduce that

$$\|\phi_j(A)B_{j+1}x\| = \|B_j x\| = \|B_{j+1}x\|.$$

This is equivalent to $B_{j+1}x \in \ker(I - \phi_j(A)^*\phi_j(A))$ or $B_{j+1}x = \phi_j(A)^*\phi_j(A) \times B_{j+1}x$. Now

$$
\begin{aligned}
&\|\phi_1(A)\cdots\phi_k(A)(I - \bar{\lambda}_1 A)x\|^2 \\
&= \|(A - \lambda_1 I)B_2 x\|^2 \\
&= \|AB_2 x\|^2 - 2\mathrm{Re}(\bar{\lambda}_1\langle AB_2 x, B_2 x\rangle) + |\lambda_1|^2\|B_2 x\|^2 \\
&= \|B_1 x\|^2 - 2\mathrm{Re}(\bar{\lambda}_1\langle AB_2 x, B_2 x\rangle) + |\lambda_1|^2\|x\|^2 \\
&= \|x\|^2 - 2\mathrm{Re}(\bar{\lambda}_1\langle AB_2 x, B_2 x\rangle) + |\lambda_1|^2\|Ax\|^2,
\end{aligned}
\tag{1}
$$

where the last equality follows from

$$\|x\| = \|B_1 x\| = \|B_2 Ax\| \leqslant \|Ax\| \leqslant \|x\|.$$

Since

$$
\begin{aligned}
\langle AB_2 x, B_2 x\rangle &= \langle A\phi_2(A)B_3 x, \phi_2(A)B_3 x\rangle \\
&= \langle AB_3 x, \phi_2(A)^*\phi_2(A)B_3 x\rangle \\
&= \langle AB_3 x, B_3 x\rangle \\
&= \cdots\cdots \\
&= \langle AB_{k+1}x, B_{k+1}x\rangle \\
&= \langle Ax, x\rangle,
\end{aligned}
$$

we obtain from (1) that

$$
\begin{aligned}
&\|\phi_1(A)\cdots\phi_k(A)(I - \bar{\lambda}_1 A)x\|^2 \\
&= \|x\|^2 - 2\mathrm{Re}(\bar{\lambda}_1\langle Ax, x\rangle) + |\lambda_1|^2\|Ax\|^2 \\
&= \|(I - \bar{\lambda}_1 A)x\|^2,
\end{aligned}
$$

completing the proof. $\quad\square$

The next lemma is the asymptotic version of the fact that, for a contraction A, $\|Ax\| = \|x\|$ if and only if x is in $\ker(I - A^*A)$.

Lemma 1.3. *Let A be a contraction and let $\{x_n\}$ be a sequence of vectors such that $\|x_n\|$ converges to 1. Then $\|Ax_n\|$ converges to 1 if and only if $\|(I - A^*A)x_n\|$ converges to 0.*

Proof. Since

$$\|(I - A^*A)x_n\|^2 = \|x_n\|^2 - 2\mathrm{Re}\langle x_n, A^*Ax_n\rangle + \|A^*Ax_n\|^2$$
$$\leqslant 2\|x_n\|^2 - 2\|Ax_n\|^2,$$

the assumptions that $\|x_n\| \to 1$ and $\|Ax_n\| \to 1$ imply that $\|(I - A^*A)x_n\| \to 0$. For the converse, if $\|x_n\| \to 1$ and $\|(I - A^*A)x_n\| \to 0$, then

$$|\langle (I - A^*A)x_n, x_n\rangle| \leqslant \|(I - A^*A)x_n\| \cdot \|x_n\| \to 0.$$

Hence

$$\|Ax_n\|^2 = \|x_n\|^2 - \langle (I - A^*A)x_n, x_n\rangle \to 1$$

as desired. $\square$

The following lemma is the essential step in the proof of Theorem 1.1(b).

Lemma 1.4. *Let A be a contraction and let $\phi_j(z) = (z - \lambda_j)/(1 - \bar{\lambda}_j z)$ with $|\lambda_j| < 1$ for $j = 1, \ldots, k$. Then $\|A\phi_2(A) \cdots \phi_k(A)\| = 1$ if and only if $\|\phi_1(A)\phi_2(A) \cdots \phi_k(A)\| = 1$.*

Proof. Let $B = \phi_2(A) \cdots \phi_k(A)$ and assume that $\|AB\| = 1$. Then there is a sequence of unit vectors $\{x_n\}$ such that $\|ABx_n\| \to 1$. Let $y_n = (I - \bar{\lambda}_1 A)x_n$. We are to show that $\|\phi_1(A)By_n\|^2 - \|y_n\|^2 \to 0$. Indeed, we have

$$\begin{aligned}
\|\phi_1(A)By_n\|^2 &- \|y_n\|^2 \\
&= \|(A - \lambda_1 I)Bx_n\|^2 - \|(I - \bar{\lambda}_1 A)x_n\|^2 \\
&= \|ABx_n\|^2 - 2\mathrm{Re}(\bar{\lambda}_1\langle ABx_n, Bx_n\rangle) + |\lambda_1|^2\|Bx_n\|^2 \\
&\quad - \|x_n\|^2 + 2\mathrm{Re}(\lambda_1\langle x_n, Ax_n\rangle) - |\lambda_1|^2\|Ax_n\|^2 \\
&= (\|ABx_n\|^2 - \|x_n\|^2) - 2\mathrm{Re}[\lambda_1(\langle Bx_n, ABx_n\rangle - \langle x_n, Ax_n\rangle)] \\
&\quad + |\lambda_1|^2(\|Bx_n\|^2 - \|Ax_n\|^2).
\end{aligned} \tag{2}$$

Since $\|x_n\| = 1$ for all n and $\|ABx_n\| \to 1$, we infer from

$$\|ABx_n\| \leqslant \|Bx_n\| \leqslant 1 \quad \text{and} \quad \|ABx_n\| = \|BAx_n\| \leqslant \|Ax_n\| \leqslant 1,$$

that $\|Bx_n\| \to 1$ and $\|Ax_n\| \to 1$. This together with $\|BAx_n\| \to 1$ implies, by Lemma 1.3, that $\|(I - B^*B)Ax_n\| \to 0$. Hence

$$\mathrm{Re}[\lambda_1(\langle Bx_n, ABx_n \rangle - \langle x_n, Ax_n \rangle)]$$
$$\leqslant |\lambda_1||\langle x_n, B^*ABx_n - Ax_n \rangle|$$
$$\leqslant |\lambda_1| \cdot \|x_n\| \cdot \|(I - B^*B)Ax_n\| \to 0.$$

Plugging all these limits into (2) yields $\|\phi_1(A)By_n\|^2 - \|y_n\|^2 \to 0$. Since

$$\|y_n\| = \|(I - \bar{\lambda}_1 A)x_n\| \geqslant \frac{1}{\|(I - \bar{\lambda}_1 A)^{-1}\|}$$

for all n, we obtain $\|\phi_1(A)B(y_n/\|y_n\|)\| \to 1$ and so $\|\phi_1(A)B\| = 1$.

To prove the converse, let $\psi(z) = (z + \lambda_1)/(1 + \bar{\lambda}_1 z)$ be the inverse function of ϕ_1 and let $\psi_j = \phi_j \circ \psi$ for $2 \leqslant j \leqslant k$. If $A_1 = \phi_1(A)$, then $\phi_j(A) = \psi_j(A_1)$ for every j, $2 \leqslant j \leqslant k$. Our assumption $\|\phi_1(A)\phi_2(A) \cdots \phi_k(A)\| = 1$ is the same as $\|A_1\psi_2(A_1) \cdots \psi_k(A_1)\| = 1$. By the implication proved in last paragraph, we obtain $\|\psi(A_1)\psi_2(A_1) \cdots \psi_k(A_1)\| = 1$ or $\|A\phi_2(A) \cdots \phi_k(A)\| = 1$ as desired. $\square$

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. To prove (a), let $\phi(z) = \phi_1(z) \cdots \phi_k(z)$, where $\phi_j(z) = (z - \lambda_j)/(1 - \bar{\lambda}_j z)$ with $|\lambda_j| < 1$ for $j = 1, \ldots, k$, and, for any x in $\ker(I - A^{k*}A^k)$, let $y_1 = (I - \bar{\lambda}_1 A)x$. Since $\|A^k x\| = \|x\|$, Lemma 1.2 implies that $\|\phi_1(A)A^{k-1}y_1\| = \|y_1\|$. Letting $y_j = (I - \bar{\lambda}_j A)y_{j-1}$, $j = 2, \ldots, k$, we apply Lemma 1.2 repeatedly to obtain $\|\phi_1(A) \cdots \phi_k(A)y_k\| = \|y_k\|$. This is the same as $y_k = (I - \bar{\lambda}_1 A) \cdots (I - \bar{\lambda}_k A)x \in \ker(I - \phi(A)^*\phi(A))$. Thus the invertible operator $X \equiv (I - \bar{\lambda}_1 A) \cdots (I - \bar{\lambda}_k A)$ maps $\ker(I - A^{k*}A^k)$ to $\ker(I - \phi(A)^*\phi(A))$, and therefore dim $\ker(I - A^{k*}A^k) \leqslant$ dim $\ker(I - \phi(A)^*\phi(A))$.

For the reverse inequality, let $\psi(z) = (z + \lambda_1)/(1 + \bar{\lambda}_1 z)$ be the inverse function of ϕ_1 and let $\psi_j = \phi_j \circ \psi$ for $2 \leqslant j \leqslant k$. If $A_1 = \phi_1(A)$, then $\phi_j(A) = \psi_j(A_1)$, $2 \leqslant j \leqslant k$. For x in $\ker(I - \phi(A)^*\phi(A))$, we have

$$\|A_1\psi_2(A_1) \cdots \psi_k(A_1)x\| = \|\phi(A)x\| = \|x\|.$$

Applying Lemma 1.2, we obtain $\|\psi(A_1)\psi_2(A_1) \cdots \psi_k(A_1)(I + \bar{\lambda}_1 A_1)x\| = \|(I + \bar{\lambda}_1 A_1)x\|$ or $\|A\phi_2(A) \cdots \phi_k(A)y_1\| = \|y_1\|$, where $y_1 = (I + \bar{\lambda}_1 A_1)x$. Applying Lemma 1.2 inductively yields $\|A^k y_k\| = \|y_k\|$ or $y_k \in \ker(I - A^{k*}A^k)$, where $y_k = (I + \bar{\lambda}_1 \phi_1(A)) \cdots (I + \bar{\lambda}_k \phi_k(A))x$. We deduce as before that dim $\ker(I - \phi(A)^*\phi(A)) \leqslant$ dim $\ker(I - A^{k*}A^k)$.

(b) can be proved by applying Lemma 1.4 k times. $\square$

2. Powers of contractions

In light of Theorem 1.1, for the discussions in the remaining two sections, we need only consider powers of contractions. Since some power A^k of a contraction

A has norm 1 implies the same for every power A^j with $1 \leqslant j < k$, it is sensible to define its *norm-one index* k_A by $\sup\{k \geqslant 0 : \|A^k\| = 1\}$. We have $0 \leqslant k_A \leqslant \infty$, and $\|A^k\| = 1$ for all k, $0 \leqslant k \leqslant k_A$, and $\|A^k\| < 1$ for $k > k_A$. Note that if the spectrum $\sigma(A)$ of a contraction A intersects the unit circle, then $k_A = \infty$ since in this case $\|A^k\| \geqslant \sup\{|z^k| : z \in \sigma(A)\} = 1$ for all k. In the following, we restrict ourselves to contractions on a finite-dimensional space which have no eigenvalue with modulus 1. Our first proposition lists some elementary properties of the norm-one index of such contractions.

Proposition 2.1. *If A is a contraction, then*

(a) $k_A = k_{A^*}$, *and*
(b) $k_{A^m} = [k_A/m]$ *for any $m \geqslant 1$, where $[x]$ denotes the integer part of the real x. If, moreover, A acts on a finite-dimensional space and has no eigenvalue of modulus 1, then*
(c) $k_A \leqslant \deg m_A - 1$, *where m_A denotes the minimal polynomial of A.*

Here (a) and (b) are trivial while (c) is essentially proved in [11, Theorem 2.1].

The next theorem provides another upper bound for the norm-one index (cf. [9, Corollary 3]).

Theorem 2.2. *If A is a contraction on a finite-dimensional space with no eigenvalue of modulus 1, then $k_A \leqslant \dim \ker(I - A^*A)$.*

Note that both of the bounds (in Proposition 2.1(c) and Theorem 2.2) are no greater than rank A. The former is in [14]; we now give the proof for the latter. Let A be a contraction on the space H. Consider the operator $A' = A|_{\ker(I - A^*A)}$ from $\ker(I - A^*A)$ to H. Since $\|Ax\| = \|x\|$ for any x in $\ker(I - A^*A)$, A' is an isometry. Hence

$$\dim \ker(I - A^*A) = \operatorname{rank} A' \leqslant \operatorname{rank} A$$

as asserted. On the other hand, the bound $\deg m_A - 1$ is unchanged under similarity of A while $\dim \ker(I - A^*A)$ is unchanged under unitary equivalence. For example, if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then both bounds are equal to 1, but for

$$B = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix},$$

which is similar to A, the former equals 1 while the latter equals 0. The bound $\dim \ker(I - A^*A)$ is not always attained as can be seen by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This is the case even for an irreducible A (A is *irreducible* if it is not unitarily equivalent to the direct sum of other operators). One such example is

$$A = \begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ & & & 0 \end{bmatrix},$$

where $0 < |a_j| \leqslant 1$ for all j; in this case, dim $\ker(I - A^*A)$ (resp., k_A) equals the number of a_j's (resp., the maximum number of consecutive a_j's with modulus 1, and hence we may have $k_A = 1$ while dim $\ker(I - A^*A)$ as large as half of n.

We include here a proof of Theorem 2.2 for completeness and also for an easy reference for the results in Section 3. The proof is based on the following lemma, which is a slight generalization of a special case of [9, Proposition 2].

Lemma 2.3. *Let A be a contraction on H and m and n be nonnegative integers with $m < n$. Then* dim $\ker(I - A^{m*}A^m) =$ dim $\ker(I - A^{n*}A^n) < \infty$ *if and only if $A = U \oplus B$, where U is unitary on a finite-dimensional space and $I - B^{m*}B^m$ is one-to-one. In this case, the dimension of the space on which U acts equals the common dimension of* $\ker(I - A^{m*}A^m)$ *and* $\ker(I - A^{n*}A^n)$.

Proof. Note that we always have $\ker(I - A^{n*}A^n) \subseteq \ker(I - A^{m*}A^m)$. Indeed, if x is in $\ker(I - A^{n*}A^n)$, then $\|A^n x\| = \|x\|$. Since $\|A^n x\| \leqslant \|A^m x\| \leqslant \|x\|$, this implies that $\|A^m x\| = \|x\|$ or $x \in \ker(I - A^{m*}A^m)$. Hence if $\ker(I - A^{m*}A^m)$ and $\ker(I - A^{n*}A^n)$ have equal finite dimensions, then they must be equal to each other. Let K be this common subspace. For any x in K, we have

$$\|x\| = \|A^n x\| \leqslant \|A^m(Ax)\| \leqslant \|Ax\| \leqslant \|x\|,$$

which yields that $Ax \in K$ and $A|_K$ is isometric. On a finite-dimensional space, $U \equiv A|_K$ is unitary, and hence $A = U \oplus B$ on $H = K \oplus K^\perp$. If x in $K^\perp$ is such that $(I - B^{m*}B^m)x = 0$, then $\|B^m x\| = \|x\|$. Hence $\|A^m(0 \oplus x)\| = \|0 \oplus x\|$, which implies that $0 \oplus x$ is in K and thus $x = 0$. This shows that $I - B^{m*}B^m$ is one-to-one as required.

The converse is trivial. $\square$

Proof of Theorem 2.2. Let $m =$ dim $\ker(I - A^*A)$. We need show that $\|A^{m+1}\| < 1$. There are two cases to consider. If any two of the dimensions of $\ker(I - A^{k*}A^k)$, $1 \leqslant k \leqslant m + 1$, are equal, then by Lemma 2.3 this common dimension must be 0 since A has no unit eigenvalue and hence dim $\ker(I - A^{m+1*}A^{m+1}) = 0$, which implies that $\|A^{m+1}\| < 1$. On the other hand, if the dimensions of $\ker(I - A^{k*}A^k)$, $1 \leqslant k \leqslant m + 1$, are all distinct, since this is a decreasing sequence, it follows that

$\dim \ker(I - A^{k*}A^k) \leqslant m - (k-1)$ and, in particular, $\dim \ker(I - A^{m+1*}A^{m+1}) \leqslant 0$. This results in $\|A^{m+1}\| < 1$, completing the proof. $\quad \square$

We conclude this section with a lower bound for the norm-one index.

Theorem 2.4. *If A is a contraction on an n-dimensional space and* $m = \dim \ker(I - A^*A)$, *then* $k_A \geqslant m/(n-m)$.

Proof. Let $M_0 = \ker(I - A^*A)$ and $M_k = AM_{k-1} \cap M_0$ for $k \geqslant 1$. We prove by induction that

$$M_k = A^k(\ker(I - A^{k+1*}A^{k+1})) \tag{3}$$

and

$$\dim M_k \geqslant (k+1)m - kn \tag{4}$$

for all $k \geqslant 0$. These are trivially true for $k = 0$. To prove (3), we assume that $M_k = N_k \equiv A^k(\ker(I - A^{k+1*}A^{k+1}))$ and proceed to show $M_{k+1} = N_{k+1}$. Indeed, if $x \in M_{k+1}$, then $x = Ay$ for some $y \in M_k$ and $x \in M_0$. Here y is also in N_k by the induction hypothesis and hence $y = A^k u$ for some u satisfying $\|A^{k+1}u\| = \|u\|$. We have $x = A^{k+1}u$ and

$$\|A^{k+2}u\| = \|Ax\| = \|x\| = \|A^{k+1}u\| = \|u\|.$$

This shows that u is in $\ker(I - A^{k+2*}A^{k+2})$ and x in N_{k+1}. Hence $M_{k+1} \subseteq N_{k+1}$. To prove the reverse containment, let $x \in N_{k+1}$. Then $x = A^{k+1}u$ for some u satisfying $\|A^{k+2}u\| = \|u\|$. Since u also satisfies $\|A^{k+1}u\| = \|u\|$, we have that $A^k u$ is in $N_k = M_k$. Thus $x = A(A^k u)$ is in AM_k. On the other hand, we also have

$$\|Ax\| = \|A^{k+2}u\| = \|u\| = \|A^{k+1}u\| = \|x\|,$$

that is, x is in M_0. We conclude that $x \in AM_k \cap M_0 = M_{k+1}$. This proves (3). For the proof of (4), we assume that $\dim M_k \geqslant (k+1)m - kn$ and proceed to show $\dim M_{k+1} \geqslant (k+2)m - (k+1)n$. Since A is isometric on M_0, so is it on M_k. Thus $\dim AM_k = \dim M_k \geqslant (k+1)m - kn$ by the induction hypothesis. Then

$$\begin{aligned}
\dim M_{k+1} &= \dim(AM_k \cap M_0) \\
&= \dim AM_k + \dim M_0 - \dim(AM_k \vee M_0) \\
&\geqslant ((k+1)m - kn) + m - n \\
&= (k+2)m - (k+1)n
\end{aligned}$$

as asserted.

If $k = k_A$, then $\ker(I - A^{k+1*}A^{k+1}) = \{0\}$ and thus $M_k = \{0\}$ by (3). Inequality (4) then implies that $0 \geqslant (k_A + 1)m - k_A n$ or $k_A \geqslant m/(n-m)$, completing the proof. $\quad \square$

3. Powers of $S(\phi)$

In this section, we consider a special class of contractions, namely, for each $n \geqslant 1$, the class $\mathscr{S}_n$ of contractions A on an n-dimensional space which have no eigenvalue with modulus 1 and satisfy $\text{rank}(I - A^*A) = 1$. One example of such contractions is the n-by-n Jordan block

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Generalizing this, we showed in [7, Corollary 1.3] that every contraction in the class $\mathscr{S}_n$ has a special upper-triangular matrix representation which is uniquely determined by its eigenvalues. In particular, an operator is in $\mathscr{S}_2$ if and only if it is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} a & (1 - |a|^2)^{1/2}(1 - |b|^2)^{1/2} \\ 0 & b \end{bmatrix},$$

where $|a|, |b| < 1$. The next theorem gives other characterizations of the operators in $\mathscr{S}_n$, one of which is in terms of the norm-one index.

Theorem 3.1. *For any contraction A on an n-dimensional space, the following conditions are equivalent*:

(a) *A is in $\mathscr{S}_n$*;
(b) $k_A = n - 1$;
(c) $\|A\| = \|A^{n-1}\| = 1$ *and* $\|A^n\| < 1$;
(d) $\text{rank}(I - A^{k*}A^k) = k$ *for all k*, $1 \leqslant k \leqslant n$;
(e) $\text{rank}(I - A^{k*}A^k) = k$ *for $k = n$ and for k equal to some k_0*, $1 \leqslant k_0 < n$.

The equivalence of (a) and (c) here has a more restricted analogue: an n-by-n matrix A is unitarily equivalent to J_n if and only if $\|A\| = \|A^{n-1}\| = 1$ and $A^n = 0$ (cf. [15, p. 352]). Also note that there is established in [2, Section 1] and [3, pp. 3052–3053] a model for contractions A on an n-dimensional space satisfying condition (c) above and having distinct eigenvalues.

Proof of Theorem 3.1. (a) $\Rightarrow$ (b). If A is in $\mathscr{S}_n$, then $\dim \ker(I - A^*A) = n - 1$. Hence $k_A \geqslant n - 1$ by Theorem 2.4. On the other hand, we also have $k_A \leqslant n - 1$ by Theorem 2.2. This proves (b).

(b) $\Leftrightarrow$ (c). This is trivial from the definition of k_A.

(c) $\Rightarrow$ (d). Assume that (c) holds and consider the dimensions $m_k \equiv \dim \ker(I - A^{k*}A^k)$, $1 \leqslant k \leqslant n$. If any two of them are equal, say, $m_i = m_j$ $(i < j)$, then, since A has no unit eigenvalue by $\|A^n\| < 1$, we deduce from Lemma 2.3 that $m_i = 0$.

This would imply that $\|A^i\| < 1$, contradicting the assumption $\|A\| = \|A^{n-1}\| = 1$. Thus the m_k's must all be distinct. Hence they are strictly decreasing with $m_1 \leqslant n - 1$. This can happen only when $m_k = n - k$ for all k, $1 \leqslant k \leqslant n$, which is the same as the assertion in (d).

(d) $\Rightarrow$ (e). This is trivial.

(e) $\Rightarrow$ (a). Assume that (e) holds. Since $\mathrm{rank}(I - A^{n*}A^n) = n$, A has no unit eigenvalue. As before, consider the dimensions $m_k = \dim \ker(I - A^{k*}A^k)$, $1 \leqslant k \leqslant k_0$. If any two of the m_k's are equal, say, $m_i = m_j$ ($1 \leqslant i < j \leqslant k_0$), then $m_i = 0$ by Lemma 2.3. This would imply $n - k_0 = m_{k_0} \leqslant m_i = 0$, a contradiction. Hence the m_k's are strictly decreasing with $m_1 \leqslant n - 1$. If $m_1 < n - 1$, then we infer from this that $m_{k_0} < n - k_0$, contradicting our assumption. Thus we must have $m_1 = n - 1$, which proves that A is in $\mathscr{S}_n$. $\square$

Contractions in $\mathscr{S}_n$ and their infinite-dimensional analogues are the so-called compressions of the shift first studied by Sarason [12] in 1967. Their properties were intensively investigated in the 1960s and 1970s resulting in the establishment of the Jordan form for the class of C_0 contractions, in which compressions of the shift play the role of building blocks. A full account of the theory can be found in [1,13]. Here we briefly sketch this functional representation for the $\mathscr{S}_n$ contractions and then relate the results in Theorem 3.1 to ones in approximation theory and even to ones involving Toeplitz and Hankel operators.

Let H^2 be the usual Hardy space on the unit circle and ϕ be a Blaschke product with zeros $\lambda_1, \ldots, \lambda_n$ in the open unit disc $\mathbb{D}$. If S denotes the unilateral shift $(Sf)(z) = zf(z)$ for f in H^2, then $S(\phi)$, the *compression of the shift* associated with ϕ, is the operator on $H^2 \ominus \phi H^2$ defined by $S(\phi) = P(S|H^2 \ominus \phi H^2)$, where P is the (orthogonal) projection from H^2 onto $H^2 \ominus \phi H^2$, or, in other words, $S(\phi)$ is the one appearing in the lower-right corner in the operator matrix

$$S = \begin{bmatrix} * & * \\ 0 & S(\phi) \end{bmatrix} \quad \text{on } H^2 = \phi H^2 \oplus (H^2 \ominus \phi H^2).$$

It is known that the dimension of $H^2 \ominus \phi H^2$ equals n and an n-dimensional operator is in $\mathscr{S}_n$ if and only if it is unitarily equivalent to $S(\phi)$ for some Blaschke product ϕ with n zeros. A linking bridge between the compressions of the shift and function theory is provided by the result of Sarason [12, Proposition 2.1] that Alg $S(\phi)$ *and* $H^\infty/\phi H^\infty$ *are isometrically isomorphic under the mapping* $f(S(\phi)) \mapsto f + \phi H^2$, where Alg $S(\phi)$ is the algebra $\{f(S(\phi)) : f \in H^\infty\}$ with the operator norm and $H^\infty/\phi H^\infty$ is the quotient algebra equipped with the quotient norm $\|f + \phi H^\infty\| = \inf\{\|f - \phi g\|_\infty : g \in H^\infty\}$ (H^∞ is the algebra of bounded analytic functions on $\mathbb{D}$). Adapting to the present situation, we have, for any two finite Blaschke products ϕ_1 and ϕ_2, the equality $\|\phi_1(S(\phi_2))\| = \inf\{\|\phi_1\bar{\phi}_2 - g\|_\infty : g \in H^\infty\}$ and, moreover, this infimum is attained for some unique rational function g_0 in $A(\mathbb{D})$ (= Banach algebra of analytic functions on $\mathbb{D}$ which have continuous extensions to $\partial\mathbb{D}$) (cf. [12, Corollary to Proposition 5.1]). This expression for the norm of $\phi_1(S(\phi_2))$ in

terms of the distance dist($\phi_1\bar{\phi}_2$, H^∞) reduces our norm-one index problem to one in approximation theory. It turns out that this latter problem has also been studied before. In 1972, Poreda showed that a continuous function f on $\partial\mathbb{D}$ is such that dist(f, $A(\mathbb{D})$) = $\|f\|_\infty$ if and only if it has a nonzero constant modulus on $\partial\mathbb{D}$ and $f(\partial\mathbb{D})$ has a strictly negative winding number with respect to the origin (cf. [10, Theorem 1] and [6, Theorem 1.3] for more general versions). In our case, this means that dist($\phi_1\bar{\phi}_2$, $A(\mathbb{D})$) = 1 if and only if the number of zeros of ϕ_1 is less than that of ϕ_2. Combining these, we obtain the equivalence of the first three conditions in the next theorem.

Theorem 3.2. *Let ϕ_1 and ϕ_2 be Blaschke products with m and n zeros* (*counting multiplicity*), *respectively. Then the following conditions are equivalent*:

(a) $\|\phi_1(S(\phi_2))\| < 1$;
(b) dist($\phi_1\bar{\phi}_2$, H^∞) < 1;
(c) $m \geqslant n$;
(d) $T_{\phi_1\bar{\phi}_2}$ *is left invertible*;
(e) $\|H_{\phi_1\bar{\phi}_2}\| < 1$.

Here (d) and (e) yield connections of our problem with properties of certain Toeplitz and Hankel operators. Recall that the *Toeplitz operator T_f* (on H^2) and Hankel operator H_f (from H^2 to $L^2 \ominus H^2$) with symbol f in L^∞ are defined by

$$T_f g = P_1(fg) \quad \text{and} \quad H_f g = P_2(fg)$$

for $g \in H^2$, where P_1 and P_2 denote the (orthogonal) projections from L^2 onto H^2 and from L^2 onto $L^2 \ominus H^2$, respectively. The proof of the equivalence of (b) and (d) is in [4, Theorem 7.30] while that of (b) and (e) from the fact that $\|H_f\| =$ dist(f, H^∞) (cf. [16, Theorem 15.14]).

We remark that parts of Theorem 3.2 are valid for more general functions ϕ_1 and ϕ_2, and the equivalence of (a), (b) and (c) in Theorem 3.1 is a consequence of Theorem 3.2. However, for finite Blaschke products, our elementary proof for Theorem 3.1 is more lucid than the general function-theoretic arguments.

References

[1] H. Bercovici, Operator Theory and Arithmetic in H^∞, American Mathematical Society, Providence, 1988.
[2] B. Cole, K. Lewis, J. Wermer, A characterization of Pick bodies, J. London Math. Soc. 48 (2) (1993) 316–328.
[3] B. Cole, J. Wermer, Isometries of certain operator algebras, Proc. Amer. Math. Soc. 124 (1996) 3047–3053.
[4] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.
[5] K. Fan, Analytic functions of a proper contraction, Math. Z. 160 (1978) 275–290.

[6] T.W. Gamelin, J.B. Garnett, L.A. Rubel, A.L. Shields, On badly approximable functions, J. Approx. Theory 17 (1976) 280–296.

[7] H.-L. Gau, P.Y. Wu, Lucas' theorem refined, Linear Multilinear Algebra 45 (1999) 359–373.

[8] P.R. Halmos, A Hilbert Space Problem Book, second ed., Springer, New York, 1982.

[9] Y.-C. Li, M.-H. Shih, The length of contractibility of compact contractions on a Hilbert space, Linear Algebra Appl., in press.

[10] S.J. Poreda, A characterization of badly approximable functions, Trans. Amer. Math. Soc. 169 (1972) 249–256.

[11] V. Pták, Lyapunov equations and Gram matrices, Linear Algebra Appl. 49 (1983) 33–55.

[12] D. Sarason, Generalized interpolation in H^∞, Trans. Amer. Math. Soc. 127 (1967) 179–203.

[13] B. Sz.-Nagy, C. Foiaş, Harmonic Analysis of Operators on Hilbert Space, North Holland, Amsterdam, 1970.

[14] W.P. Wardlaw, Problem 1179, Math. Mag. 56 (1983) 326.

[15] P.Y. Wu, A numerical range characterization of Jordan models, Linear Multilinear Algebra 43 (1998) 351–361.

[16] N. Young, An Introduction to Hilbert Space, Cambridge University Press, Cambridge, 1988.